eLOQC[a] IV

Efficient Computation: Failure Models and Codes

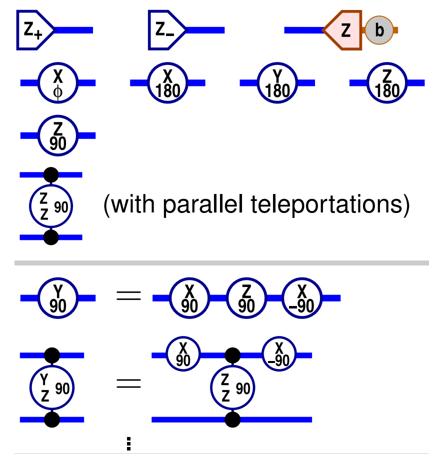
- How the postselected gates fail.
- Stabilizer codes for the Z-measurement error model.
- Simple encoded operations.

[a] Efficient Linear Optics Quantum Computation [1]

knill@lanl.gov, http://www.lanl.gov/~knill

Failure Model for Ideal eLOQC Gates

- Detected Z-meas. error models

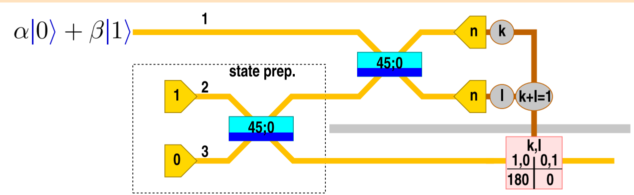


Physical	Encoded
No error	No error
No error	No error
No error	Z, prob. f
Z, prob. f; indep.; Z, prob. f	Z, prob. f; indep.; Z, prob. f
No error	Y, prob. f
Y, prob. f; indep.; Z, prob. f	Y, prob. f; indep.; Z, prob. f

(• Sequential teleportation is less destructive.)

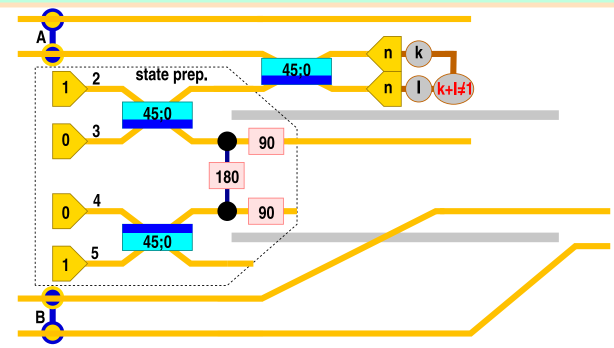
Failure Behavior of the Teleportation Schemes

- Tel_1:



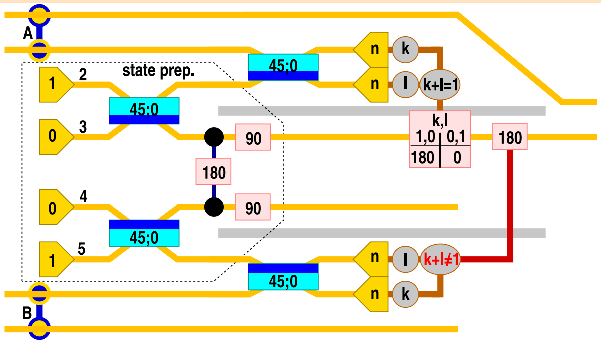
  - $k+l=0$: Input was $|0\rangle$.
  - $k+l=2$: Input was $|1\rangle$.
  - Failure of Tel_1 is measurement of $|0\rangle$ vs. $|1\rangle$. Probability of $\mathbf{n}$-measurement error: $1/2$.
- Generalization to Tel_k: [p = number of photons detected.]
  - $p=0$: Input was $|0\rangle$. $p=k+1$: Input was $|1\rangle$.
  - Failure of Tel_k is measurement of $|0\rangle$ vs. $|1\rangle$. Probability of $\mathbf{n}$-measurement error: $1/(k+1)$.

Failure Modes for $(ZZ)_{90^\circ}$: Sequential



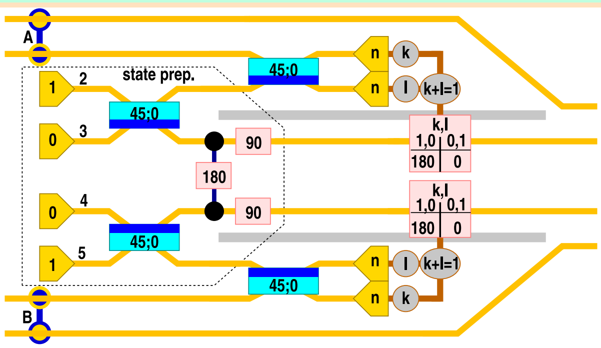
- Sequential teleportation:
  Teleport A.
  - Probability f, measure $(Z)_A$.
    Effect: As if measured before $(ZZ)_{90^\circ}$ was attempted.

Failure Modes for $(ZZ)_{90^\circ}$: Sequential



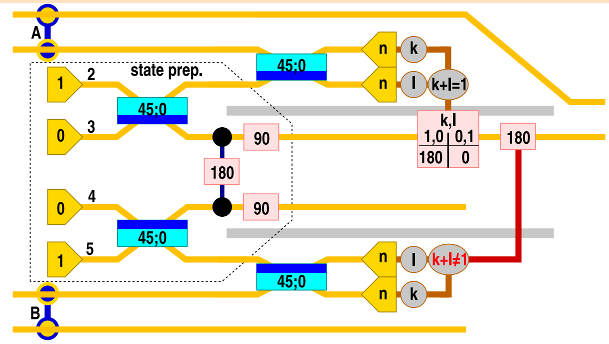
- Sequential teleportation:
  Teleport A successfully. Teleport B.
  - Probability f, measure $(Z)_B$.
    Fix it so that: As if measured after $(ZZ)_{90^\circ}$ was attempted.

Failure Modes for $(ZZ)_{90^\circ}$: Sequential



- Sequential teleportation:
  Teleport A successfully. Teleport B successfully.
- Success probability: $(1-f)^2$.

Failure Modes for $(ZZ)_{90^\circ}$: Parallel



- Parallel teleportation:
  - Probability $(1-f)f$, measure $(Z)_B$ but not $(Z)_A$.
    Fix it so that: As if measured after $(ZZ)_{90^\circ}$ was attempted.

  ⋮
- Independent measurement errors, as if after the operation.

A Z-Measurement Correcting Code

- $X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- Logical states for code $\mathcal{X}_2$:

$$\begin{aligned} |\mathtt{0}\rangle_{L(1,2)} &= \frac{1}{\sqrt{2}}\left(|\mathtt{00}\rangle_{12} + |\mathtt{11}\rangle_{12}\right) \\ |\mathtt{1}\rangle_{L(1,2)} &= \frac{1}{\sqrt{2}}\left(|\mathtt{01}\rangle_{12} + |\mathtt{10}\rangle_{12}\right) \end{aligned}$$

- Stabilizer for $\mathcal{X}_2$: I.e. Pauli products U s.t. $U|\psi\rangle_L = \lambda_U|\psi\rangle_L$.
  $\{II = \mathbb{I}, XX = (X)_1(X)_2\}$, *generated* by $\{XX\}$.
- Projector for $\mathcal{X}_2$:
  $P_{\mathcal{X}_2} = \frac{1}{2}(II + XX)$.

Properties of $\mathcal{X}_2$

- Possible errors, which happened is known:

$$\begin{array}{rclrcl} II,\ P_{0*} &=& \frac{1}{2}(II+ZI), & P_{1*} &=& \frac{1}{2}(II-ZI),\\ P_{*0} &=& \frac{1}{2}(II+IZ), & P_{*1} &=& \frac{1}{2}(II-IZ). \end{array}$$

- Effects on states, by example:

$$\begin{array}{c} \alpha|0\rangle_L \\ + \\ \beta|1\rangle_L \end{array} = \left.\begin{array}{c} \frac{\alpha}{\sqrt{2}}\big(|00\rangle+|11\rangle\big) \\ + \\ \frac{\beta}{\sqrt{2}}\big(|01\rangle+|10\rangle\big) \end{array}\right\} \xrightarrow{P_{0*}} \left\{\begin{array}{c} \alpha|00\rangle \\ + \\ \beta|01\rangle \end{array}\right\} \xrightarrow{U_{0*}} \left\{\begin{array}{c} \alpha|0\rangle_L \\ + \\ \beta|1\rangle_L \end{array}\right.$$

$$\xrightarrow{P_{*1}} \left\{\begin{array}{c} \alpha|11\rangle \\ + \\ \beta|01\rangle \end{array}\right\} \xrightarrow{U_{*1}} \left\{\begin{array}{c} \alpha|0\rangle_L \\ + \\ \beta|1\rangle_L \end{array}\right.$$

- Z-measurement errors can be corrected.

Stabilizer Codes

- $S = \{U_1, U_2, \ldots\}$: Commuting set of Pauli products.

  Examples: $S_0 = \{Z\}$,
  $S_1 = \{XXI, IXX\}$,
  $S_2 = \{XXXI, IYYY, ZIZZ\}$

- *Stabilizer code* $\mathcal{C}(S, \lambda)$: Common eigenspace of S, character λ.

  Examples:
  $$\begin{array}{rcl} \mathcal{C}(S_0,1) &=& \mathrm{span}\big(|0\rangle\big)\\ \mathcal{C}(S_1,1) &=& \mathrm{span}\big(|000\rangle+|110\rangle+|011\rangle+|101\rangle,\\ && |001\rangle+|010\rangle+|100\rangle+|111\rangle\big) \end{array}$$

- Projector for $\mathcal{C}(S,1)$: $P_S = \frac{1}{2}(I+U_1)\,\frac{1}{2}(I+U_2)\ldots$.
  - Let: $|\psi_p\rangle = \frac{1}{2}(I+U)|\psi\rangle$. Then:
  $$\begin{array}{rcl} U|\psi_p\rangle &=& U\left(\frac{1}{2}(I+U)\right)|\psi\rangle\\ &=& \frac{1}{2}(U+U^2)|\psi\rangle\\ &=& \frac{1}{2}(U+I)|\psi\rangle\\ &=& |\psi_p\rangle \end{array}$$

When is Detected $(I \pm U)/2$ Correctable?

- $S = \{U_1, \ldots, U_k\}$ Pauli products, commuting; $\mathcal{C} = \mathcal{C}(S,1)$; U a Pauli product.

- **Theorem 1:** Detected $(\mathbb{I} \pm U)/2$ is correctable if U anti-commutes with U_k for some k (i.e. $U \notin S^\perp$ by definition).

  Proof:
  - Let $|\psi\rangle \in \mathcal{C}$, $iV = UU_k$.
  - Then V is a Pauli product.
  - $$\begin{array}{rcl} \frac{1}{2}(\mathbb{I}\pm U)|\psi\rangle &=& \frac{1}{2}(\mathbb{I}\pm U)\frac{1}{2}(\mathbb{I}+U_k)|\psi\rangle\\ &=& \frac{1}{4}(\mathbb{I}\pm U+U_k\pm UU_k)|\psi\rangle\\ &=& \frac{1}{4}(\mathbb{I}\pm U+U_k\pm iV)|\psi\rangle\\ &=& \frac{1}{4}(\mathbb{I}\pm iVU_k+U_k\pm iV)|\psi\rangle\\ &=& \frac{1}{4}(\mathbb{I}\pm iV)(\mathbb{I}+U_k)|\psi\rangle\\ &=& \frac{1}{\sqrt{2}}e^{\pm iV\pi/4}|\psi\rangle \end{array}$$
  - Correct error by applying $e^{\mp iV\pi/4}$.

When is Detected $(I \pm U)/2$ Correctable?

- $S = \{U_1, \ldots, U_k\}$ Pauli products, commuting; $\mathcal{C} = \mathcal{C}(S,1)$; U a Pauli product.

- **Theorem 1:** Detected $(\mathbb{I} \pm U)/2$ is correctable if U anti-commutes with U_k for some k (i.e. $U \notin S^\perp$ by definition).
  - Also ok: $U \propto$ a product of the U_k (i.e. $U \in \langle S\rangle$ by definition).
  - $(\mathbb{I} \pm U)/2$ is not correctable otherwise.
  - Def. U is *detectable* by S if $U \in \langle S\rangle$ or $U \notin S^\perp$.

- Examples:
  $$\left.\begin{array}{rcl} S_1 &=& \{XXI, IXX\}\\ S_2 &=& \{XXX\} \end{array}\right\} \text{ for } Z\text{-measurement errors.}$$
  $$S_3 = \{XXXX, YYYY\} \quad \text{for any one-qubit meas. errors.}$$
  $$\dim \mathcal{C}(S_1) = 2^1, \dim \mathcal{C}(S_2) = 2^2, \dim \mathcal{C}(S_3) = 2^2.$$

Correcting Multiple Measurement Errors

- $S=\{U_1,\ldots,U_k\}$ Pauli products, commuting; $\mathcal{C}=\mathcal{C}(S,1)$; V_i commuting Pauli products.
- **Theorem 2:** Detected $\prod_{i=1}^{n}((\mathbb{I}\pm V_i)/2)$ is correctable if each $\prod_j V_{i_j}$ is detectable by S.

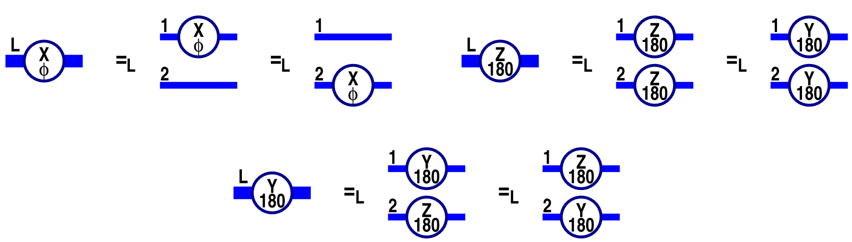
  Proof: Can assume that each $V_i \notin S^{\perp}$.
  - Construct $W_i \in \langle S\rangle$ such that: W_i anticommutes with V_i, W_i commutes with V_j for $j \neq i$.
    - By induction: Given desired $W_1,\ldots W_{k-1}$.
    - Choose $W_k \in S$ anticommuting with V_k.
    - For each $l<k$, if W_k anticommutes with V_l, multiply W_k by W_l. Final W_k is as required.
  - Apply method in proof of Thm. 1 to each $(\mathbb{I}\pm V_i)/2$, W_i.
- The converse also holds.

Encoded Operations for $\mathcal{X}_2$

- $\mathcal{X}_2=\mathcal{C}(\{XX\},1)$: $|0\rangle_L=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$, $|1\rangle_L=\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)$
- The following hold when restricted to $\mathcal{X}_2$:

$$\begin{array}{llllllllll}(I)_L & =_L & II & =_L & XX & \quad (Z)_L & =_L & ZZ & =_L & -YY\\ (X)_L & =_L & XI & =_L & IX & \quad (Y)_L & =_L & YZ & =_L & ZY\end{array}$$

- Error-free encoded rotations:



Encoded Operations for $\mathcal{X}_2$

- $\mathcal{X}_2=\mathcal{C}(\{XX\},1)$: $|0\rangle_L=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$, $|1\rangle_L=\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)$
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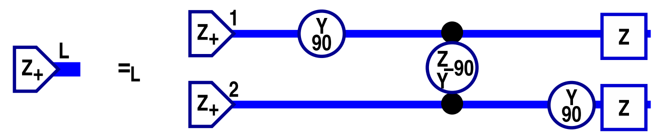
- Encoded state preparation, error-free if successful:



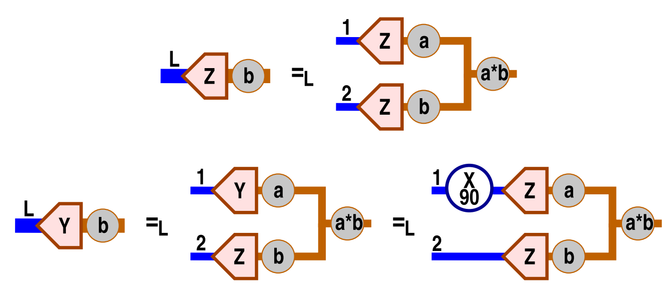
Stabilizer: $ZI \longrightarrow -YY$
$IZ \longrightarrow ZZ$

Encoded Operations for $\mathcal{X}_2$

- $\mathcal{X}_2=\mathcal{C}(\{XX\},1)$: $|0\rangle_L=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$, $|1\rangle_L=\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)$
- The following hold when restricted to $\mathcal{X}_2$:

$$\begin{array}{llllllllll}(I)_L & =_L & II & =_L & XX & \quad (Z)_L & =_L & ZZ & =_L & -YY\\ (X)_L & =_L & XI & =_L & IX & \quad (Y)_L & =_L & YZ & =_L & ZY\end{array}$$

- Encoded measurements, error free:

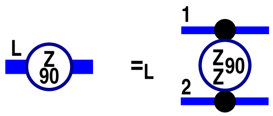


Encoded Operations for $\mathcal{X}_2$

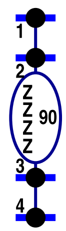
- $\mathcal{X}_2 = \mathcal{C}(\{XX\}, 1)$: $|0\rangle_{\mathsf{L}} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), |1\rangle_{\mathsf{L}} = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$
- The following hold when restricted to $\mathcal{X}_2$:

$$\begin{array}{lllll} (I)_{\mathsf{L}} & =_{\mathsf{L}} & II & =_{\mathsf{L}} & XX \\ (X)_{\mathsf{L}} & =_{\mathsf{L}} & XI & =_{\mathsf{L}} & IX \end{array} \qquad \begin{array}{lllll} (Z)_{\mathsf{L}} & =_{\mathsf{L}} & ZZ & =_{\mathsf{L}} & -YY \\ (Y)_{\mathsf{L}} & =_{\mathsf{L}} & YZ & =_{\mathsf{L}} & ZY \end{array}$$

- Other encoded operations:
  - Logical Z_{90°:



    - Measurement errors require recovery attempts.
  - Logical $(ZZ)_{90^\circ}$: By teleportation. In principle:



References

[1] E. Knill, R. Laflamme, and G. Milburn. A scheme for efficient linear optics quantum computation. *Nature*, 409:46–52, 2001.

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