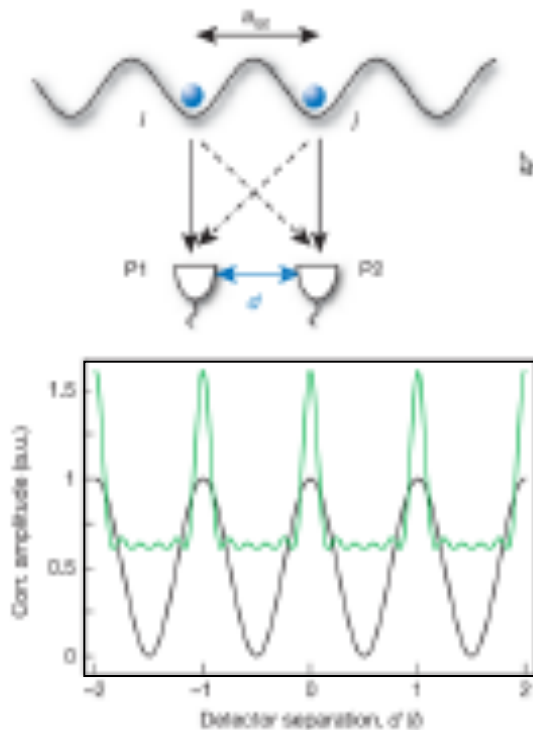


Quench dynamics of interacting bosons in 1-dimension

Natan Andrei



Hanbury-Brown Twiss effect

I Bloch et al.



with:



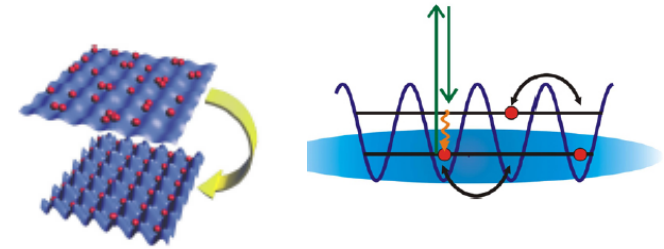
Deepak Iyer – Rutgers U

Quenching and Time Evolution

- Prepare an isolated quantum many-body system in state $|\Phi_0\rangle$, typically eigenstate of H_0
- At $t = 0$ turn on interaction H_1 , and evolve system with $H = H_0 + H_1$:

$$|\Phi_0, t\rangle = e^{-iHt} |\Phi_0\rangle$$

- Many experiments: cold atom systems, nano-devices, molecular electronics, photonics
- New technologies, old questions



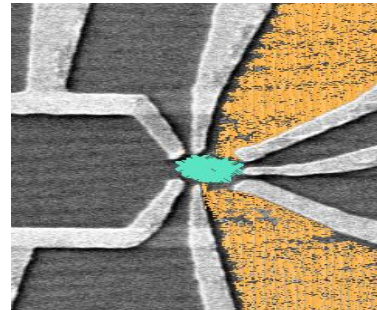
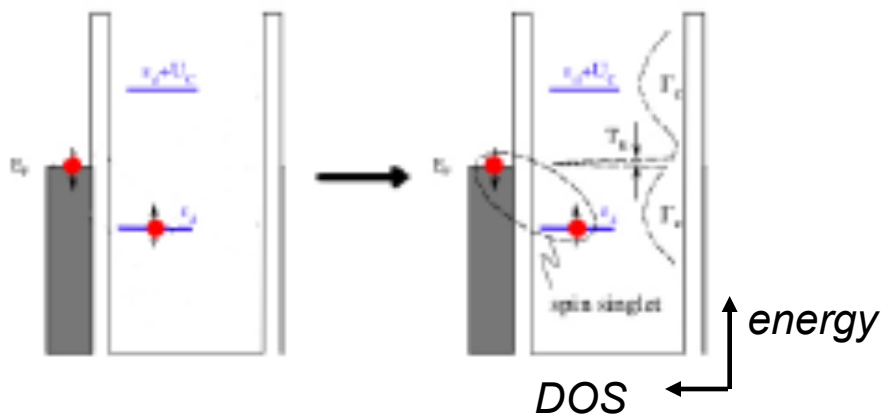
Questions: (as an introduction)

- Time evolution of observables $\langle O(t) \rangle = \langle \Phi_0, t | O | \Phi_0, t \rangle$
- Evolution of correlation functions in quenched systems

$$\langle \Phi_0 | A(t + \tau) B(t) | \Phi_0 \rangle = \langle \Phi_0, t | A(\tau) B | \Phi_0, t \rangle$$

- Dynamics of evolution of the Kondo resonance in a quantum dot: Anderson model

Quench at $t = 0$: couple dot to leads



Measure time evolution of the Kondo peak.

- Time resolved photo emission spectroscopy
- Time dependent current

Closed systems: quenching – long time limit, thermalization

- **Manifestation of interactions in time evolution dynamics**

The subject of this talk: bosons in 1-d

Time evolution and statistical mechanics:

- **Long time limit and thermalization:**

- is there a limit $\bar{O} = \lim_{t \rightarrow \infty} \langle O(t) \rangle$?
- is there a density operator ρ such that $\bar{O} = \text{tr}(\rho O)$?

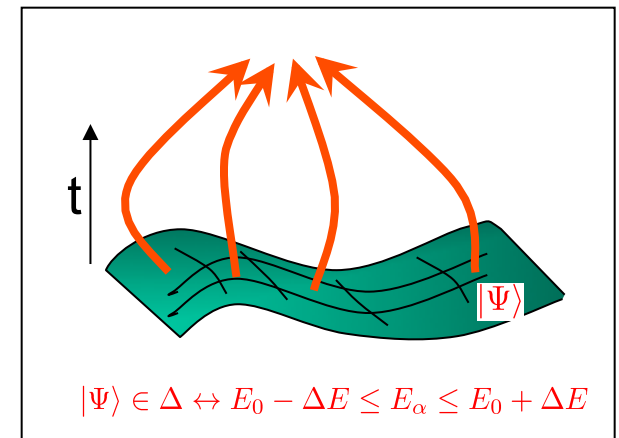
Does it depend on $E_0 = \langle \Phi_0 | H | \Phi_0 \rangle$ not on $|\Phi_0\rangle$

- **Scenarios of thermalization (Rigol et al)**

- Diagonal matrix elements of physical operators $A_{\alpha\alpha} = \langle \psi_\alpha | A | \psi_\alpha \rangle$ do not fluctuate much around constant energy surface (ETH-eigenstate thermalization hypothesis, Deutsch 92, Srednicki 94)
- Overlaps $|C_\alpha|^2 = |\langle \psi_\alpha | \Phi_0 \rangle|^2$ do not fluctuate on the energy surface for reasonable IC
- Both fluctuate but are uncorrelated

- **Thermalization, Integrability, Non-Boltzmannian ensembles, Rigol, Cardy, Cazalilla..**

If conservation laws are present – how do they affect dynamics of thermalization?

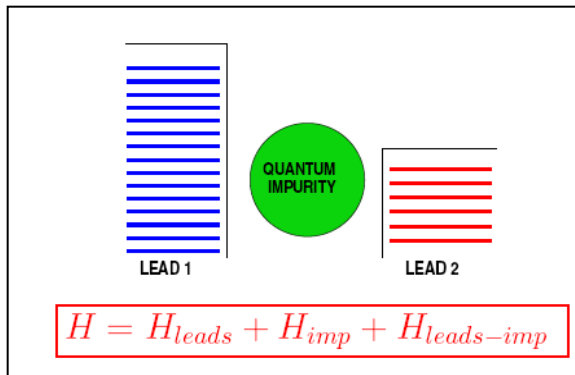


Open systems: quenching and non-thermalization, transport

Nonequilibrium currents

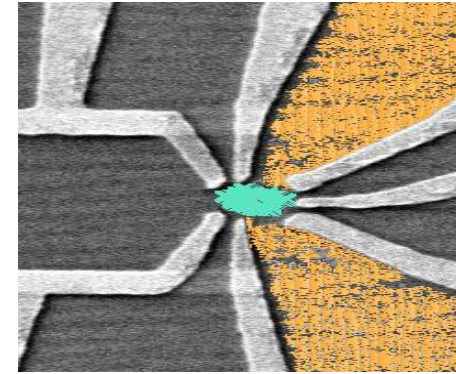
Goldhaber-Gordon *et al*, Conenwett *et al*, Schmid *et al*

- **Two baths or more**
time evolution in a nonequilibrium set up



Interplay - strong correlations and nonequilibrium

Quench
or
Keldysh



- $t \leq 0$, leads decoupled, system described by: ρ_o
- $t = 0$, couple leads to impurity
- $t \geq 0$, evolve with $H(t) = H_0 + H_1$
- **What is the time evolution of the current $\langle I(t) \rangle$?**
- **Asymptotic limit?**
- **Under what conditions is there a steady state? Dissipation?**
- **Steady state – is there a non thermal ρ_s ?**
- **New effects out of equilibrium? New scales? Phase transitions, universality?**

Quenching in 1-d systems

Physical Motivation:

- Natural dimensionality of many systems:
 - wires, waveguides, optical traps, edges
- Impurities: Dynamics dominated by s-waves, reduces to 1D system
- Many experimental realizations: Cold atom traps, nano-systems..

Special features of 1- d : theoretical

- Strong quantum fluctuations for any coupling strength
- Powerful mathematical methods:
 - RG methods, Bosonization, CFT methods, **Bethe Ansatz approach**
- **Bethe Ansatz approach: allows complete diagonalization of H**
- **Experimentally realizable:** *Hubbard model, Kondo model, Anderson model, Lieb-Liniger model, Sine-Gordon model, Heisenberg model, Richardson model..*
- **BA \longrightarrow Quench dynamics of many body systems? Exact!**

Others approaches: Keldysh, t-DMRG, t-NRG, t-RG

Much work in context of Luttinger Liquid: Cazalilla et al, Mitra et al

Time Evolution and the Bethe Ansatz

- A given state $|\Phi_0\rangle$ can be formally time evolved in terms of a complete set of energy eigenstates $|F^\lambda\rangle$

$$|\Phi_0\rangle = \sum_\lambda |F^\lambda\rangle \langle F^\lambda | \Phi_0\rangle \longrightarrow |\Phi_0, t\rangle = e^{-iHt} |\Phi_0\rangle = \sum_\lambda e^{-i\epsilon_\lambda t} |F^\lambda\rangle \langle F^\lambda | \Phi_0\rangle$$

If H integrable \rightarrow eigenstates $|F^\lambda\rangle$ are known via the Bethe-Ansatz

- Use the Bethe Ansatz to study quenching and evolution
- New technology is necessary:
 - **Standard approach:** PBC \rightarrow Bethe Ansatz eqns \rightarrow spectrum \rightarrow thermodynamics
 - **Non equilibrium entails more difficulties:**
 - i. Compute overlaps
 - ii. Sum over complete basis
 - iii. Take limits

Some progress was made - J. S. Caux et al, V. Gritsev et al

The Bethe Ansatz - Review

- General N - particle state

$$|F^\lambda\rangle_N = \int d^N x F^\lambda(\vec{x}) \prod_{j=1}^N \psi^\dagger(x_j) |0\rangle$$

- Wave function very complicated in general

- The BA -wave function much simpler -

Product of single particles wave functions $f_\lambda(x)$ and S-matrices S_{ij} ,

i. divide configuration space into $N!$ regions Q , $\{x_{Q1} \leq \dots, \leq x_{QN}\}$

ii particles interact only when crossing: inside a region product of single particle wave funct.

iii. assign amplitude A^Q to region Q

iv. amplitudes related by S-matrices S_{ij} (e.g. $A^{132} = S^{23} A^{123}$)

$$\rightarrow F^\lambda(\vec{x}) = \sum_{Q \in S_N} A^Q \prod_j f_{\lambda_{Qj}}(x_j)$$

- v. do it consistently: **Yang-Baxter relation**

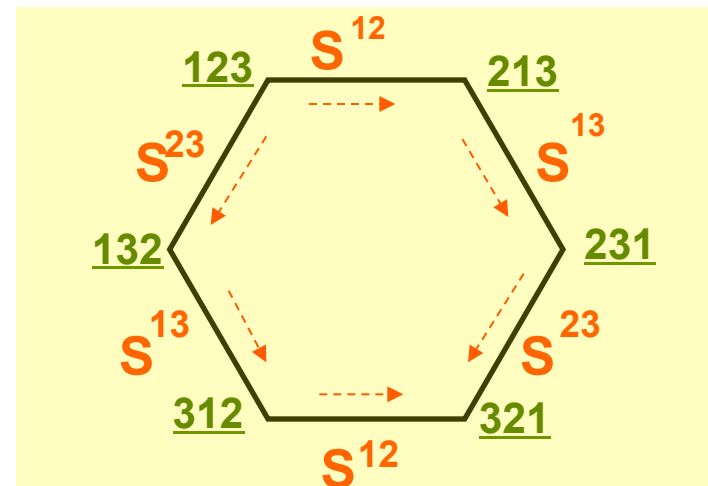
$$S^{12} S^{13} S^{23} = S^{23} S^{13} S^{12}$$

Example:

$$H = - \sum_{j=1}^N \partial_{x_j}^2 + c \sum_{i < j} \delta(x_i - x_j)$$

$$f_\lambda(x) \sim e^{i\lambda x}$$

$$S_{ij}(\lambda_i - \lambda_j) = \frac{\lambda_i - \lambda_j + ic}{\lambda_i - \lambda_j - ic}$$



The contour representation

Instead of $|\Phi_0\rangle = \sum_{\lambda} |F^{\lambda}\rangle \langle F^{\lambda}|\Phi_0\rangle$ introduce (directly in infinite volume):

Contour representation of $|\Phi_0\rangle$

$$|\Phi_0\rangle = \int_{\gamma} d^N \lambda |F^{\lambda}\rangle \langle F^{\lambda}|\Phi_0\rangle$$

V. Yudson, sov. phys. *JETP* (1985)

Computed S-matrix of Dicke model

with: $|F^{\lambda}\rangle$ Bethe eigenstate

$|F^{\lambda}\rangle$ obtained from Bethe eigenstate by setting $S = I$ - easier to calculate

γ contour in momentum space $\{\lambda\}$ determined by **pole structure** of $S(\lambda_i - \lambda_j)$

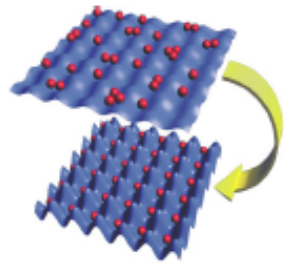
Note: in the infinite volume limit momenta $\{\lambda\}$ are not quantized
- no Bethe Ansatz equations, $\{\lambda\}$ free parameters

then:

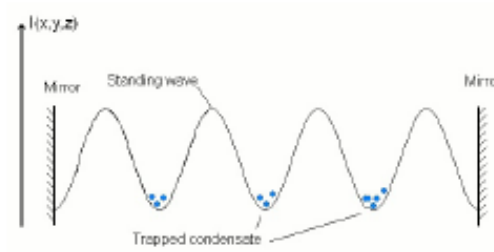
$$|\Phi_0, t\rangle = \int_{\gamma} d^N \lambda e^{-iE(\lambda)t} |F^{\lambda}\rangle \langle F^{\lambda}|\Phi_0\rangle$$

Boson Systems - experiments

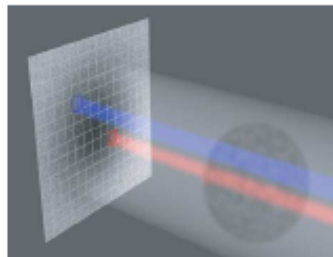
Bosons in optical traps



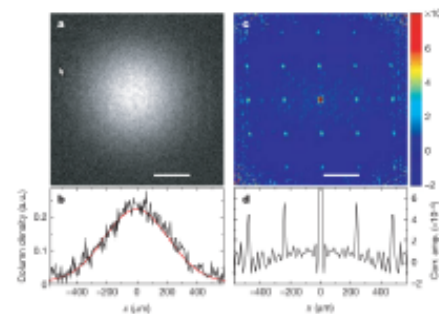
Superfluid Mott insulator transition



Mott insulator – initial condition



Imaging of density cloud using a CCD



Density and noise correlation functions

Bloch et al (Nature 2005, Rev Mod Phys 2008)

Interacting bosonic system

Bosons in a 1-d with short range interactions

$$H = - \int dx b^\dagger(x) \partial^2 b(x) + c \int dx b^\dagger(x) b(x) b^\dagger(x) b(x)$$

c - coupling constant

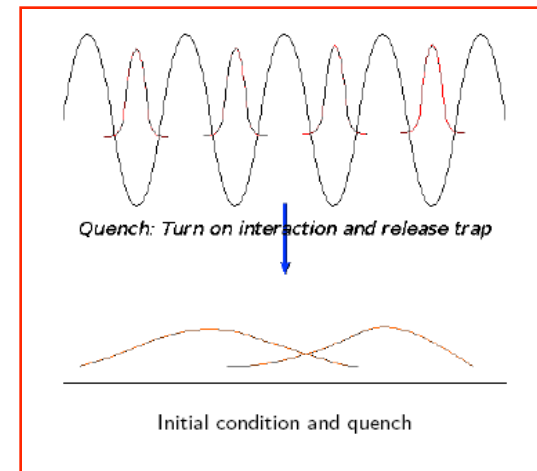
$c > 0$ repulsive

$c < 0$ attractive

Equivalently:

$$H = - \sum_{j=1}^N \partial_{x_j}^2 + c \sum_{i < j} \delta(x_i - x_j)$$

- Initial condition I : bosons in a periodic optical lattice



- Initial condition II : bosons in a trap - condensate



Bosonic system – BA solution

The N-boson eigenstatestate (Lieb-Linniger '67)

$$|\lambda_1, \dots, \lambda_N\rangle = \int^y \prod_{i < j} Z_{ij}^y(\lambda_i - \lambda_j) \prod_j e^{i\lambda_j y_j} b^\dagger(y_j) |0\rangle$$

- Eigenstates labeled by Momenta

$$\lambda_1, \dots, \lambda_N$$

- i. - *satisfy BA eqns*, if PBC imposed,
- *unconstrained*, in open space
- ii. real, - for $c > 0$
- complex “n-strings” - for $c < 0$



- The 2-particle S-matrix

$$S_{ij}(\lambda_i - \lambda_j) = \frac{\lambda_i - \lambda_j + ic}{\lambda_i - \lambda_j - ic}$$



Dynamic factor:

$$Z_{ij}^y(\lambda_i - \lambda_j) = \frac{\lambda_i - \lambda_j - ic \operatorname{sgn}(y_i - y_j)}{\lambda_i - \lambda_j - ic}$$

Note: poles of the S-matrix at: $\lambda_i = \lambda_j + ic$

- The energy eigenvalues

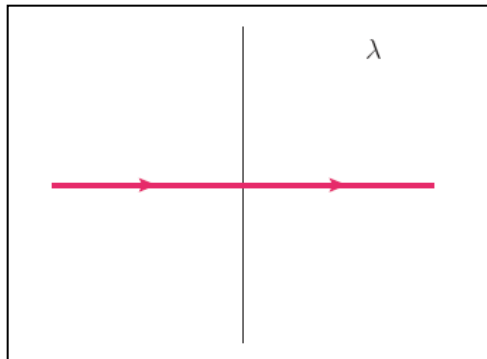
$$H|\lambda_1, \dots, \lambda_N\rangle = \sum_j \lambda_j^2 |\lambda_1, \dots, \lambda_N\rangle$$

bosonic system: contour representation

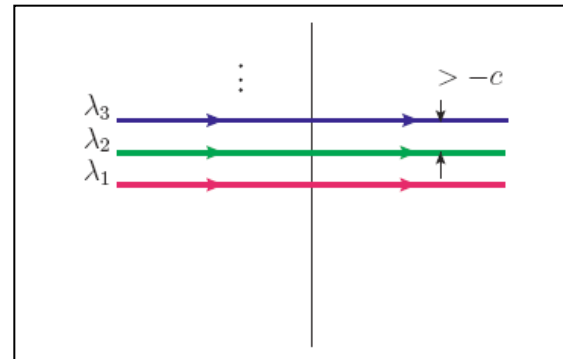
“Central theorem”

denote: $\theta(\vec{x}) = \theta(x_1 > x_2 > \dots > x_N)$

$$\begin{aligned}
 |\Phi_0\rangle &= \int_x \Phi_0(\vec{x}) b^\dagger(x_N) \cdots b^\dagger(x_1) |0\rangle = \\
 &= \int_{x,y} \int_\lambda \theta(\vec{x}) \Phi_0(\vec{x}) \prod_{i<j} \frac{\lambda_i - \lambda_j - ic \operatorname{sgn}(y_i - y_j)}{\lambda_i - \lambda_j - ic} \prod_j e^{i\lambda_j(y_j - x_j)} b^\dagger(y_j) |0\rangle
 \end{aligned}$$



Repulsive $c > 0$



Attractive $c < 0$,

contour accounts for strings, bound states

It time evolves to:

$$|\Phi_0, t\rangle = \int_x \int_y \int_\lambda \theta(\vec{x}) \Phi_0(\vec{x}) \prod_{i<j} \frac{\lambda_i - \lambda_j - ic \operatorname{sgn}(y_i - y_j)}{\lambda_i - \lambda_j - ic} \prod_j e^{-i\lambda_j^2 t} e^{i\lambda_j(y_j - x_j)} b^\dagger(y_j) |0\rangle$$

- Expression contains full information about the dynamics of the system

Keldysh

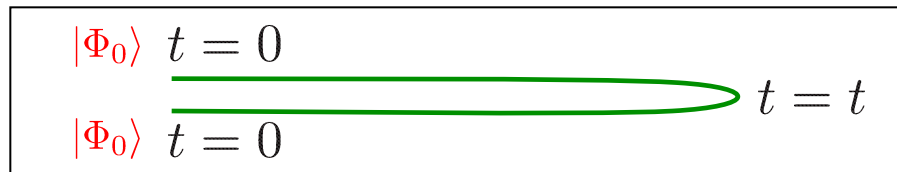
- Time evolution of expectation values:

$$O_{\Phi_0}(t) = \langle \Phi_0 | e^{iHt} \hat{O} e^{-iHt} | \Phi_0 \rangle = \langle \Phi_0, t | \hat{O} | \Phi_0, t \rangle$$

Non-perturbative Keldysh:

$$= \int \mathcal{D} b^* \mathcal{D} b \hat{O} e^{-i \int_C [S_0(b, b^*) + S_I(b, b^*)] dt}$$

carried out on the Keldysh contour C , with separate fields for the top and bottom lines:



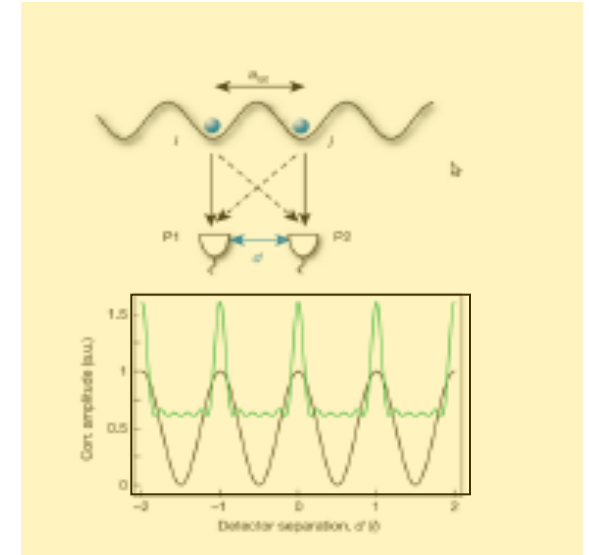
What to calculate?

- We shall study:

1. Evolution of the density

$$C_1(x, t) = \langle \rho(x, t) \rangle \quad \text{Time Of Flight experiment}$$

competition between quantum broadening and attraction



Hanbury-Brown Twiss effect

Measure: $C_2(x_1, x_2, t)$

- two sources: originally stars

Free bosons $C_2(x, -x) \sim \cos x$

Free Fermions $C_2(x, -x) \sim -\cos x$

- Two bosons:

Similar, but time dependent

- Many bosons:

More structure: main peaks, sub peaks

Effects of interactions?

2. Evolution of noise correlation

$$C_2(x_1, x_2; t) = \frac{\langle \rho(x_1, t) \rho(x_2, t) \rangle}{\langle \rho(x_1, t) \rangle \langle \rho(x_2, t) \rangle} - 1$$

time dependent Hanbury-Brown Twiss effect

- repulsive bosons evolve into fermions
- attractive bosons evolve to a condensate

Evolution of a bosonic system: density

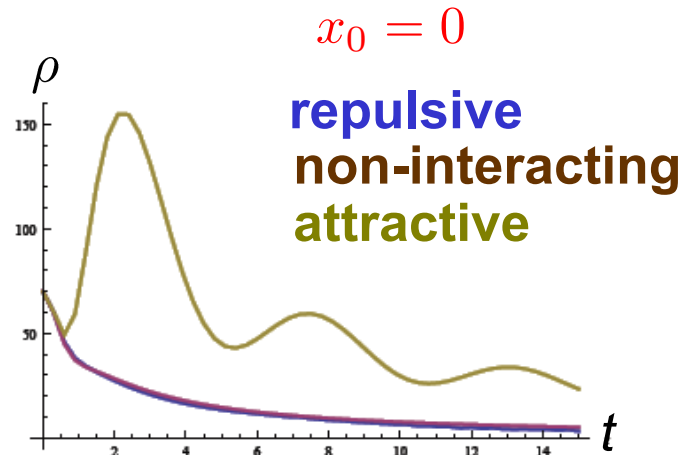
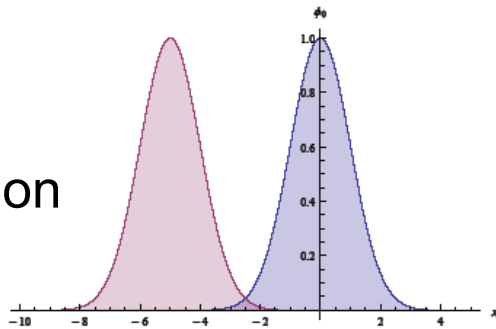
Density evolution:

(Time of flight experiment)

$$\langle \rho(x_0, t) \rangle = \langle \Phi_0(t) | b^\dagger(x_0) b(x_0) | \Phi_0(t) \rangle$$

Two bosons

Initial condition



$$|\Phi_0\rangle = \frac{1}{(\pi\sigma^2)^{\frac{1}{2}}} \int_x e^{-\frac{(x_1)^2}{2\sigma^2}} e^{-\frac{(x_2+a)^2}{2\sigma^2}} b^\dagger(x_1) b^\dagger(x_2) |0\rangle$$



$$|\Phi_0(t)\rangle_2 = \int_y \frac{e^{i\frac{(y_1-x_1)^2}{4t}} + i\frac{(y_2-x_2)^2}{4t}}{4\pi it} \left(1 - c\sqrt{\pi it} \theta(y_2 - y_1) e^{\frac{i}{8t}\alpha^2(t)} \operatorname{erf}\left(\frac{i-1}{4} \frac{i\alpha(t)}{\sqrt{t}}\right) \right) b^\dagger(y_1) b^\dagger(y_2) |0\rangle$$

with $\alpha(t) = 2ct - i(y_1 - x_1) - i(y_2 - x_2)$

Attractive: Time period of oscillations $T = \frac{4\pi}{c^2}$
- independent of IC

Repulsive: almost coincides with free boson **Broadening**

Attractive: competition between **attraction** and **Broadening**

Emergence of an asymptotic Hamiltonian

Long time asymptotics - repulsive:

- **Bosons turn into fermions as time evolves (for any $c > 0$)** (cf. Buljan et al. '08)

$$\begin{aligned}
 |\Phi_0, t\rangle &= \int_x \int_y \int_\lambda \theta(\vec{x}) \Phi_0(\vec{x}) \prod_{i < j} \frac{\lambda_i - \lambda_j - ic \operatorname{sgn}(y_i - y_j)}{\lambda_i - \lambda_j - ic} \prod_j e^{-i\Sigma_j \lambda_j^2 t - \lambda_j(y_j - x_j)} \prod_j b^\dagger(y_j) |0\rangle \\
 &= \int_x \int_y \int_\lambda \theta(\vec{x}) \Phi_0(\vec{x}) \prod_{i < j} \frac{\lambda_i - \lambda_j - ic\sqrt{t} \operatorname{sgn}(y_i - y_j)}{\lambda_i - \lambda_j - ic\sqrt{t}} e^{-i\Sigma_j \lambda_j^2 - \lambda_j(y_j - x_j)/\sqrt{t}} \prod_j b^\dagger(y_j) |0\rangle \\
 &\rightarrow \int_x \int_y \int_\lambda \theta(\vec{x}) \Phi_0(\vec{x}) e^{-i\Sigma_j \lambda_j^2 - \lambda_j(y_j - x_j)/\sqrt{t}} \prod_{i < j} \operatorname{sgn}(y_i - y_j) \prod_j b^\dagger(y_j) |0\rangle \\
 &= e^{-iH_0^f t} \int_{x,k} \mathcal{A}_x \theta(\vec{x}) \Phi_0(\vec{x}) \prod_j c^\dagger(x_j) |0\rangle.
 \end{aligned}$$

\mathcal{A}_x antisymmetrizer

where

$$H_0^f = - \int_x c^\dagger(x) \partial^2 c(x)$$

- **In the long time limit repulsive bosons for any $c > 0$ propagate under the influence of Tonks – Girardeau Hamiltonian (hard core bosons=free fermions)**
- **The state equilibrates, does not thermalize**
- **valid independently of Φ_0**
- **Scaling argument fails for attractive bosons (instead, they condense to a bound state)**

Evolution of a bosonic system: saddle point app

Corrections to long time asymptotics -

Stationary phase approx at large times (carry out λ - integration)

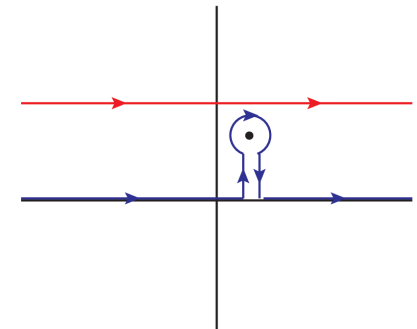
- **Repulsive** – only stationary phase contributions (on real line); (cf. Lamacraft 2011)

$$\phi\left(\xi \equiv \frac{y}{2t}, x, t\right) = S_\xi \frac{1}{(4\pi it)^{\frac{N}{2}}} \prod_{i < j} \frac{\xi_i - \xi_j - ic \operatorname{sgn}(\xi_i - \xi_j)}{\xi_i - \xi_j - ic} e^{\sum_j i\xi_j^2 t - i\xi_j x_j}$$

- **Attractive** – contributions from stationary phases and poles. For two particles:

Pole contributions from deformation of contours – formation of bound states

$$\phi(\xi, x, t) = S_\xi \left[\frac{1}{4\pi it} \frac{\xi_1 - \xi_2 - ic \operatorname{sgn}(\xi_1 - \xi_2)}{\xi_1 - \xi_2 - ic} e^{\sum_j i\xi_j^2 t - i\xi_j x_j} + \frac{2c\theta(\xi_2 - \xi_1)}{\sqrt{4\pi it}} e^{i\xi_1^2 t - i\xi_1 x_1 - i(\xi_1 - ic)^2 t + i(\xi_1 - ic)(2t\xi_2 - x_2)} \right]$$



- repulsive correlations depend on $\xi = \frac{y}{2t}$ only (light cone propagation)
- attractive correlations maintain t also dependence (bound states provide additional scales)

Evolution of a bosonic system

Long time asymptotics:

- **General expression – repulsive**

$$|\Phi_0, t\rangle = \int_x \int_y \theta(\vec{x}) \Phi_0(\vec{x}) \prod_{i < j} \frac{\xi_i - \xi_j - ic \operatorname{sgn}(\xi_i - \xi_j)}{\xi_i - \xi_j - ic} \prod_j \frac{1}{\sqrt{4\pi it}} e^{i\xi_j^2 t - i\xi_j x_j} b^\dagger(y_j) |0\rangle$$

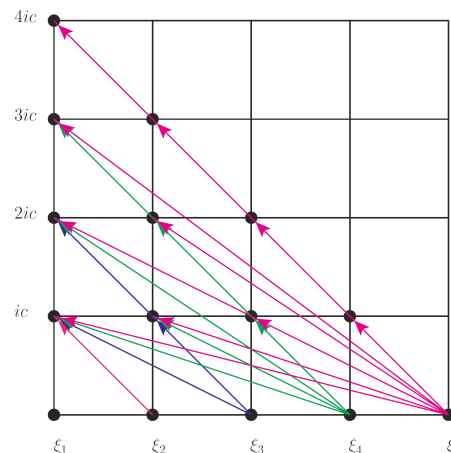
function of $\xi = y/2t$ only, light-like propagation

*Exp: Bloch et al
Nature 2012*

- **General expression – attractive (poles and bound states)**

$$|\Phi_0, t\rangle = \int_x \int_y \theta(\vec{x}) \Phi_0(\vec{x}) \sum_{\xi_j^* = \xi_j, \xi_i^* + ic, i < j} \prod_{i < j} \frac{\xi_i^* - \xi_j^* + ic \operatorname{sgn}(\xi_i - \xi_j)}{\xi_i^* - \xi_j^* + ic} \prod_j \frac{1}{\sqrt{4\pi it}} e^{-i(\xi_j^*)^2 t + i\xi_j^* (2t\xi_j - x_j)} b^\dagger(y_j) |0\rangle$$

Pole contributions follow recursive pattern:



Pattern corresponds to successive formation and contributions of bound states

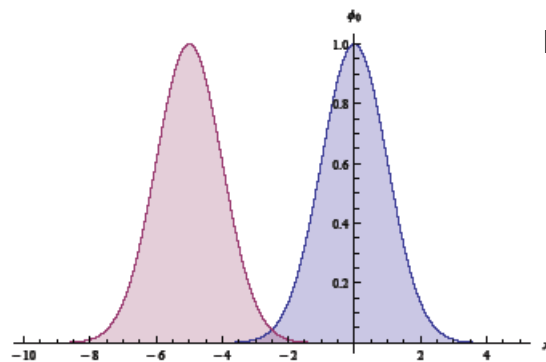
Evolution of repulsive bosons into fermions: HBT

Long time asymptotics - repulsive:

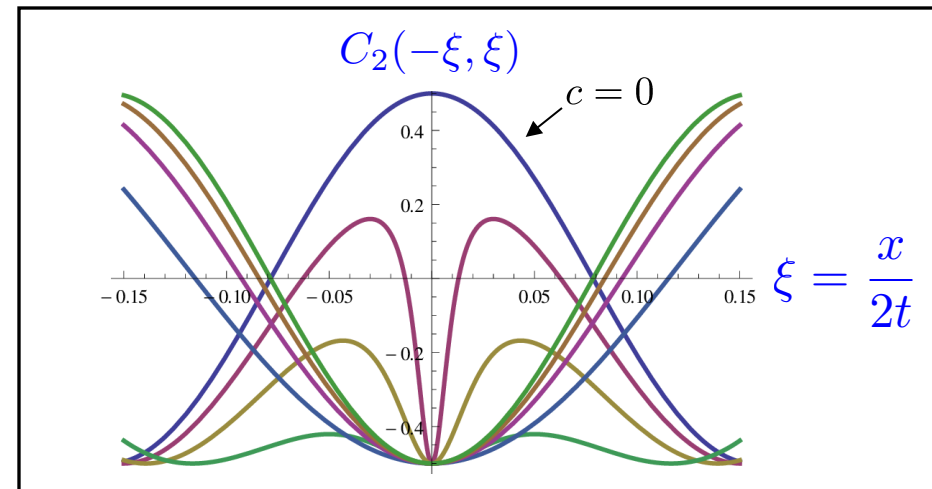
- Bosons turn into fermions as time evolves (for any $c > 0$)
- Can be observed in the noise correlations: (dependence on t only via $\xi = x/2t$)

$$C_2(x_1, x_2, t) \rightarrow C_2(\xi_1, \xi_2) = \frac{\langle \rho(\xi_1) \rho(\xi_2) \rangle}{\langle \rho(\xi_1) \rangle \langle \rho(\xi_2) \rangle} - 1,$$

$$c/a = 0, .3, \dots, 4$$



Fermionic correlations evolve

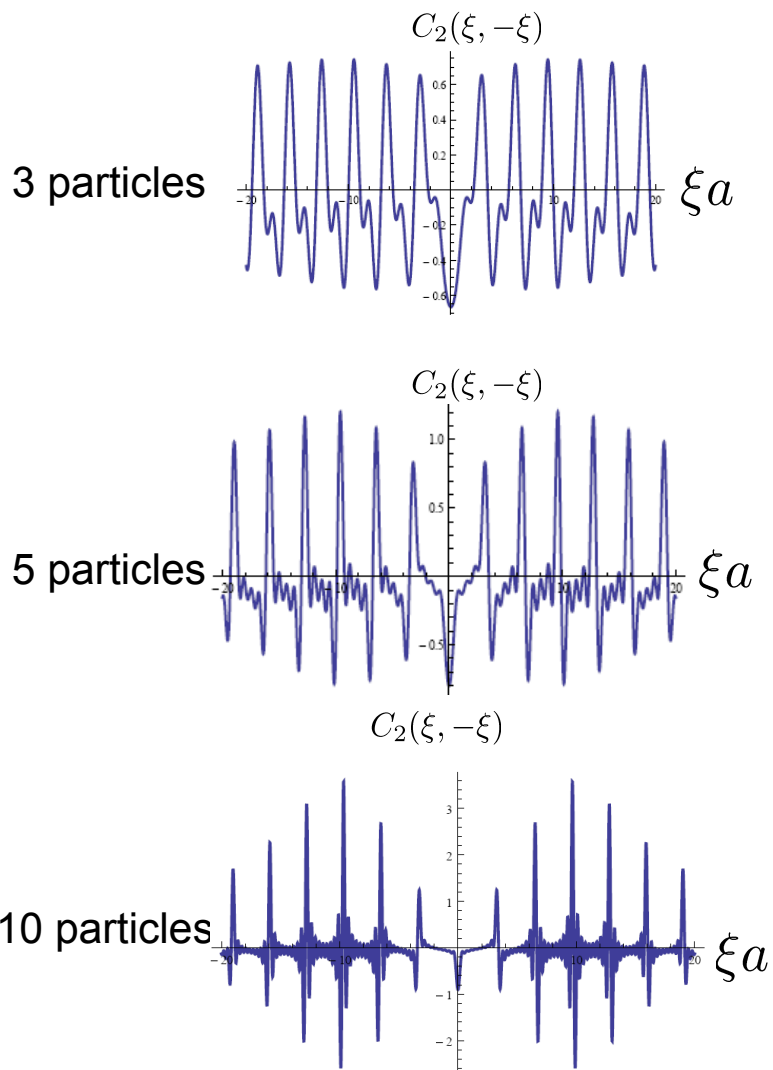


- Fermionic dip develops for any repulsive interaction on time scale set by $1/c^2$

Evolution of a bosonic system: noise correlations

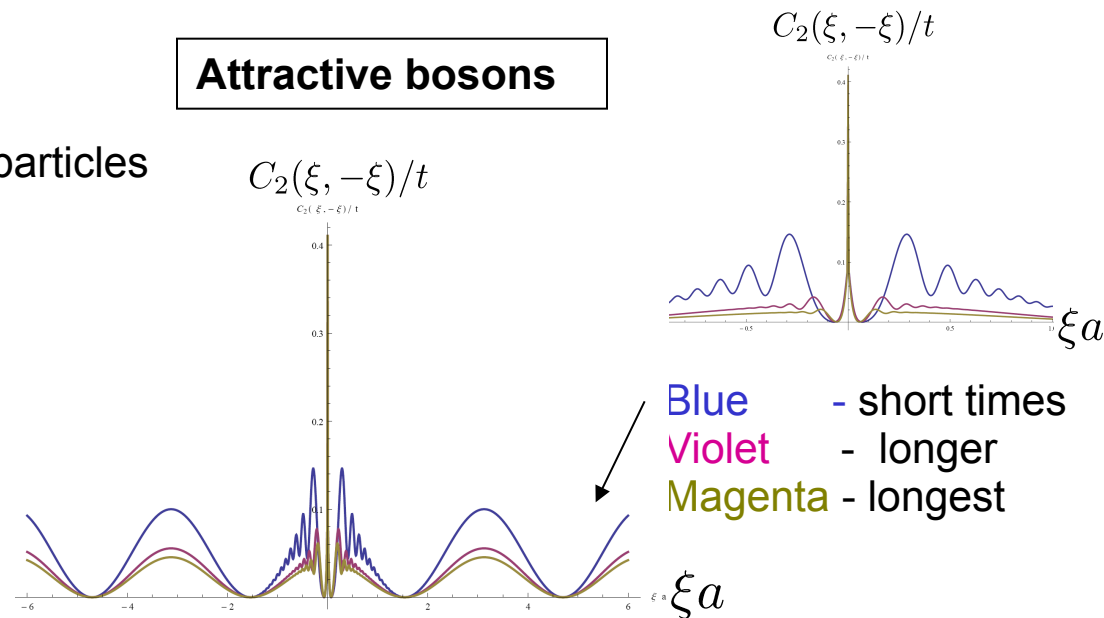
Noise correlations – starting from a lattice

Repulsive bosons



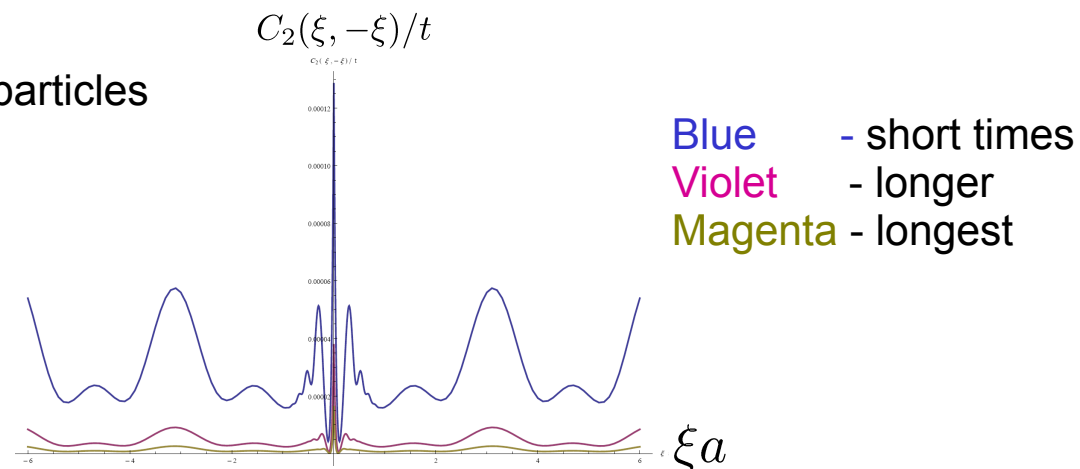
Attractive bosons

2 particles



central peaks increase with time
- weight in the bound states
increases

3 particles



peaks diffuse – momenta redistribute

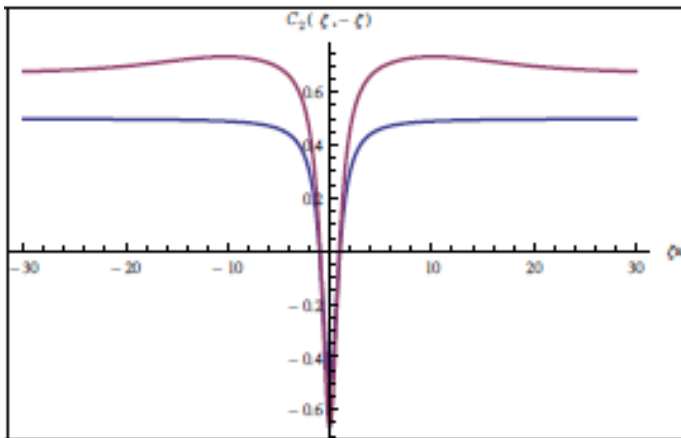
Fermionic dip as $\xi \rightarrow 0$

Structure emerges at $\xi a = \sigma$

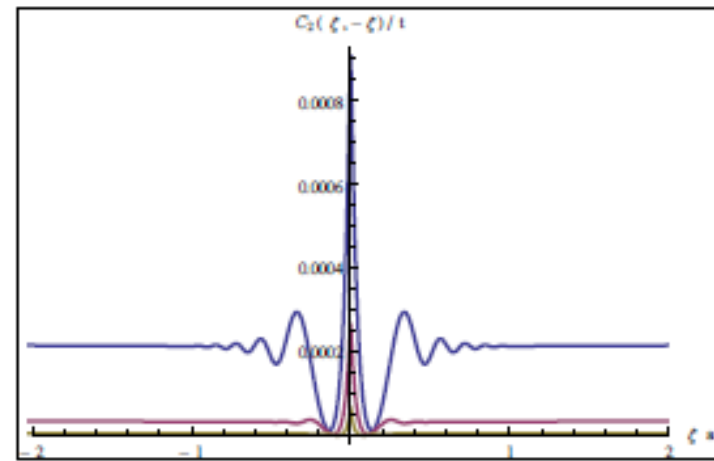
Evolution of a bosonic system: noise correlation

Noise correlations – starting from a condensate

Repulsive bosons



Attractive bosons



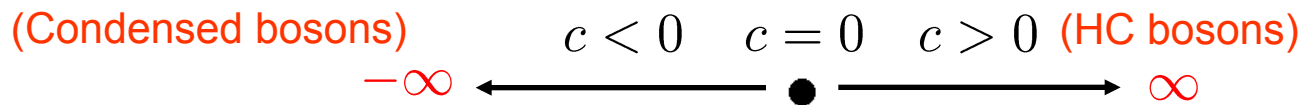
Two (blue) and three bosons,

Three bosons, at times: $tc^2 = 20; 40; 60$

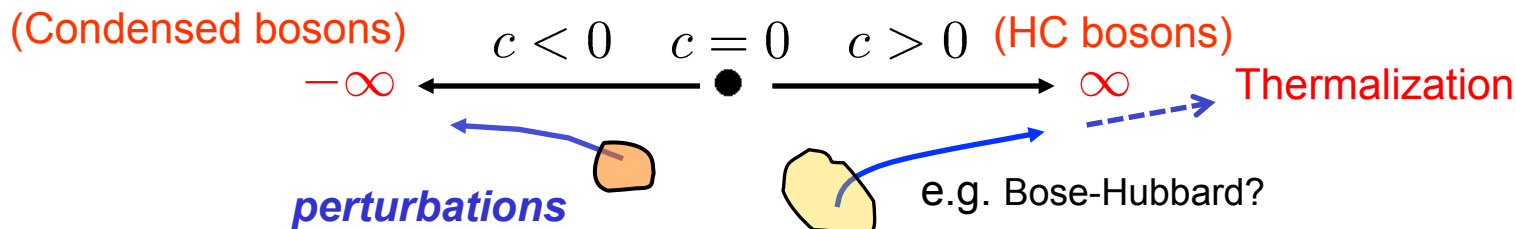
Time evolution “Renormalization Group”

“Dynamic” RG interpretation

- Universality out of equilibrium
- Can view time evolution as RG flow $t \sim \ln(D_0/D)$
 - As time evolves the weight of eigenstate contributions varies, time successively “integrates out” high energy states



- Are there “basins of attraction” for perturbations flowing to dynamic fixed points



- Is there a “thermalization” scale, beyond TG (pre-thermalization?) scale?

Evolution of a bosonic system

Conclusions:

- Does not need the spectrum of Hamiltonian or normalized eigenstates
- Takes into account existence of bound states w/o dealing with large sums over strings
- Asymptotics calculable for *repulsive* and *attractive* interactions in the *Lieb-Liniger* model

To do list:

- Generalize to other integrable models: Heisenberg model (in progress, with Deepak Iyer), Anderson model (Deepak Iyer, Paata Kakashvili), Lieb-Liniger + impurity (Huijie Guan)
- Time evolution at finite volume, finite density (in progress, with Deepak Iyer)
- Time evolution at finite temperatures (under discussion)
- Study approach to nonequilibrium steady state (in progress, with P. Kakashvili)
- Numerical tests of *dynamic RG hypothesis* (in progress, with P. Schmitteckert, t-DMRG)
- Generalize to correlation functions (open)

Big Questions:

- What drives thermalization of pure states? Canonical typicality, entanglement entropy (Lebowitz, Tasaki, Short...)
- General principles, variational? F-D theorem out-of-equilibrium? Heating? Entanglement?
- What is universal? RG Classification?

