A Rough Introduction to Quantum Feedback Control

Howard Wiseman

Centre for Quantum Dynamics, School of Science, Griffith University, Brisbane

Quantum Feedback: Past, Present and Future

- First experiments in optics c. 1985 (Yamamoto and co-workers).

- In recent years, experiments have developed rapidly:
  ∗ Freezing a conditional entangled atom-cavity state in cavity QED (Orozco & co, PRL, 2002)
  ∗ Near optimal phase estimation by a real-time adaptive detection (Mabuchi & co, PRL, 2002)
  ∗ Coherent-state discrimination (Geremia & co, Science, 2007).

- Possibilities in the relatively near future include:
  ∗ Cooling to a ground-state in nanomechanics and atom optics
  ∗ Poducing deterministic spin-squeezing
  ∗ Estimating a continuously varying phase on a squeezed beam.

Outline

1. Theory: Conditional quantum evolution and feedback
2. Experiment: Freezing a conditional entangled atom-cavity state
3. Quantum master equations.
4. The Preferred Ensemble Fact.
5. Proving the Fact by example: Linear phase-space dynamics.
6. Realizing the Preferred Ensembles: Gaussian measurements.
7. Application to Quantum Feedback Control.
8. Markovian Quantum Feedback Control.

1. The Theory: Open Quantum Systems

By interacting with its environment, a quantum system becomes entangled with it. e.g. $|\Psi\rangle = \alpha|e\rangle|\mathbf{0}\rangle + \sum_k \beta_k|g\rangle|1_k\rangle$.

Ignoring the environment = tracing over it: $\rho(t) = \mathrm{Tr}_{\mathrm{env}}[|\Psi(t)\rangle\langle\Psi(t)|]$.

Under the Markovian approximation, get *Lindblad evolution*:

$$
\begin{aligned}
|\psi(t)\rangle\langle\psi(t)| \to \rho(t+dt) &= (1+\mathcal{L}dt)|\psi(t)\rangle\langle\psi(t)|, \\
\text{where for example } \mathcal{L}\rho &\equiv -i[\hat{H},\rho] + \gamma[\hat{c}\rho\hat{c}^\dagger - \tfrac{1}{2}\hat{c}^\dagger\hat{c}\rho - \tfrac{1}{2}\rho\hat{c}^\dagger\hat{c}],
\end{aligned}
$$

where $\hat{c}$ is the system lowering operator (e.g. $|g\rangle\langle e|$).

Photon Detection

But we can rewrite this to $O(dt)$ as

$$\rho(t+dt) = P_0(dt)|\psi_0(t+dt)\rangle\langle\psi_0(t+dt)| + P_1(dt)|\psi_1(t+dt)\rangle\langle\psi_1(t+dt)|,$$

where

$$\begin{aligned} |\psi_0(t+dt)\rangle &= \left(1 - i\hat{H}dt - \tfrac{1}{2}\gamma\hat{c}^\dagger\hat{c}dt\right)|\psi(t)\rangle/\sqrt{P_0(dt)} \\ |\psi_1(t+dt)\rangle &= \sqrt{\gamma dt}\,\hat{c}|\psi(t)\rangle/\sqrt{P_1(dt)} \end{aligned}$$

and $P_1(dt) = 1 - P_0(dt)$ is the probability for a photon to be detected. $\Longrightarrow$ $|\psi_r(t+dt)\rangle$ $(r = 0,1)$ is the *conditioned* system state, the observer's *state of knowledge*, following a *quantum trajectory*:

$|\psi_0(t+dt)\rangle \approx |\psi(t)\rangle$ (no detection $\Longrightarrow$ smooth evolution)
$|\psi_1(t+dt)\rangle \not\approx |\psi(t)\rangle$ (detection $\Longrightarrow$ quantum jump).

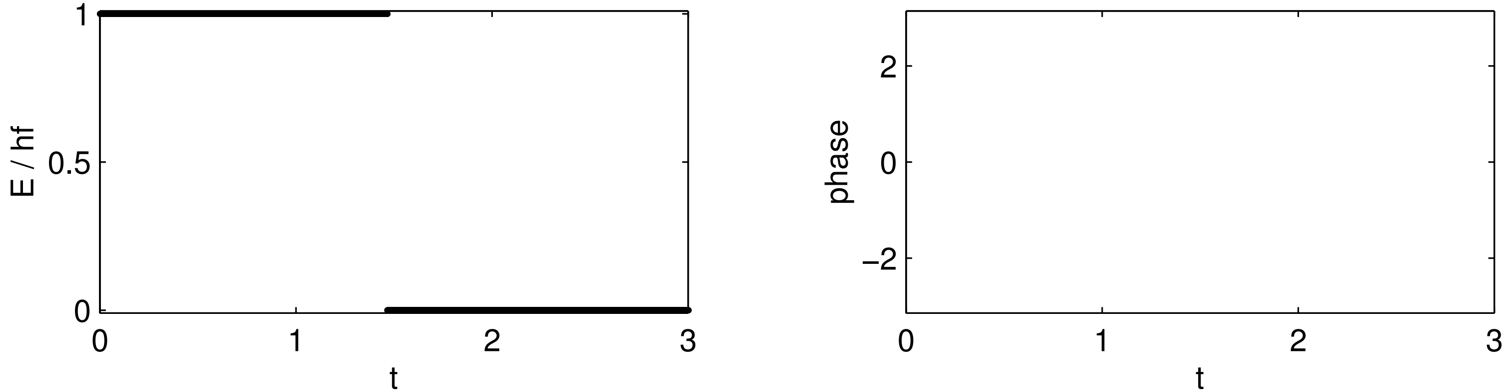
Can I ignore this “state of knowledge” psycho-babble?

Can I ignore this "state of knowledge" psycho-babble?

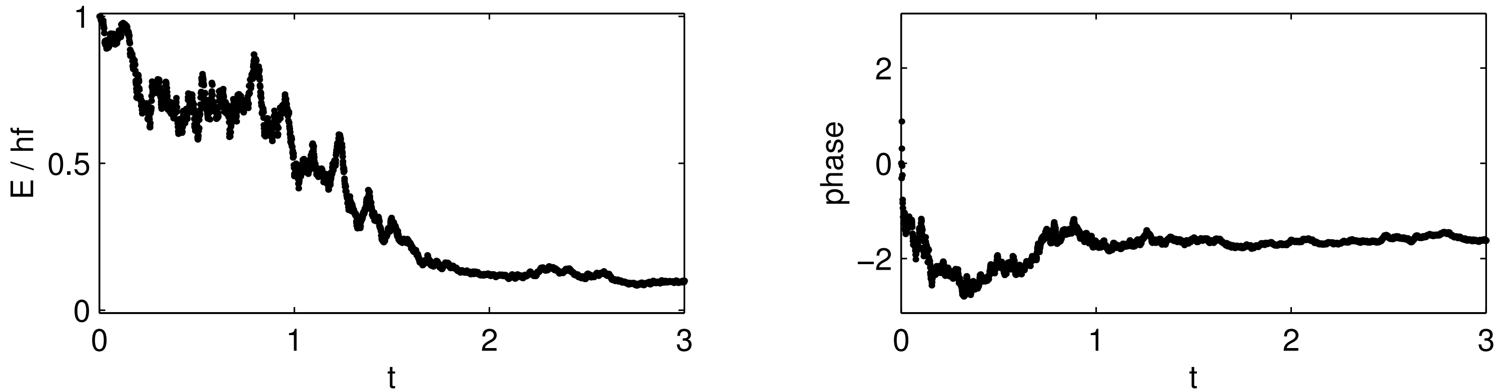
1. **No**: Different ways of monitoring the environment (e.g. homodyne detection) lead to different sorts of quantum trajectories.
   (This will be important later.)

Example: Decay of an Excited State Atom — Quantum Trajectories

Direct Detection (Avalanche Photodiode):



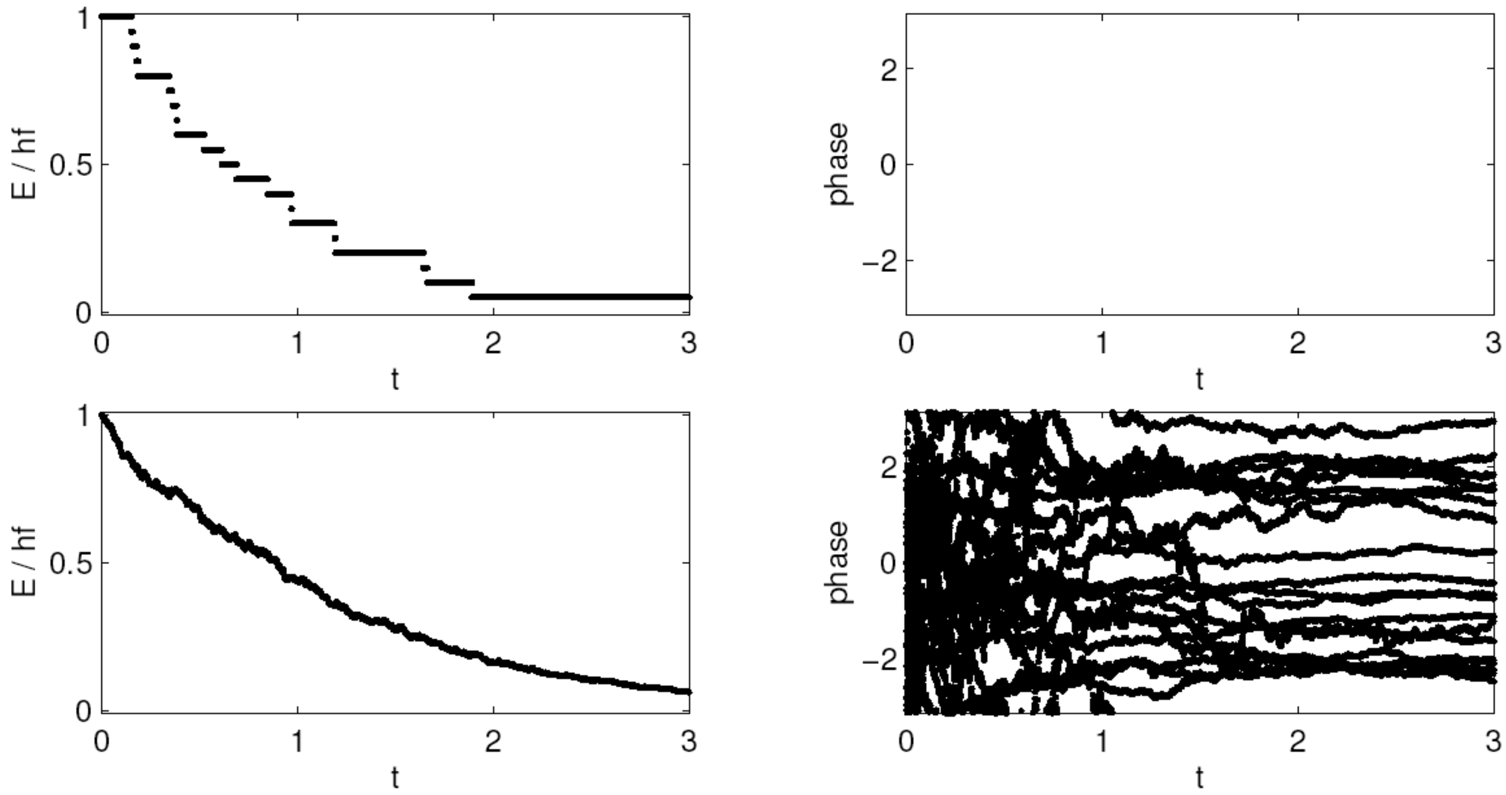
Heterodyne Detection (Laser and Photoreceiver):



Example: Decay of an Excited State Atom — Ensemble Average Evolution

Direct Detection (Avalanche Photodiode):

Heterodyne Detection (Laser and Photoreceiver):



Can I ignore this “state of knowledge” psycho-babble?

1. **No**: Different ways of monitoring the environment (e.g. homodyne detection) lead to different sorts of quantum trajectories.
   (This will be important later.)

Can I ignore this "state of knowledge" psycho-babble?

1. **No**: Different ways of monitoring the environment (e.g. homodyne detection) lead to different sorts of quantum trajectories.
   (This will be important later.)

2. **No**: The state $|\psi_c\rangle$ (c for conditioned) is very useful for quantum feedback (Belavkin 1980s; Wiseman & Milburn 1993; Doherty & Jacobs 1999):

   $|\psi_c\rangle$ is the observer's knowledge (her whole knowledge and nothing but her knowledge) about the system.

   Therefore *by definition*[1] it is the optimal basis for controlling it:

$$\text{optimal } \hat{H}_{\text{fb}}(t) = \text{function of } |\psi_c\rangle.$$

[1] Except see Matt James' work on risk-sensitive control.

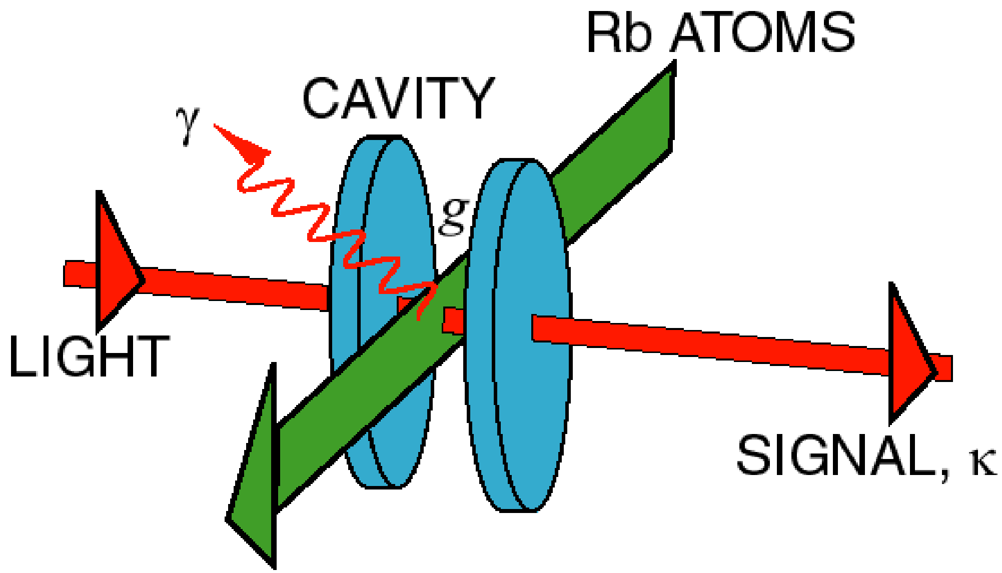
2. “Deep” quantum feedback

Until recently, all quantum feedback experiments have been in a regime of small quantum noise.
This allows linearization and a semiclassical treatment.

The exception is the Cavity QED experiment at SUNY Stony-Brook:
[Smith, Reiner, **Orozco**, Kuhr, and Wiseman, PRL (2002)].
This took quantum feedback into the “deep quantum regime”:

$$g > \kappa, \gamma_\perp \ : \ \begin{cases} g & = \ d\sqrt{\omega / 2\varepsilon_0 \hbar V_{\rm mode}} = \text{single-photon Rabi frequency,} \\ \kappa & = \ \text{cavity amplitude decay rate,} \\ \gamma_\perp & = \ \text{atomic transverse decay rate.} \end{cases}$$

Here, feedback can only be understood using quantum trajectories for the conditioned quantum state $|\psi_c\rangle$.



$$\frac{g}{2\pi} = 4.7\ \text{MHz} \qquad \frac{\kappa}{2\pi} = 3.6\ \text{MHz} \qquad \frac{\gamma_\perp}{2\pi} = 3.0\ \text{MHz}$$

For simplicity we assume at most one atom in the cavity at any time.

Weak Driving Limit

In limit of weak driving $\varepsilon \ll \kappa$, $\rho_{\rm ss} = |\psi_{\rm ss}\rangle\langle\psi_{\rm ss}| + O(\lambda^3)$, where

$$|\psi_{\rm ss}\rangle \propto |0,g\rangle + \lambda\left(|1,g\rangle - \frac{2g}{\gamma}|0,e\rangle\right) + \lambda^2\left(\zeta_0\frac{1}{\sqrt{2}}|2,g\rangle - \theta_0\frac{2g}{\gamma}|1,e\rangle\right)$$

where $\lambda = \varepsilon/(\kappa + g^2/\gamma) \ll 1$, and $\zeta_0, \theta_0 \sim 1$ depend on $g \sim \gamma \sim \kappa$. This occurs because

1. The rate of jumps (emissions from atom or cavity) is $O(\kappa\lambda^2)$.

2. The after-jump state is $|0\rangle + O(\lambda)$ which $= |\psi_{\rm ss}\rangle + O(\lambda)$.

3. After a jump, the system relaxes back to $|\psi_{\rm ss}\rangle$ at a rate $O(\kappa)$.

i.e. excursions from $|\psi_{\rm ss}\rangle$ are $O(\lambda)$ and are there for $O(\lambda^2)$ of the time.

Conditioning on a Cavity Emission

When a detection occurs, $|\psi_{ss}\rangle$ collapses to $\hat{a}|\psi_{ss}\rangle$, then evolves as

$$|\psi_{c}(\tau)\rangle = |0,g\rangle + \lambda\left(\zeta(\tau)|1,g\rangle - \theta(\tau)\frac{2g}{\gamma}|0,e\rangle\right) + O(\lambda^2),$$

where θ and ζ describe the amplitude of the atomic dipole and cavity field. Note that *both* amplitudes change ($\zeta_0 \neq 1, \theta_0 \neq 1$) after the detection of a photon from the cavity *because* $|\psi_{ss}\rangle$ *is entangled*.

After the jump, these amplitudes obey the equations describing coupled driven damped harmonic oscillators (in the rotating frame):

$$\begin{aligned} d_\tau\theta(\tau) &= -\frac{\gamma}{2}\theta(\tau) - g\zeta(\tau), \\ d_\tau\zeta(\tau) &= -\kappa\zeta(\tau) + g\theta(\tau) + \varepsilon/\lambda \end{aligned}$$

Measuring the Conditional Transients

Using two photodetectors, we can measure the autocorrelation function for the photocurrent $I(t) \propto \sum_n \delta(t - t_n)$

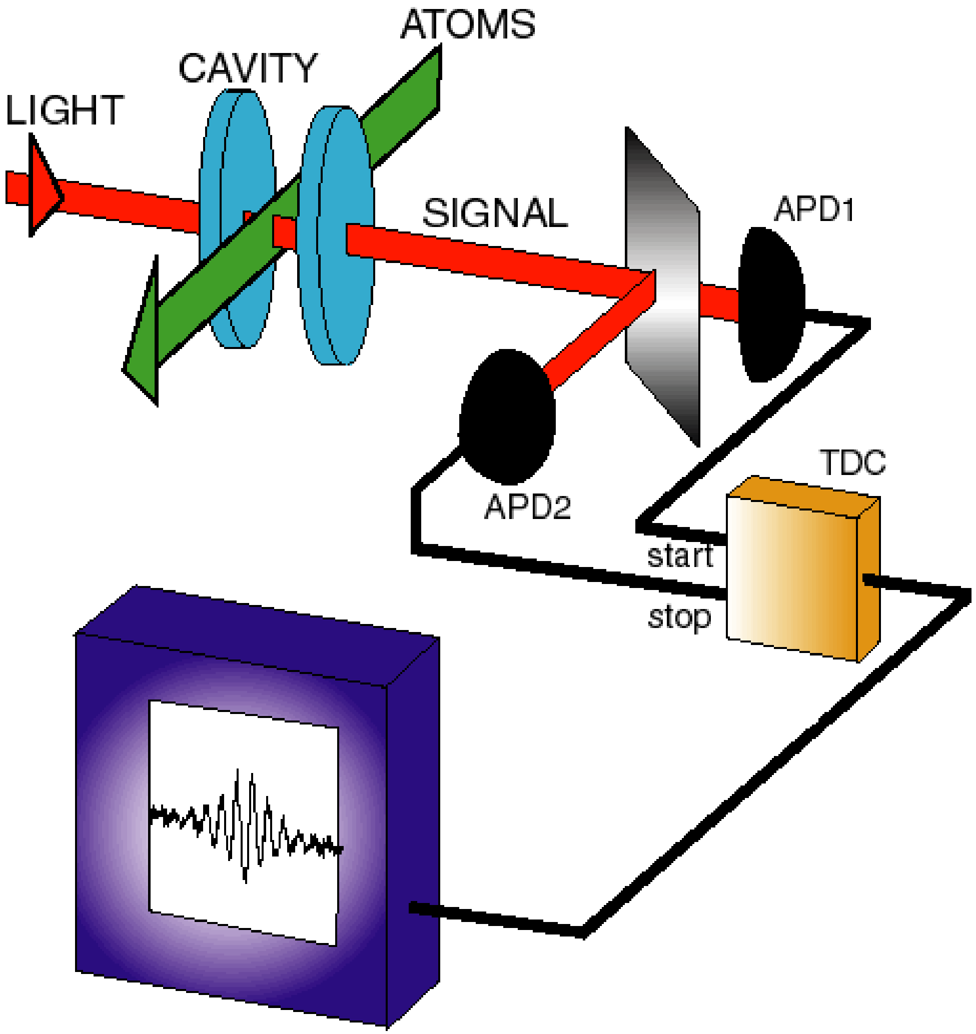
$$g^{(2)}(\tau) \;=\; \frac{\langle I(t+\tau)I(t)\rangle_{\rm ss}}{\langle I(t)\rangle_{\rm ss}^2} = \frac{\langle I(t+\tau)\rangle_{\rm c}}{\langle I\rangle_{\rm ss}},$$

where c means "given a detection at time t in steady state".

For $\tau > 0$, we can use the conditional state to get

$$g^{(2)}(\tau) \simeq \frac{|\langle 1,g|\psi_{\rm c}(\tau)\rangle|^2}{|\langle 1,g|\psi_{\rm ss}\rangle|^2} \;=\; [\zeta(\tau)]^2.$$

By symmetry, $g^{(2)}(-\tau) = g^{(2)}(\tau)$.


ATOMS
CAVITY
LIGHT
SIGNAL
APD1
APD2
TDC
start
stop

Freezing a Conditional Transient

If we choose a time $\tau = T$ such that $\zeta(T) = \theta(T)$ then

$$|\psi_{c}(T)\rangle = |0,g\rangle + \lambda\left(\zeta(T)|1,g\rangle - \theta(T)\frac{2g}{\gamma}|0,e\rangle\right) + O(\lambda^2),$$

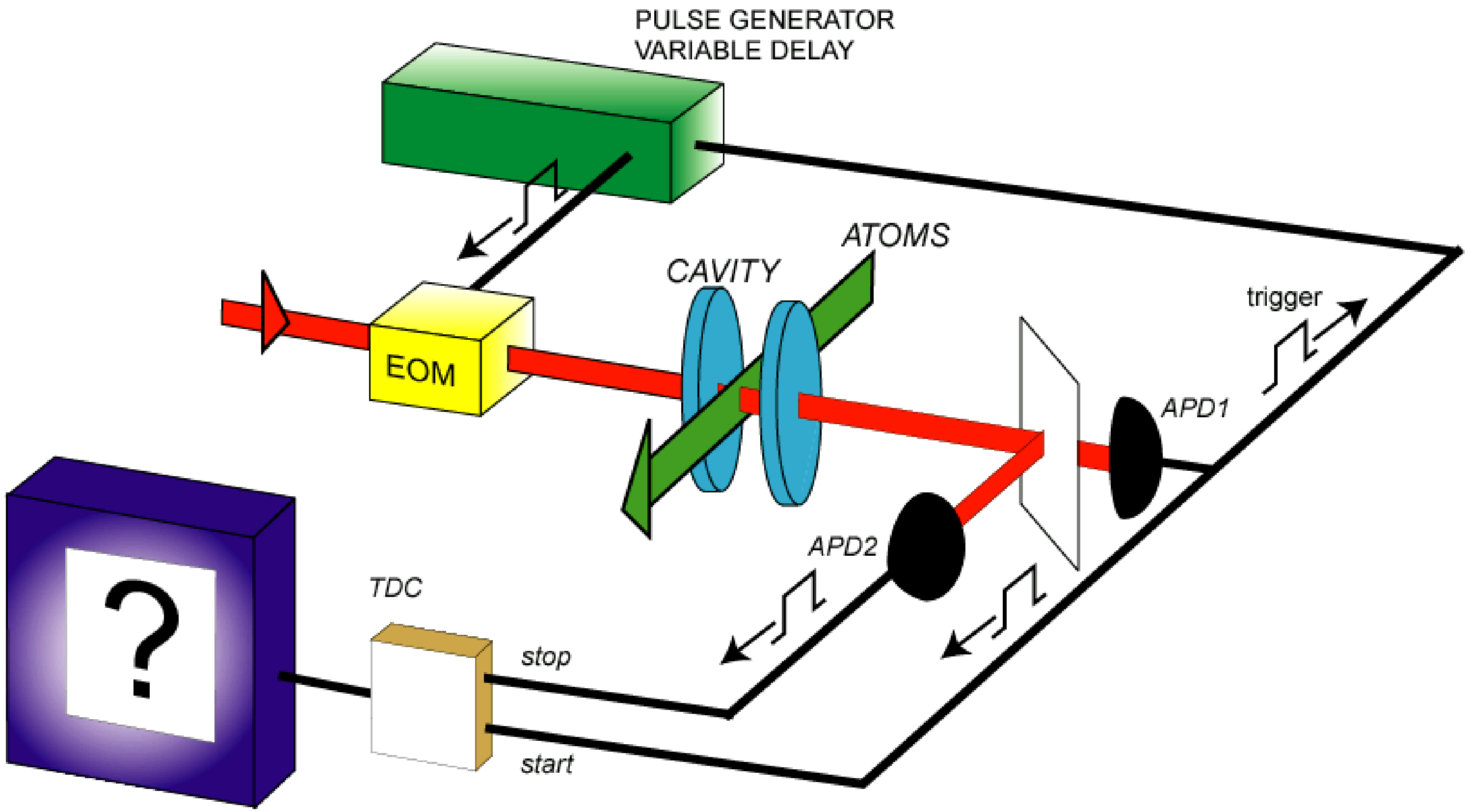
is, to order λ, of the form of $|\psi_{ss}\rangle$,

$$|\psi_{ss}\rangle \simeq |0,g\rangle + \lambda'\left(|1,g\rangle - \frac{2g}{\gamma}|0,e\rangle\right) + O(\lambda'^2),$$

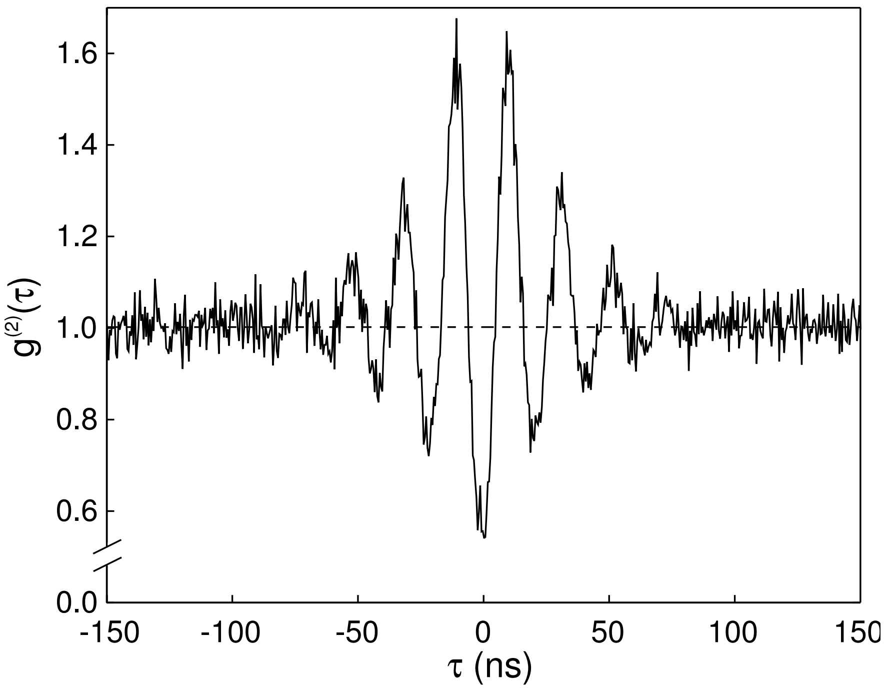
but with a different mean field, $\lambda' = \zeta(T)\lambda$.

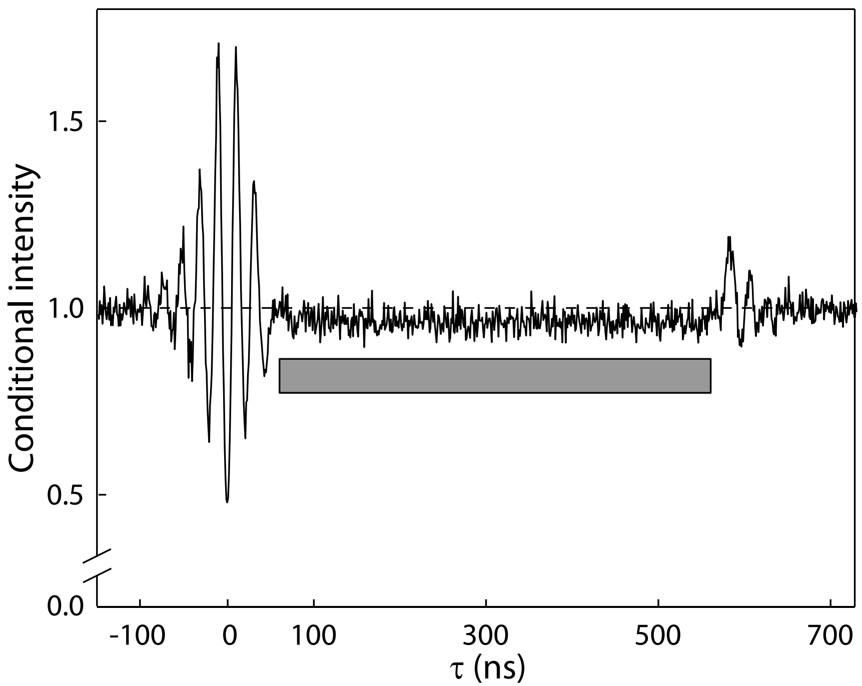
We can freeze this state if, at $\tau = T$, we change the driving ε to $\varepsilon \times \zeta(T)$. By changing the driving back to ε at a later time, we *release* the state.

Trigger the intensity-step with a fluctuation (photon) and measure the time evolution of the intensity as in $g^{(2)}(\tau)$.



Experimental Results (New: PRA, 2004)





With feedback (right), the ring down is stopped when the feedback is applied at $T = 45$ns, reducing ε by about 1.3%.

The oscillations resume, with the same amplitude and phase, when the driving is returned to its original value, 500 ns later.

3. Quantum master equations

In many situations in atomic, optical, and (increasingly) solid-state physics, the system is small and is coupled weakly to a large bath at effectively zero temperature.

The system and bath entangle, so tracing over (ignoring) the bath makes an initial system state $|\psi(0)\rangle$ evolve into a mixed state $\rho(t)$.

Often this decoherence process can be described by a *quantum master equation* (QME):

$$\hbar\dot{\rho} = \mathcal{L}\rho \tag{1}$$

where $\mathcal{L}$ is the Liouvillian superoperator.

We consider master equations with a *unique*, *mixed* steady state

$$\rho_{\rm ss} = \lim_{t\to\infty} e^{\mathcal{L}t/\hbar}|\psi(0)\rangle\langle\psi(0)|. \tag{2}$$

Unravelling quantum master equations

It is not always appropriate to ignore the bath — often it can be measured, yielding information about the system.

If a master equation can be derived then the bath can be measured repeatedly, much faster than any relevant system rate *without invalidating the master equation*.

If this monitoring is perfect, then this produces a *pure* conditioned system state $|\psi_c(t)\rangle$.

We say the stochastic evolution for $|\psi_c(t)\rangle$ *unravels* the QME:

$$\mathrm{E}[|\psi_c(t)\rangle\langle\psi_c(t)|] = \rho(t) = \exp[\mathcal{L}t/\hbar]|\psi(0)\rangle\langle\psi(0)|. \quad (3)$$

Different ways of measuring the bath (such as photon counting, homodyne detection, *etc.*) lead to different unravellings.

4. Physically Realizable Ensembles

For a system with steady state $\rho_{\rm ss}$, an equivalent ensemble of pure states is any ensemble $E = \left\{ \wp^E{}_k, |\phi^E{}_k\rangle \right\}$ satisfying

$$\rho_{\rm ss} = \sum_k \wp_k^E |\phi_k^E\rangle\langle\phi_k^E|. \tag{4}$$

Such an ensemble is physically realizable (PR) iff some unravelling of the master equation $\hbar\dot{\rho} = \mathcal{L}\rho$ gives a stochastically evolving $|\psi_{\rm c}(t)\rangle$, such that in the long-time limit $\wp_k^E$ is the proportion of time for which $|\psi_{\rm c}(t)\rangle = |\phi_k^E\rangle$. (By construction, this guarantees Eq. (4) will hold.)

Preferred Ensemble Fact (Wiseman & Vaccaro, PRL, 2001):

Some ensembles (the preferred ones) are physically realizable, while others are not.

Schrödinger/Hughston-Josza-Wootters Theorem

If

$$\rho = \mathrm{Tr}_{\mathrm{bath}}\left[|\Psi\rangle\langle\Psi|\right] = \sum_k \wp_k |\phi_k\rangle\langle\phi_k|$$

then there is a way to measure the bath such that the system state conditioned on the measurement result is $|\phi_k\rangle$ with probability $\wp_k$.

This seems to say that any ensemble $\{\wp_k, |\phi_k\rangle\}$ is PR. But remember our definition of PR needs the ensemble to be *continuously realized*, so that $\wp_k \propto$ the amount of time the system has state $|\phi_k\rangle$.

That is, if at some time t the ensemble is realized by measuring the bath, it must be realized again at time $t+\tau$ by measuring the bath in the interval $[t, t+\tau)$ for any τ.

5. Quantum systems in Phase Space

These provide a practical application of the PEFact.

In place of a classical point $\mathbf{x} = (q, p)^\top$ in phase space, we have the canonically conjugate pair $\hat{\mathbf{x}} = (\hat{q}, \hat{p})^\top$ which obey $[\hat{p}, \hat{q}] = -i\hbar$. This implies the Schrödinger-Heisenberg uncertainty relation

$$V_q V_p - C_{xp}^2 \geq \hbar^2/4. \tag{5}$$

Introducing the following matrices

$$\Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } V = \begin{pmatrix} V_q & C_{qp} \\ C_{qp} & V_p \end{pmatrix}, \tag{6}$$

we can write this as a Linear Matrix Inequality (LMI):

$$V + i\hbar\Sigma/2 \geq 0. \tag{7}$$

Gaussian States and Linear Systems

A *Gaussian state* is fully specified by $\langle \hat{\mathbf{x}} \rangle$ and V, and is so-called because it has a Gaussian Wigner function:

$$W(\mathbf{x}) = (\det[2\pi V])^{-1/2} \exp[-(\mathbf{x} - \langle \hat{\mathbf{x}} \rangle)^\top (2V)^{-1} (\mathbf{x} - \langle \hat{\mathbf{x}} \rangle)]. \tag{8}$$

These states are interesting in the context of open quantum systems because for systems with linear phase-space dynamics[2] the master equation $\dot{\rho} = \mathcal{L}\rho$ has a Gaussian state as its solution.

The evolution can be expressed as the *moment equations*

$$d\langle \hat{\mathbf{x}} \rangle / dt = A \langle \hat{\mathbf{x}} \rangle \tag{9}$$

$$dV/dt = AV + VA^\top + D. \tag{10}$$

[2] i.e. where the Heisenberg equations of motion are linear (and have only Gaussian noise).

Stationary Mixed States

Provided that the drift matrix A is stable, the stationary solution $\rho_{\rm ss}$ has $\langle \hat{\mathbf{x}} \rangle_{\rm ss} = \mathbf{0}$, and $V_{\rm ss}$ given by

$$AV_{\rm ss} + V_{\rm ss}A^{\top} + D = 0. \tag{11}$$

If the system is not driven then it may relax to a steady state $\rho_{\rm ss}$ that is pure, and the uncertainty relation $V_{\rm ss} + i\hbar\Sigma/2 \geq 0$ is saturated. (Half of the eigenvalues of the LHS vanish.)

We are interested in a driven system where $\rho_{\rm ss}$ is *mixed*. That is, where $V_{\rm ss}$ is larger than required by the uncertainty relation.

Example: Optical Parametric Oscillator

For example, an optical parametric oscillator at threshold can be described by the QME

$$\hbar\dot{\rho} = -i[\chi(\hat{q}\hat{p}+\hat{p}\hat{q})/2, \rho] + \mathcal{D}[\hat{q}+i\hat{p}]\rho \tag{12}$$

Here $\hat{q} \propto \hat{a}+\hat{a}^\dagger$ and $\hat{p} \propto -i\hat{a}+i\hat{a}^\dagger$ are *quadratures* of an optical mode in the cavity. The free Hamiltonian $\hbar\omega\hat{a}^\dagger\hat{a}$ has been eliminated by working in a rotating frame. χ parametrizes the strength of two-photon driving at frequency 2ω, and *squeezes* the p-quadrature.

The system goes unstable for $\chi > 1$. For definiteness we take $\chi = 1$. Then

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}, D = \hbar\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, V_{\rm ss} = \hbar\begin{pmatrix} \infty & 0 \\ 0 & 1/4 \end{pmatrix}. \tag{13}$$

Uniform Gaussian Ensembles

Recall that we are interested in pure state ensembles $\{\wp_k^E, |\phi_k^E\rangle\}$ such that $\rho_{ss} = \sum_k \wp_k^E |\phi_k^E\rangle\langle\phi_k^E|$.

A uniform Gaussian ensemble is one for which all $|\phi_k^E\rangle$ are Gaussian *with the same covariance matrix* W, but with different means $\langle\hat{\mathbf{x}}\rangle = \bar{\mathbf{x}}$. We call such an ensemble a W-ensemble, and we can replace the index k by $\bar{\mathbf{x}}$. Thus $\wp_k^E \to \wp_{\bar{\mathbf{x}}}^W$ is a Gaussian in $\bar{\mathbf{x}}$ such that

$$\rho_{ss} = \int d\bar{\mathbf{x}} \wp_{\bar{\mathbf{x}}}^W |\phi_{\bar{\mathbf{x}}}^W\rangle\langle\phi_{\bar{\mathbf{x}}}^W|. \tag{14}$$

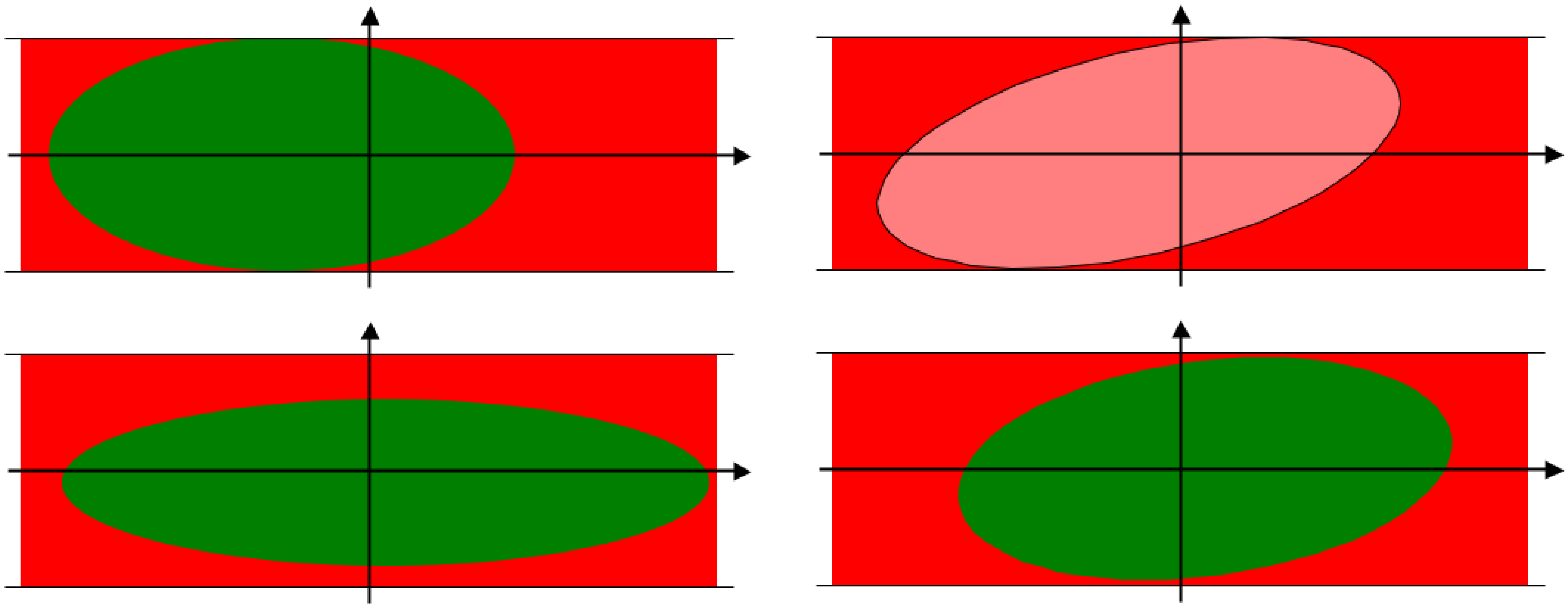
The only condition is that W should fit inside V_{ss}. That is, the LMI

$$V_{ss} - W \geq 0. \tag{15}$$

Example W-ensembles: the OPO at threshold

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}, \; D = \hbar \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; V_{\rm ss} = \hbar \begin{pmatrix} \infty & 0 \\ 0 & 1/4 \end{pmatrix} \tag{16}$$

Vertical axis is p, horizontal axis is q, and area of the ellipse is $\propto \hbar$.



Which Ensembles are Physically Realizable?

Recall that for an ensemble to be PR then if it is realized at time t, it should be realizable again at time $t+dt$.

That is, not only must W fit inside V_{ss} of ρ_{ss} (that is, $V_{ss} - W \geq 0$), but also W must fit inside the covariance matrix of $e^{\mathcal{L}dt/\hbar}|\phi_{\bar{\mathbf{x}}}^{W}\rangle\langle\phi_{\bar{\mathbf{x}}}^{W}|$:

$$\left[W + dt(AW + WA^{\top} + D)\right] - W \geq 0. \tag{17}$$

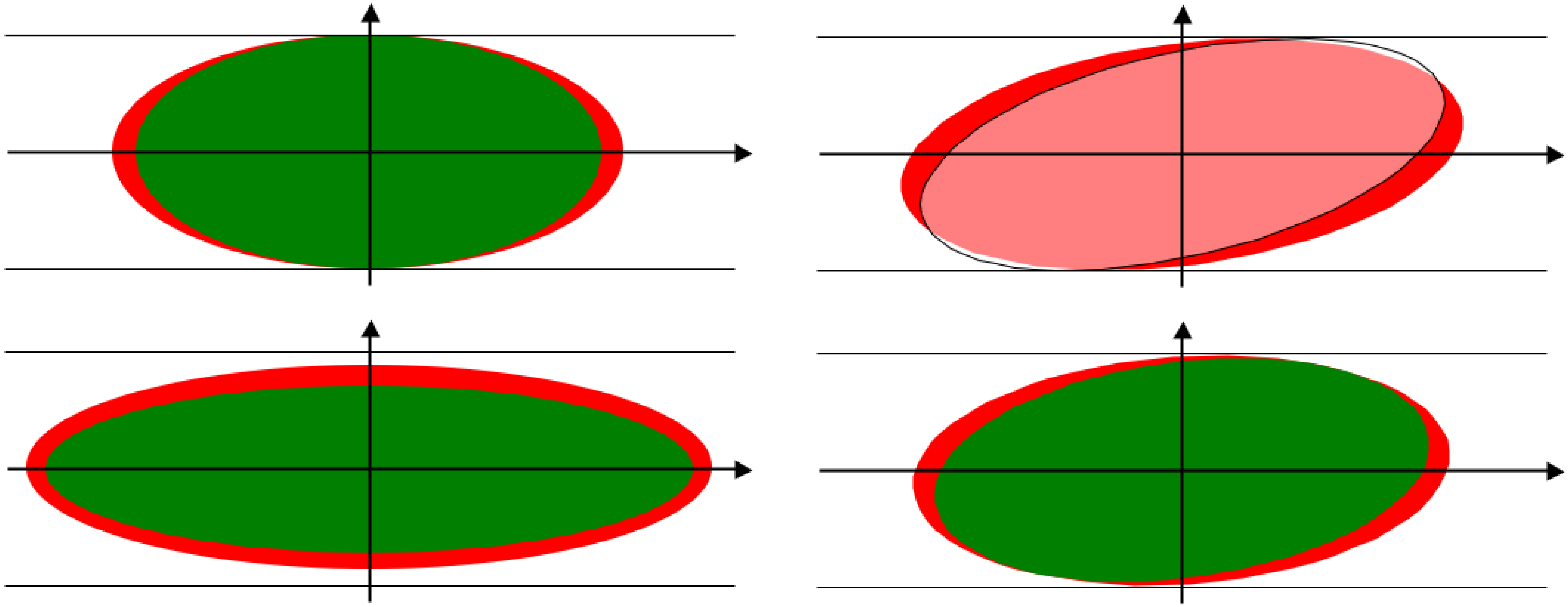
The condition for a W-ensemble to be PR is the LMI

$$AW + WA^{\top} + D \geq 0, \tag{18}$$

which implies $V_{ss} - W > 0$.

Example Ensembles: OPO at threshold

$$V = W + dt(AW + WA^\top + D),\ dt = \text{ (ahem) } 0.2 \tag{19}$$



The PEFact: Only some ensembles are physically realizable.

6. Realizing the Realizable: LG Systems

For Linear systems, a W-ensemble will be realized provided

- All baths coupled to the system are effectively at zero temperature.
- All baths coupled to the system are monitored with unit efficiency.
- The measurement outcomes are currents with Gaussian noise.

e.g. perfect homodyne detection of the output beam of the OPO.

A different unravelling (e.g. a different local oscillator phase) will result in a different W-ensemble. And any W-ensemble such that

$$AW + WA^{\top} + D \geq 0 \,, \quad W + i\hbar\Sigma/2 \geq 0 \tag{20}$$

can be realized by *some* unravelling of this type.

7. Application to Feedback Control: LQG systems

Feedback control of a quantum system means altering the dynamics of the system based upon past measurement results for some purpose such as minimizing some “cost function”.

Typically, this means inducing a feedback Hamiltonian $\hat{H}_{\rm fb}(t)$ that depends upon the stochastic measurement record for times $< t$.

In general this is a very difficult problem, but not for LQG control, where Q means having a cost function J that is Quadratic in the relevant variables. For simplicity, say

$$J = \left\langle \hat{\mathbf{x}}^\top P \, \hat{\mathbf{x}} \right\rangle_{\rm ss} = \lim_{t\to\infty} {\rm E}\left[\langle \psi_{\rm fb}(t) | \hat{\mathbf{x}}^\top P \, \hat{\mathbf{x}} | \psi_{\rm fb}(t) \rangle \right] . \quad (21)$$

Here $|\psi_{\rm fb}(t)\rangle$ is the conditional system state *that includes the effect of the feedback control*.

How the feedback works

We constrain the feedback Hamiltonian to be of the form

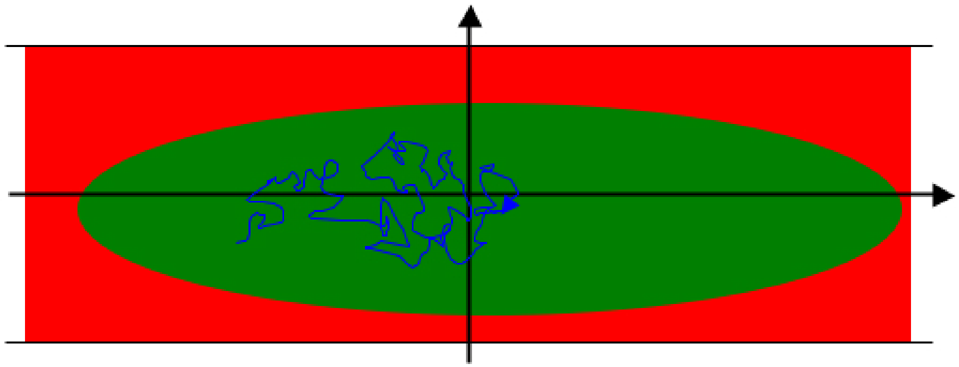
$$\hat{H}_{\rm fb}(t) = \hat{\mathbf{x}}^{\top}\Sigma\mathbf{u}(t). \tag{22}$$

This cannot change the "shape" (W) of the conditioned state, but can arbitrarily move around the centroid in phase space $\bar{\mathbf{x}} = (\bar{q}, \bar{p})^{\top}$, with

$$d\bar{\mathbf{x}}/dt = A\bar{\mathbf{x}} + \text{noise}(t) + \mathbf{u}(t). \tag{23}$$

For *any* cost function, the optimal control is simply to choose $\mathbf{u}(t)$ to pin the centroid at $\bar{\mathbf{x}} = \mathbf{0}$, so that

$$|\psi_{\rm fb}(t)\rangle \to |\phi^{W}_{\bar{\mathbf{x}}:=\mathbf{0}}\rangle. \tag{24}$$



How the feedback works

We constrain the feedback Hamiltonian to be of the form

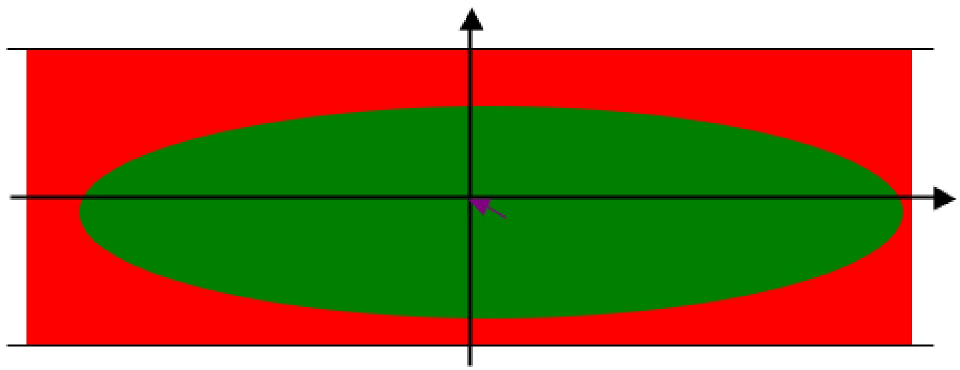
$$\hat{H}_{\rm fb}(t) = \hat{\mathbf{x}}^{\top}\Sigma\mathbf{u}(t). \quad (22)$$

This cannot change the "shape" (W) of the conditioned state, but can arbitrarily move around the centroid in phase space $\bar{\mathbf{x}} = (\bar{q}, \bar{p})^{\top}$, with

$$d\bar{\mathbf{x}}/dt = A\bar{\mathbf{x}} + \text{noise}(t) + \mathbf{u}(t). \quad (23)$$

For *any* cost function, the optimal control is simply to choose $\mathbf{u}(t)$ to pin the centroid at $\bar{\mathbf{x}} = \mathbf{0}$, so that

$$|\psi_{\rm fb}(t)\rangle \to |\phi^{W}_{\bar{\mathbf{x}}:=\mathbf{0}}\rangle. \quad (24)$$



How the feedback works

We constrain the feedback Hamiltonian to be of the form

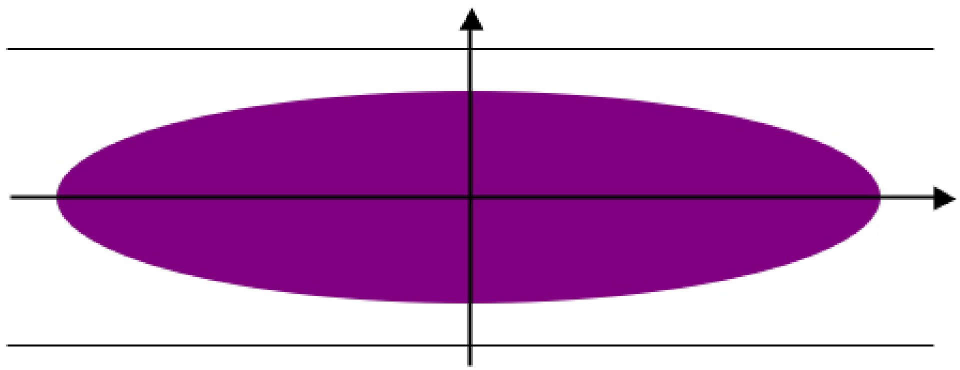
$$\hat{H}_{\rm fb}(t) = \hat{\mathbf{x}}^\top \Sigma \mathbf{u}(t). \quad (22)$$

This cannot change the “shape” (W) of the conditioned state, but can arbitrarily move around the centroid in phase space $\bar{\mathbf{x}} = (\bar{q}, \bar{p})^\top$, with

$$d\bar{\mathbf{x}}/dt = A\bar{\mathbf{x}} + \text{noise}(t) + \mathbf{u}(t). \quad (23)$$

For *any* cost function, the optimal control is simply to choose $\mathbf{u}(t)$ to pin the centroid at $\bar{\mathbf{x}} = \mathbf{0}$, so that

$$|\psi_{\rm fb}(t)\rangle \to |\phi^W_{\bar{\mathbf{x}}:=\mathbf{0}}\rangle. \quad (24)$$



Finding the Best Unraveling

Under this ideal feedback where $|\psi_{\rm fb}(t)\rangle \to |\phi^{W}_{\bar{\mathbf{x}}:=\mathbf{0}}\rangle$, the cost is

$$J = \lim_{t\to\infty} \mathrm{E}\left[\langle\psi_{\rm fb}(t)|\hat{\mathbf{x}}^{\top} P\, \hat{\mathbf{x}}|\psi_{\rm fb}(t)\rangle\right] = \mathrm{tr}[PW]. \tag{25}$$

We wish to minimize this *linear* function of W subject to the LMIs

$$AW + WA^{\top} + D \geq 0\,, \;\; W + i\hbar\Sigma/2 \geq 0. \tag{26}$$

This is precisely the form of a *semi-definite program* for which there are efficient numerical algorithms to solve.

Finding the unravelling that gives W can also be found efficiently.

Example: The OPO at threshold

Say the aim is to produce a stationary state where $q = p$ as nearly as possible. A suitable cost function to be minimized is

$$J = \left\langle (\hat{q} - \hat{p})^2 \right\rangle_{\rm ss}. \tag{27}$$

This is of the form $J = \left\langle \hat{\mathbf{x}}^\top P\, \hat{\mathbf{x}} \right\rangle_{\rm ss}$ with

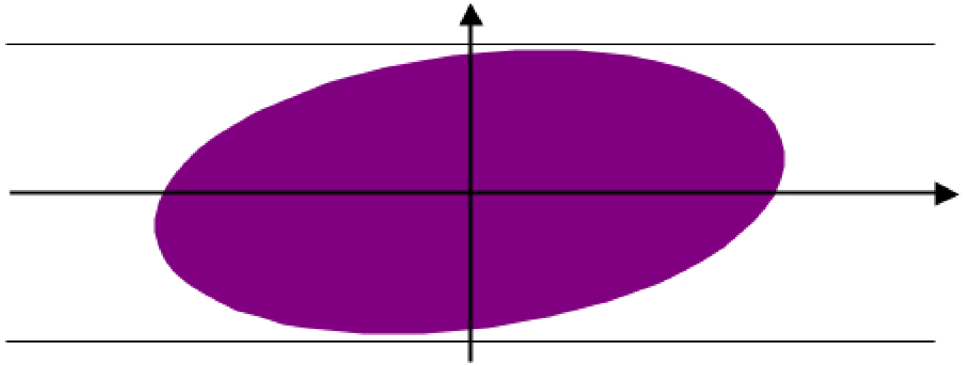
$$P = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{28}$$

Minimizing $J = \mathrm{Tr}[WP]$ subject to $AW + WA^\top + D \geq 0$ and $W + i\hbar\Sigma/2 \geq 0$ yields

$$J \approx \hbar 1.12 \;\; \text{for} \;\; W = \frac{\hbar}{2}\begin{pmatrix} 2.26 & 0.248 \\ 0.248 & 0.469 \end{pmatrix}. \tag{29}$$

Example: The OPO at threshold (cont.)

The feedback stabilized state $|\phi^{W}_{\bar{\mathbf{x}}:=\mathbf{0}}\rangle$ is the PR "sheared state" we have seen before:



We find the optimal unraveling is by homodyne detection with local oscillator phase $\theta \approx 0.278\pi$.

This could not have been predicted; in particular it is different from $\theta = 0.25\pi$ which corresponds to measuring the quadrature $\hat{q} - \hat{p}$ whose deviation $\left\langle (\hat{q} - \hat{p})^2 \right\rangle_{\rm ss}$ one is trying to minimize.

Further Successes and Failures

The theory can be easily generalized (HMW & Doherty, PRL 2005) to systems with:

- arbitrarily many degrees of freedom. That is, $\mathbf{x} = (q_1, p_1, q_2, p_2, \cdots)$.
- arbitrary cost functions that are time-integrals of quadratic functions of $\mathbf{x}(t)$ and $\mathbf{u}(t)$.
- limitations on which degrees of freedom can be controlled.
- limitations of the efficiency with which any bath can be monitored.

Unfortunately, it is not obvious how to determine *efficiently* the optimal unravelling when the detection is not perfect.

Describing imperfect detection?

For perfect detection, the conditioned state is pure: $\rho_{\rm c}(t) = |\psi_{\rm c}\rangle\langle\psi_{\rm c}|$.
Real detectors are not perfect. e.g. photoreceivers have efficiency η.
Wiseman & Milburn (1993): stochastic master equation (SME) [3]

$$d\rho_{\rm c}(t) = dt\{-i[\hat{H},\rho_{\rm c}(t)] + \mathcal{D}[\hat{c}]\rho_{\rm c}(t)\} + \sqrt{\eta}\, dW(t)\mathcal{H}[\hat{c}]\rho_{\rm c}(t).$$

$\rho_{\rm c}(t)$ is the state conditioned on the noise $dW(t)$, the *innovation*. That is, the unpredictable part of of the homodyne photocurrent $J(t)$:

$$dW(t)/dt = J(t) - \sqrt{\eta}\,{\rm Tr}[(\hat{c}+\hat{c}^\dagger)\rho_{\rm c}(t)].$$

Here $\rho_{\rm c}(t)$ will be impure in general, but is still useful for understanding and designing feedback control.

[3] $\mathcal{D}[\hat{c}]\rho \equiv \hat{c}\rho\hat{c}^\dagger - \{\hat{c}\hat{c}^\dagger,\rho\}$, $\mathcal{H}[\hat{c}]\rho \equiv \hat{c}\rho + \rho\hat{c}^\dagger - {\rm Tr}\left[\hat{c}\rho+\rho\hat{c}^\dagger\right]\rho$, ${\rm E}[dW(t)] = 0$, $dW^2 = dt$.

8. Quantum Feedback Control in General

The first quantum feedback experiments (Walker and Jakeman, Machida and Yamamoto) were described theoretically using *Heisenberg equations*. In general this is tractable only for *linear* (or linearized) systems.

By contrast, the SME allows feedback to be treated for any system, at least numerically, (Wiseman, 1994), just by adding

$$d_{\mathrm{fb}}\rho_{\mathrm{c}}(t) = -i[\hat{H}_{\mathrm{fb}}(t), \rho_{\mathrm{c}}(t)]dt$$

where $\hat{H}_{\mathrm{fb}}(t) =$ arbitrary Hermitian functional of $J(s)$ for $s < t$.

Markovian Quantum Feedback

Say the feedback is based on the current *just measured*. For example

$$
\begin{aligned}
d\rho_c(t) &= dt\{-i[\hat{H}+\hat{H}_{\rm fb}(t),\rho_c(t)]+\mathcal{D}[\hat{c}]\rho_c(t)\}+\sqrt{\eta}dW(t)\mathcal{H}[\hat{c}]\rho_c(t) \\
J(t) &= \sqrt{\eta}\,{\rm Tr}[(\hat{c}+\hat{c}^\dagger)\rho_c(t)]+dW(t)/dt \\
\hat{H}_{\rm fb}(t) &= \hat{F}\lim_{h(\tau)\to\delta(\tau)}\int_0^\infty J(t-\tau)h(\tau)d\tau
\end{aligned}
$$

In this Markovian limit, a deterministic master equation can be derived (Wiseman & Milburn, 1993)

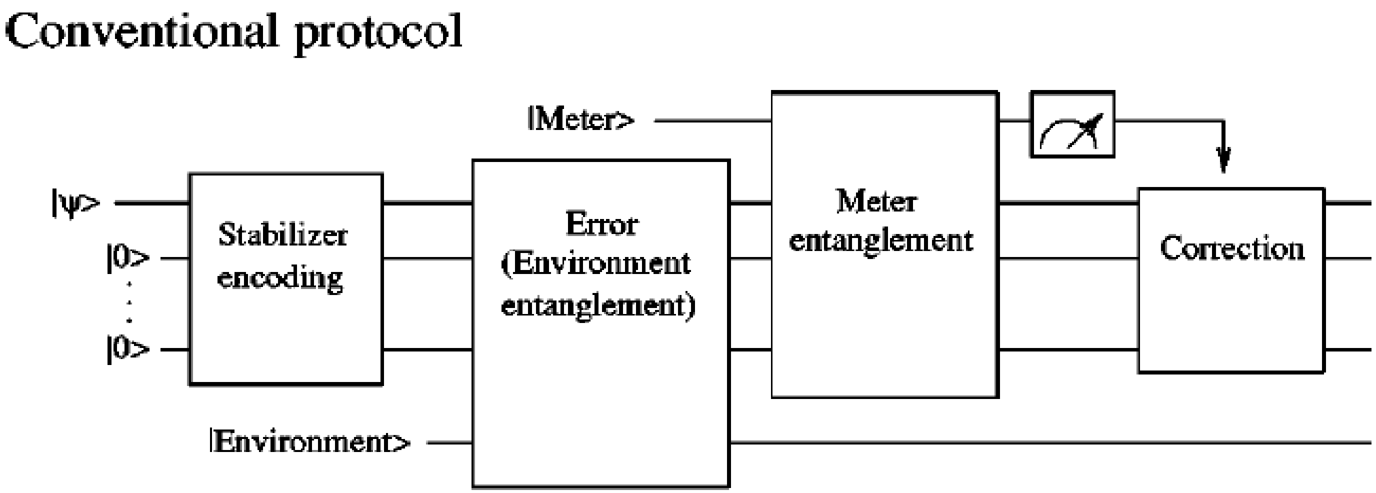
$$
\dot{\rho} = -i[\hat{H},\rho]+\mathcal{D}[\hat{c}]\rho - i\sqrt{\eta}[\hat{F},\hat{c}\rho+\rho\hat{c}^\dagger]+\mathcal{D}[\hat{F}]\rho
$$

This enormously simplifies the theoretical description of feedback.

Application 1: Quantum Error Correction

Quantum information (QI) can be stored in qubits (2-level quantum systems). If each qubit is coupled to a bath, the QI will be lost. Unlike classically, this is true *even if the baths are monitored*.

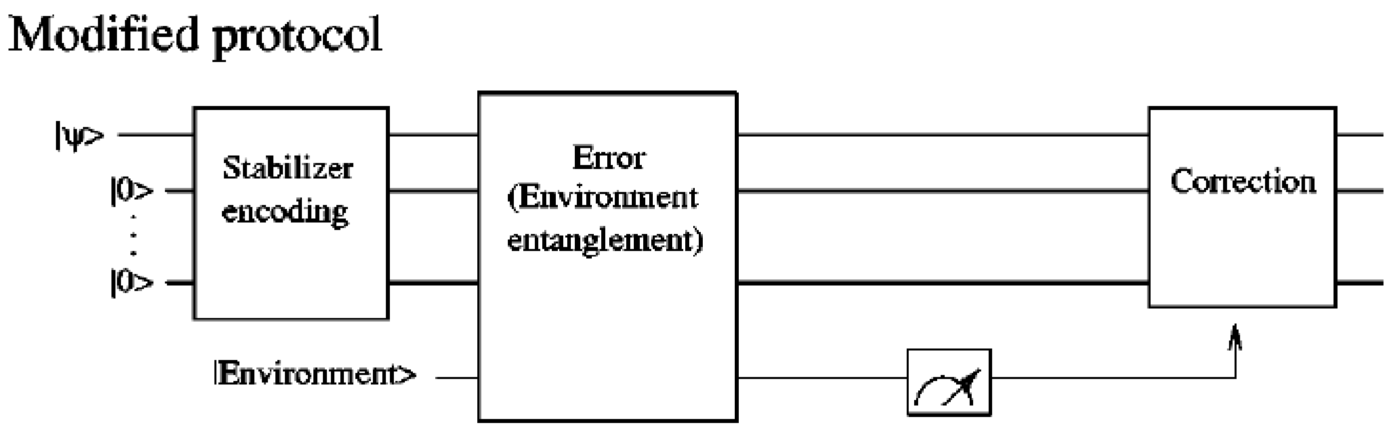
The conventional solution is *quantum encoding and error correction*, and in standard schemes (no monitoring of the baths), at least five *physical* qubits are required to encode one *logical* qubit.



Quantum Error Correction (cont.)

However, if the baths are monitored, then it is possible to encode such that quantum error correction can be done via Markovian feedback. (Ahn, Wiseman & Milburn, 2003).

This requires only $N+1$ *physical* qubits to encode N *logical* qubits.



Even if each qubit has multiple environments, only $N+3$ *physical* qubits to encode N *logical* qubits (Ahn,Wiseman & Jacobs, 2004).

Application 2: Deterministic Spin Squeezing

Time standards are kept by measuring the y-rotation of the Bloch vector of two-level atoms = (pseudo)spins relative to the RF "clock".

This requires a many-atom state of well-defined spin J_x. You might think it would be optimal to prepare every atom with $\sigma_x = 1$.

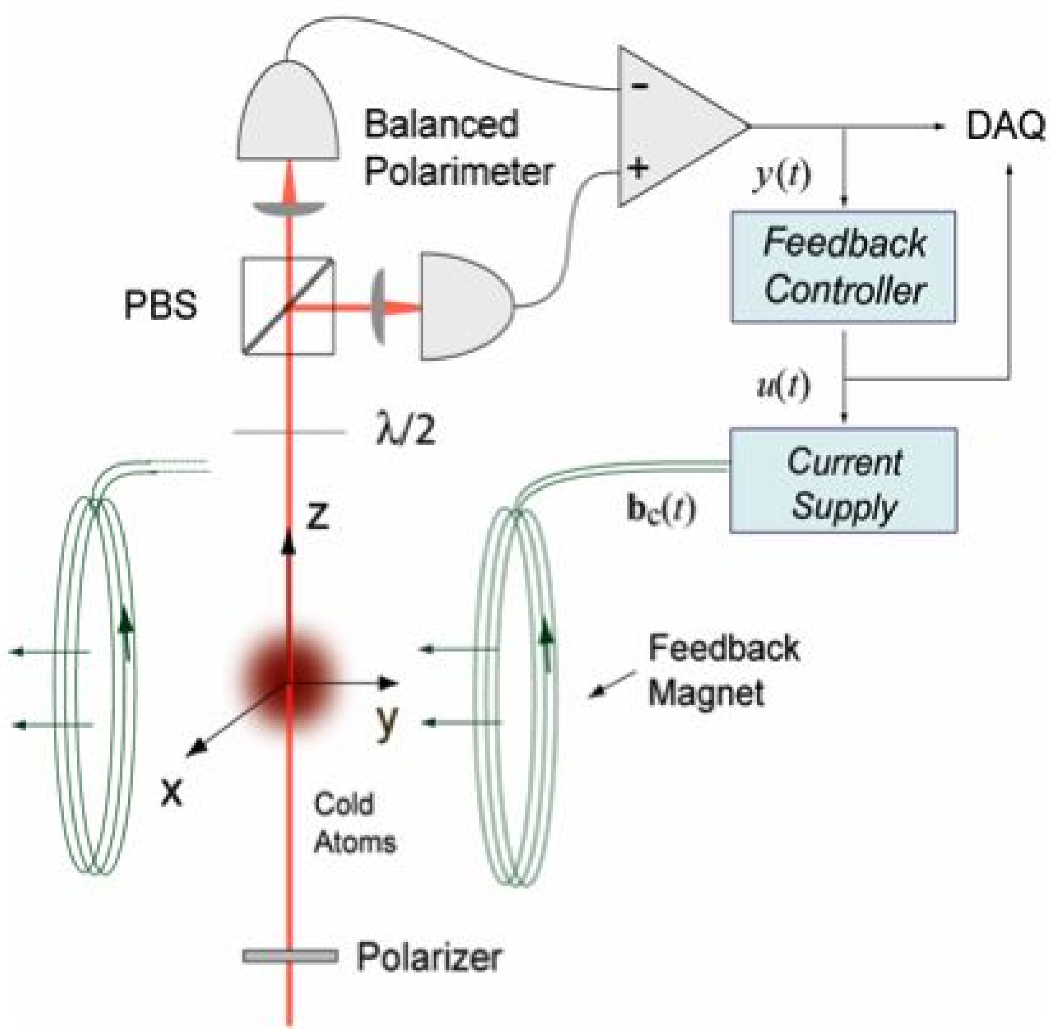
In fact the optimal state is *entangled*. This reduces ΔJ_z at the expense of ΔJ_y, hence is called *spin squeezing*.

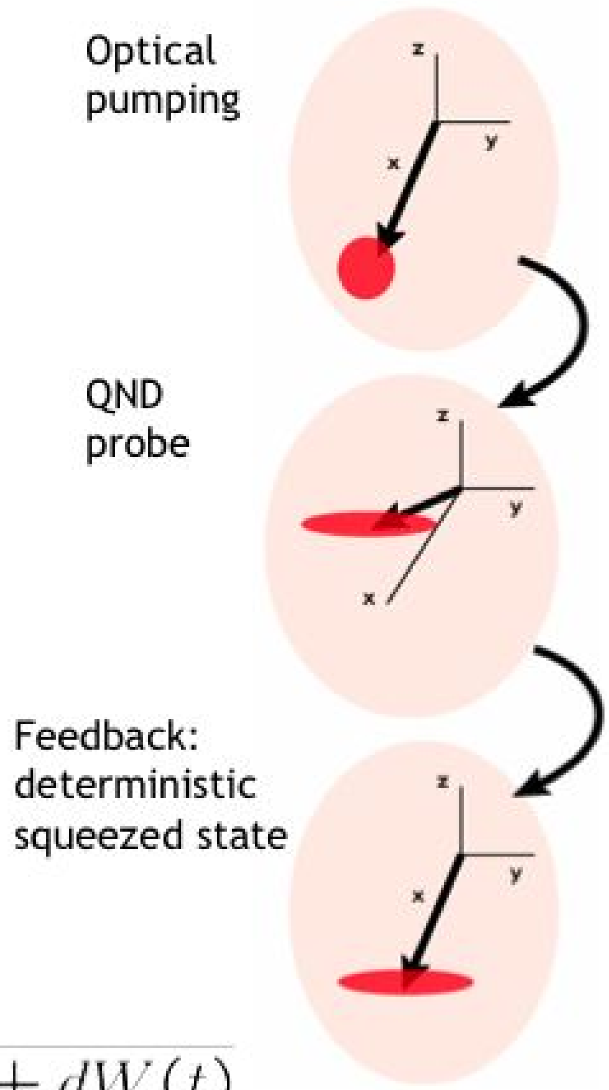
This can be achieved *deterministically* by measuring the collective J_z and using Markovian feedback

Steps towards this have been taken by the Mabuchi group.

Deterministic preparation of spin-squeezed states

(L.K. Thomsen, S. Mancini, and H.M. Wiseman: PRA **65**, 061801(R) (2002))





$$I_c(t)dt = 2\sqrt{M}\langle \hat{J}_Z(t)\rangle_c dt + dW(t)$$