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Manipulation of vortex motion in quantum lattice models

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Outline:

Introduction
- Quantum statistics and anyons
- Topological phases and quantum computation
- Toric code

Kitaev honeycomb lattice model
- Symmetries on torus
- Finite size effects on torus
- Vortex/anyon manipulation

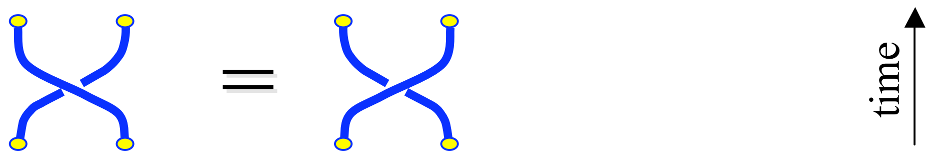
Quantum statistics

Configuration space of **n** indistinguishable particles in **d** dimensional space excluding diagonal points D:

$$M_n = (R^{nd} - D)/S_n$$

In (3+1) dimensions, the configuration space is simply connected; quantum mechanics permits only two kinds of statistics:

Exchanging particles in 3D space belongs to the permutation group S_n



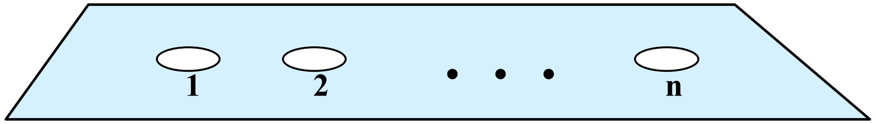
Bose-Einstein statistics: $\chi_+ (\sigma) = +1$

Fermi-Dirac statistics: $\chi_- (\sigma) = +1$ (even) or -1 (odd permutations)

Anyons

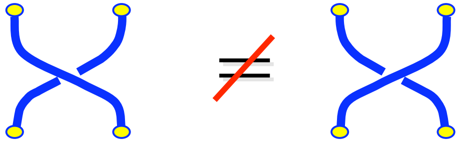
• particles with fractional statistics in (2+1) dimensional quantum mechanics

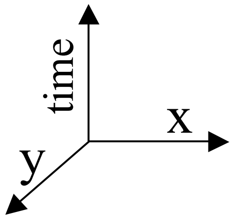
The configuration space of **n** indistinguishable particles in **2** dimensional space excluding diagonal points is multiply connected



Leinaas and Myrheim'77
Wilczek'82

Exchanging particles on a plane is not anymore an element of permutation group





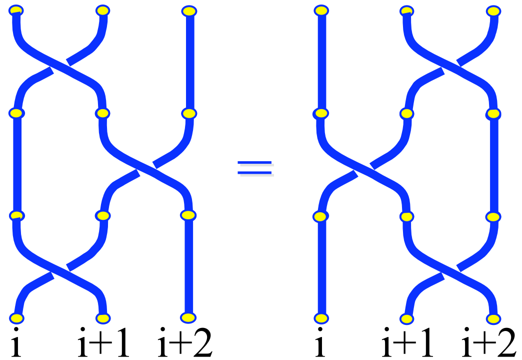
it is braiding, an element of a braid group!

Braid group B_n

A braid group for n strands (particles) has n generators $\{1, \sigma_1, \ldots, \sigma_{n-1}\}$ which satisfy:

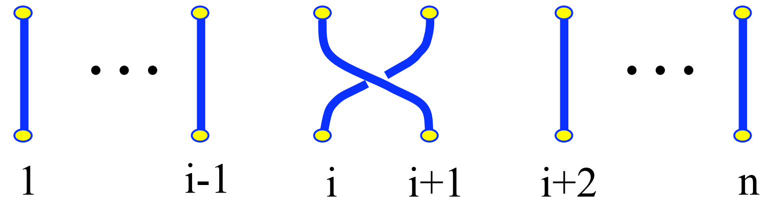
$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{for } |j - i| > 1$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$



Yang-Baxter equation

Example: σ_i



Artin, Ann. Math. 48, 101 (1947)

One-dimensional irreps of B_n correspond to abelian fractional statistics:

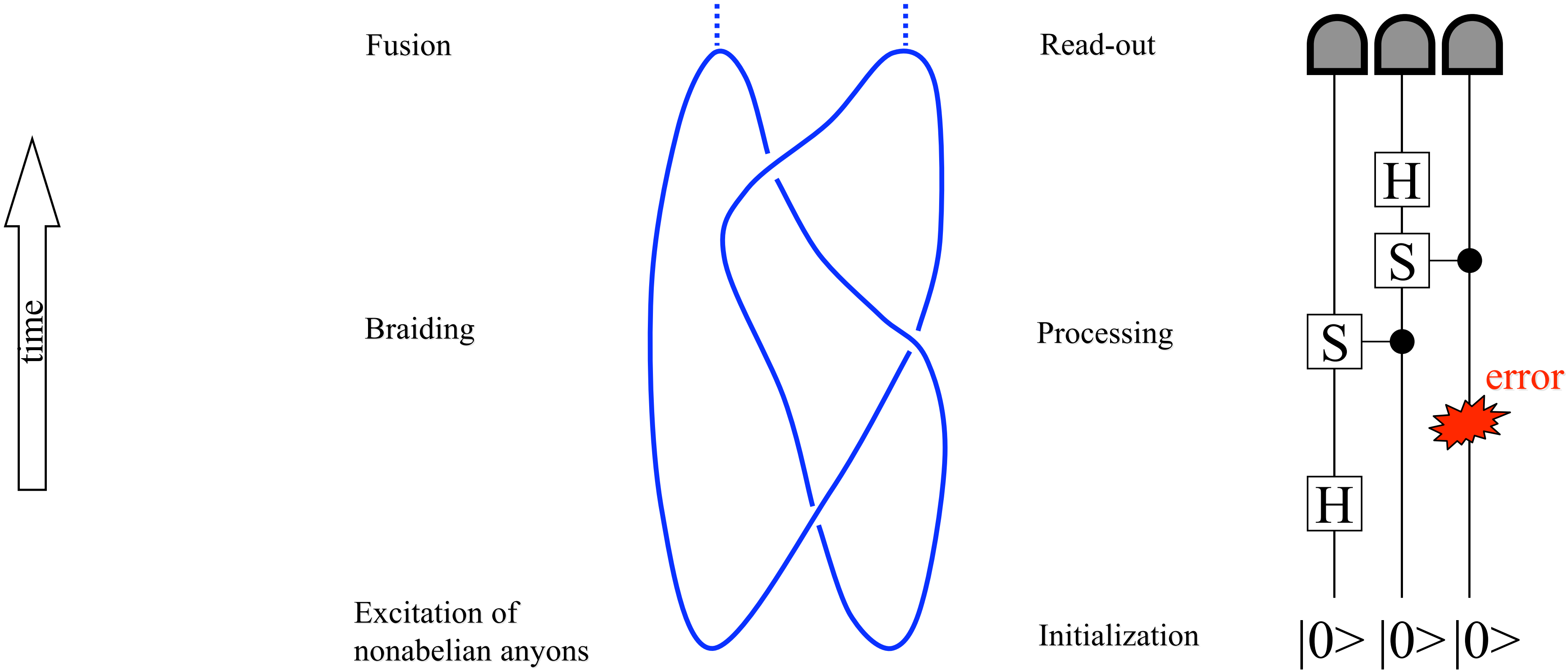
$$\chi_\theta\ (\sigma) = e^{i\theta} \qquad \text{from } U(1)$$

Higher dimensional irreps correspond to nonabelian fractional statistics:

$$\chi_\theta\ (\sigma) = e^{i\theta\Lambda} \qquad \text{e.g. from } SU(2)$$

Topological quantum computation

- quantum computing where fault-tolerance is naturally built into quantum computing hardware
- unique model of quantum computation which inspires new quantum algorithms

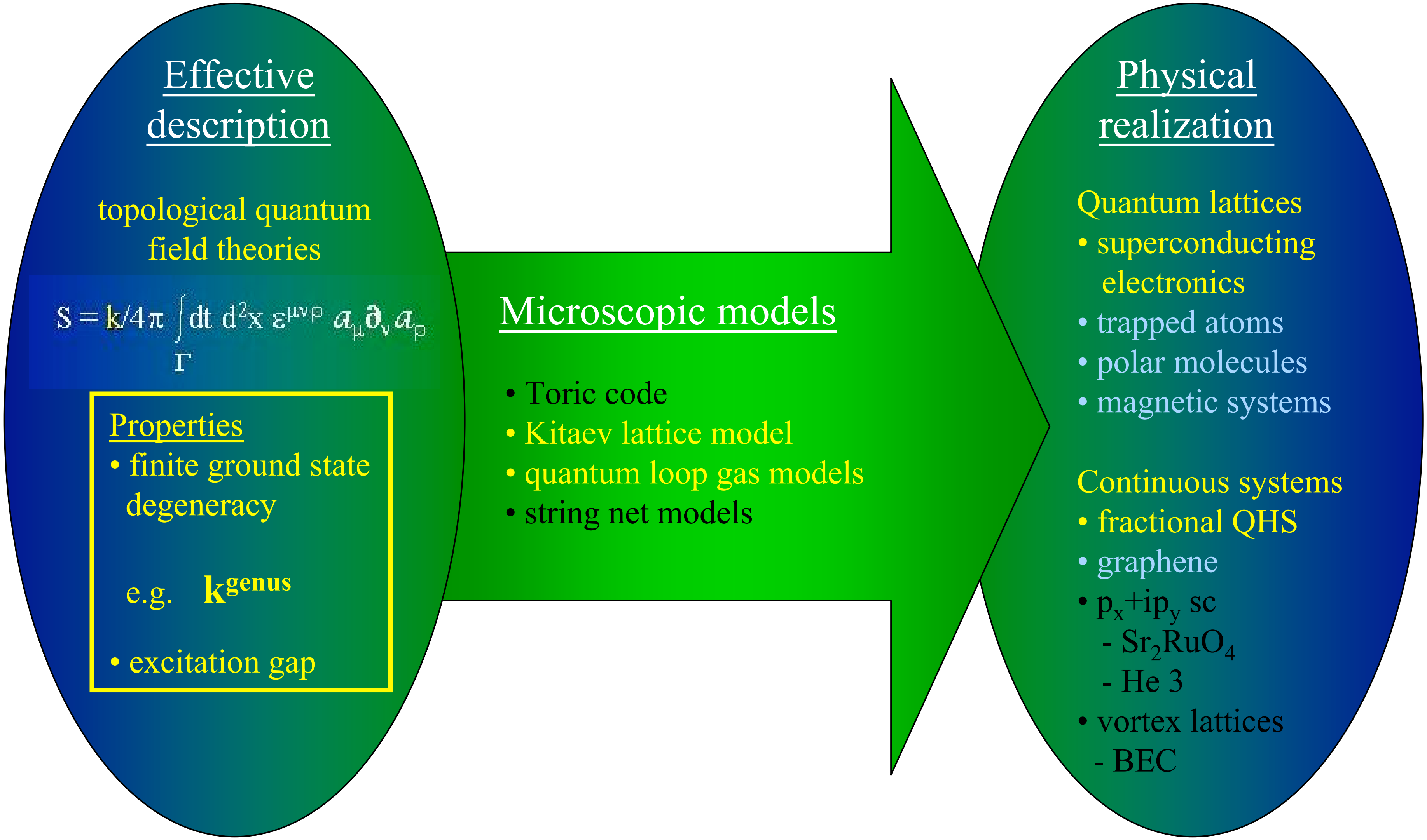


Topological phase

a ground state (possibly degenerate) of a certain gapped 2D many-body quantum system

- states within topological phase depend only on topology and are decoupled from local errors
- spectral gap exponentially suppresses thermal excitation of stray anyons and thus non-local errors

Topological phases



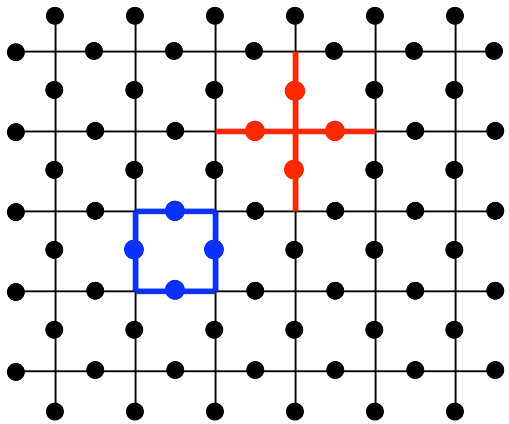
Topological phases

TQFT/CFT	*Realization*	*Application*
Abelian		
• $Z_2 \times Z_2$ theory (1, e, m, em)	Toric code Kitaev honeycomb model (phase A) Freedman loop gas model at k=1 Fendley quantum loop gas at k=2 etc.	Topological quantum memory
Non- abelian		
• $SU(2)_2$ Ising theory $(1, \sigma, \varepsilon)$ $\varepsilon \times \varepsilon = 1$ $\varepsilon \times \sigma = \sigma$ $\sigma \times \sigma = 1 + \varepsilon$	Kitaev honeycomb model (phase B in magnetic field) • neutral atoms in optical lattices • lattices of polar molecules • arrays of Josephson junctions Quantum Hall state at $\nu = 5/2$ $p_x + ip_y$ superconductors	Quantum computation with partial topological protection
• $SU(2)_3$ Z_3 Parafermion theory $SO(3)_3$ Fibonacci $(1, \tau)$ $\tau \times \tau = 1 + \tau$	Quantum Hall state at $\nu = 12/5$ Fendley loop gas model at k=3	Topological quantum computation
• Other general theories	Levin-Wen string nets Fendley loop gas models	

Toric code

Spin-1/2 particles on edges of a square lattice

$$H_{\text{eff}} = -J_{\text{eff}}\left(\sum_{\text{vertices}} Q_s + \sum_{\text{plaquettes}} Q_p\right)$$



$$Q_s = \Pi_{vertex}\sigma^x$$

$$Q_p = \Pi_{plaquette}\sigma^z \qquad [Q_s, Q_p] = 0$$

Toric code

• is exactly solvable

• exhibits abelian topological phase, specifically

- its ground state is 4-fold degenerate on torus, and
- the system has a robust spectral gap to the first excited state at thermodynamic limit

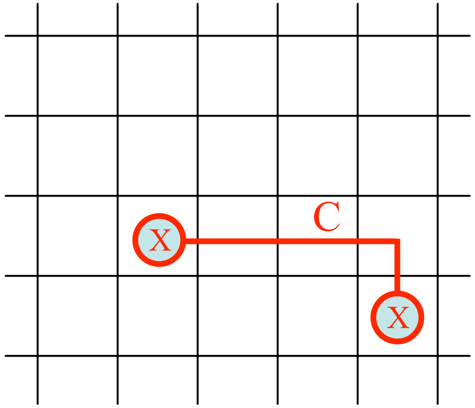
Particles of the toric code

$$H_{\text{eff}} = -J_{\text{eff}} \left(\sum_{\text{vertices}} Q_s + \sum_{\text{plaquettes}} Q_p \right)$$

$Q_s = \Pi_{vertex} \sigma^x$ $Q_p = \Pi_{plaquette} \sigma^z$

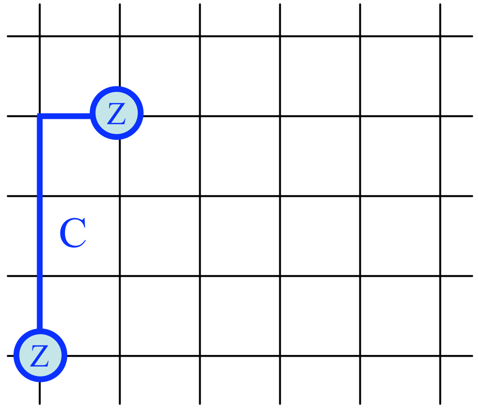
Magnetic charges m

$Q_s = \Pi_C \, \sigma^x$



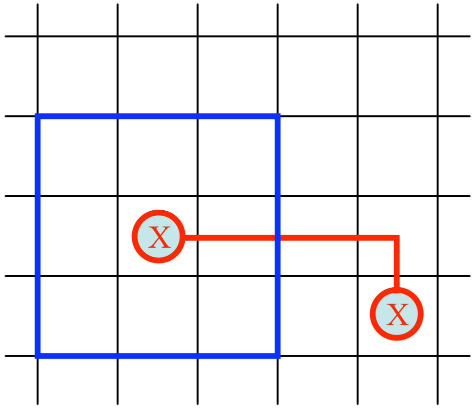
Electric charges e

$Q_s = \Pi_C \, \sigma^z$



Composite particle me

Braiding



e and e (or m and m) particles braid as bosons

e and m particles are semions, i.e. anyons with the statistical phase *i*

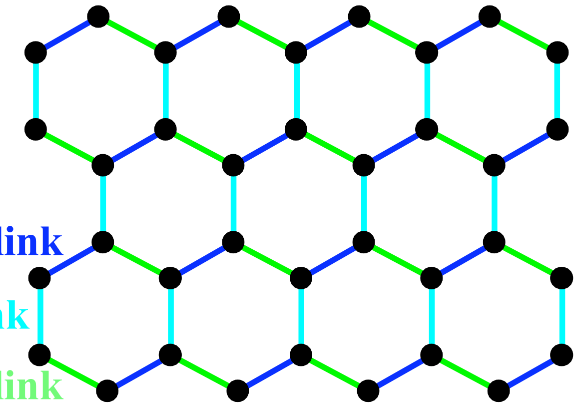
e-m composite particle is fermion

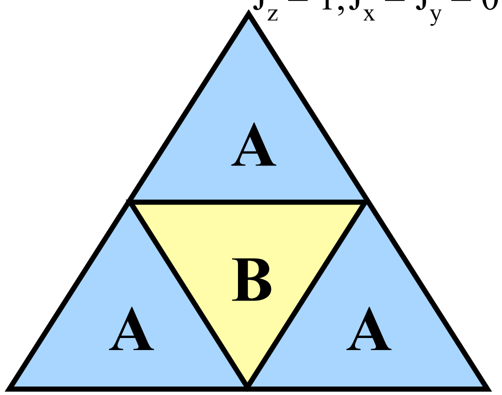
Kitaev honeycomb lattice model

$$H_0 = J_x \sum_{\substack{i,j \\ \text{x-link}}} \sigma^x_i \sigma^x_j + J_y \sum_{\substack{i,j \\ \text{y-link}}} \sigma^y_i \sigma^y_j + J_z \sum_{\substack{i,j \\ \text{z-link}}} \sigma^z_i \sigma^z_j$$

$$= \Sigma_\alpha J_\alpha \Sigma_{i,j} \sigma^\alpha_i \sigma^\alpha_j = \Sigma_\alpha J_\alpha \Sigma_{i,j} K^\alpha_{ij}$$

α -link:





Phase diagram:

- **phase A** - can be mapped perturbatively onto Toric code with particles (1, e, m, em);
- **phase B** - gapless.

In magnetic field:

$$H = H_0 + \Sigma_i \Sigma_{\alpha=x,y,z} B_\alpha \sigma_{\alpha,i}$$

- phase B acquires a gap and becomes non-abelian topological phase of Ising type

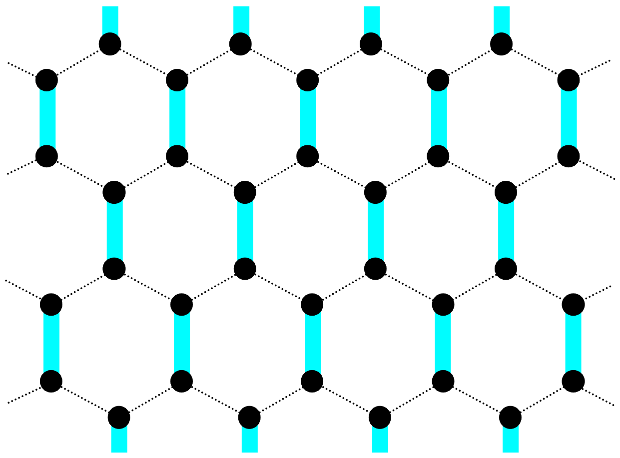
A.Y.Kitaev, Ann. Phys. 321, 2 (2006).

Mapping abelian phase onto Toric code

$J_z >> J_y, J_x$

$$H_D = -J_z \sum_{z-links} \sigma_j^z \sigma_k^z,$$ "dimers"

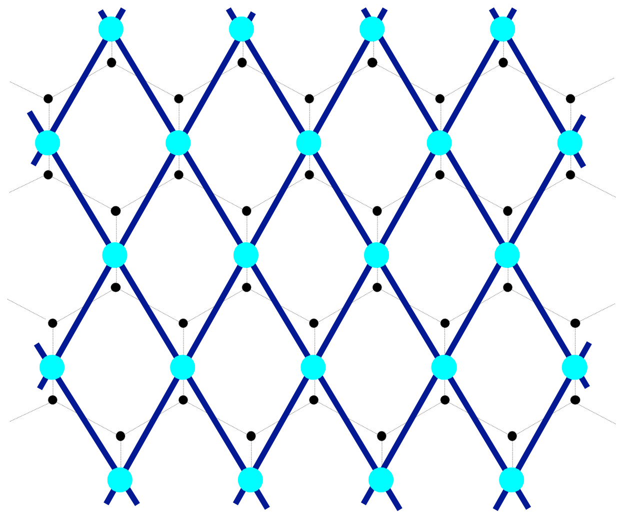
$$V = -J_x \sum_{x-links} \sigma_j^x \sigma_k^x - J_y \sum_{y-links} \sigma_j^y \sigma_k^y$$



Effective spins

- are formed by ferromagnetic ground states of $-J_z\sigma_j^z\sigma_k^z$

$$|\uparrow\rangle_{eff} = |\uparrow\uparrow\rangle \quad |\downarrow\rangle_{eff} = |\downarrow\downarrow\rangle$$



A.Y.Kitaev, *Fault-tolerant quantum computation by anyons*,
Ann. Phys. 303, 2 (2003).

Mapping abelian phase onto Toric code

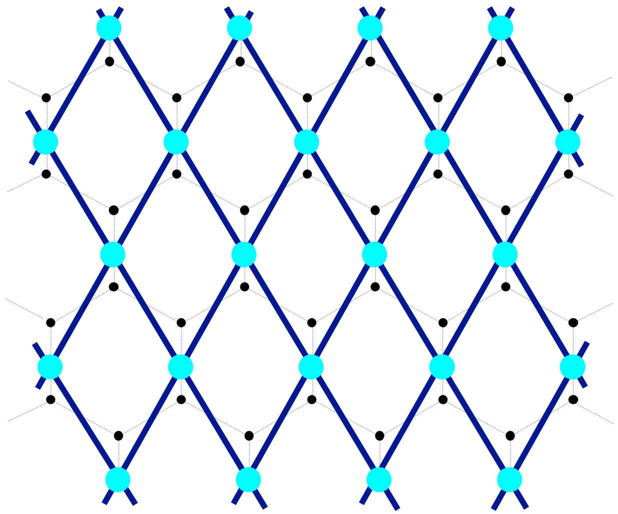
Effective Hamiltonian

first non-constant term of perturbation theory occurs on the 4th order

$$H_{\text{eff}} = -\frac{J_x^2 J_y^2}{16|J_z|^3} \sum_p Q_p,$$

$$Q_p = \sigma^y_{\text{left}(p)} \sigma^y_{\text{right}(p)} \sigma^z_{\text{up}(p)} \sigma^z_{\text{down}(p)}$$

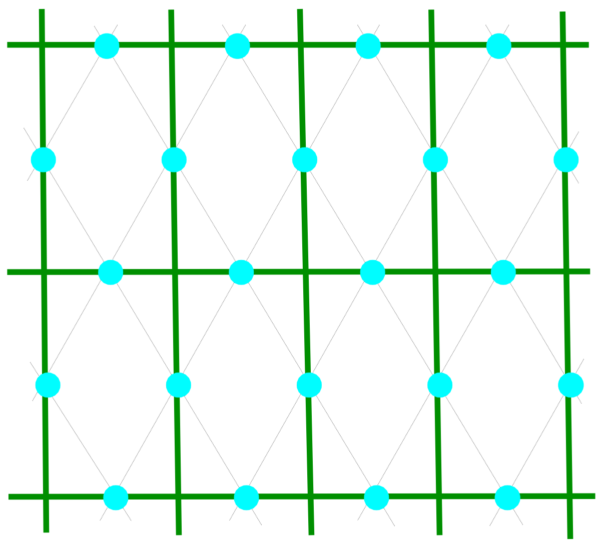
defined on the square lattice with effective spins on the vertices



Toric code

the effective Hamiltonian is unitarily equivalent to toric code on the green lattice

$$H_{\text{eff}} = -J_{\text{eff}} \left(\sum_{\text{vertices}} Q_s + \sum_{\text{plaquettes}} Q_p \right)$$



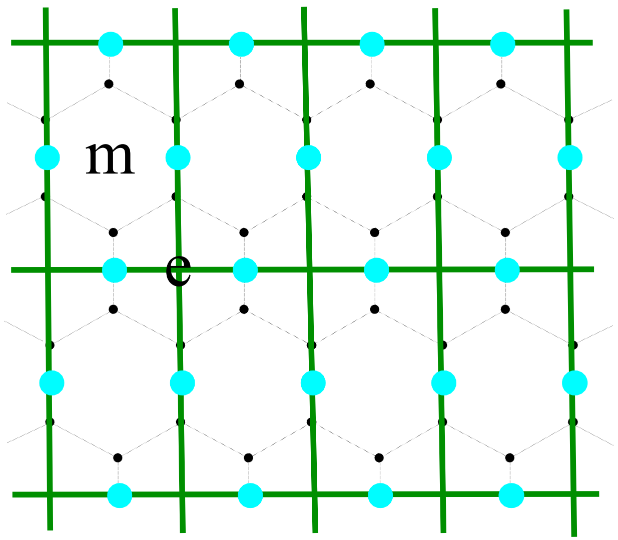
A.Y.Kitaev, *Fault-tolerant quantum computation by anyons*, Ann. Phys. 303, 2 (2003).

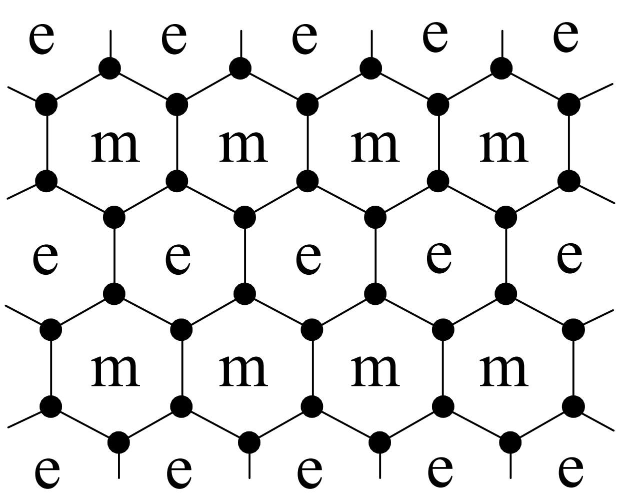
Mapping abelian phase onto Toric code

Toric code particle types

• magnetic charges live on plaquettes of the toric code lattice

• electric charges live on its vertices



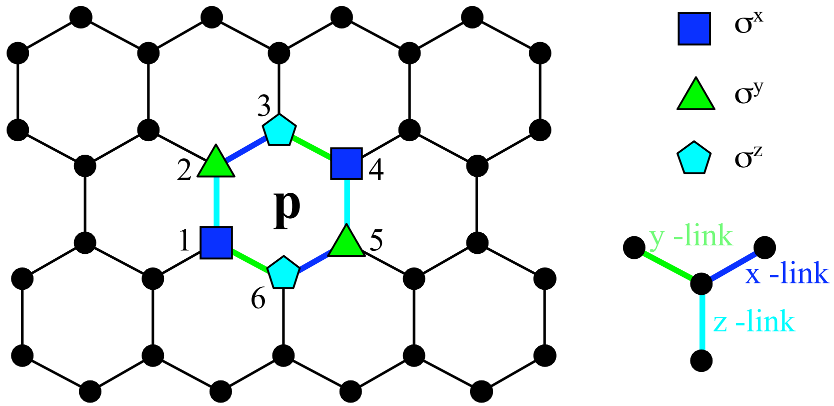


A.Y.Kitaev, *Fault-tolerant quantum computation by anyons*,
Ann. Phys. 303, 2 (2003).

Vortex operators

$$W_p = \sigma^x_1 \sigma^y_2 \sigma^z_3 \sigma^x_4 \sigma^y_5 \sigma^z_6 =$$

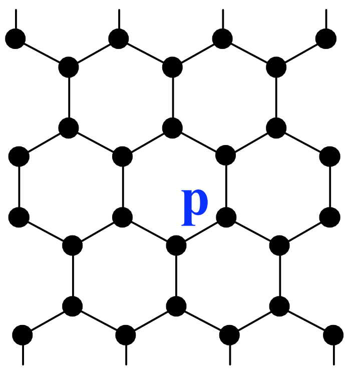
$$= K^z_{1,2} K^x_{2,3} K^y_{3,4} K^z_{4,5} K^x_{5,6} K^y_{6,1}$$



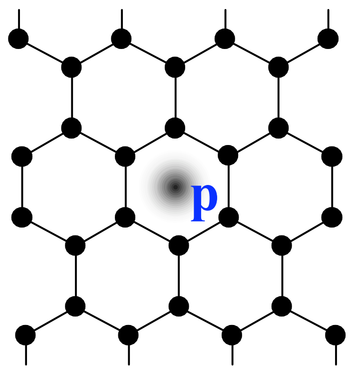
$$[H_0, W_p] = 0 \qquad (K^\alpha_{k,k+1})^2 = 1 \qquad K^\beta_{k+1,k+2} K^\alpha_{k,k+1} = - K^\alpha_{k,k+1} K^\beta_{k+1,k+2}$$

$$H_0 |n\rangle = E_n |n\rangle$$

$w_p = \langle n|W_p|n\rangle = +1$



$w_p = \langle n|W_p|n\rangle = -1$



Vortex sectors

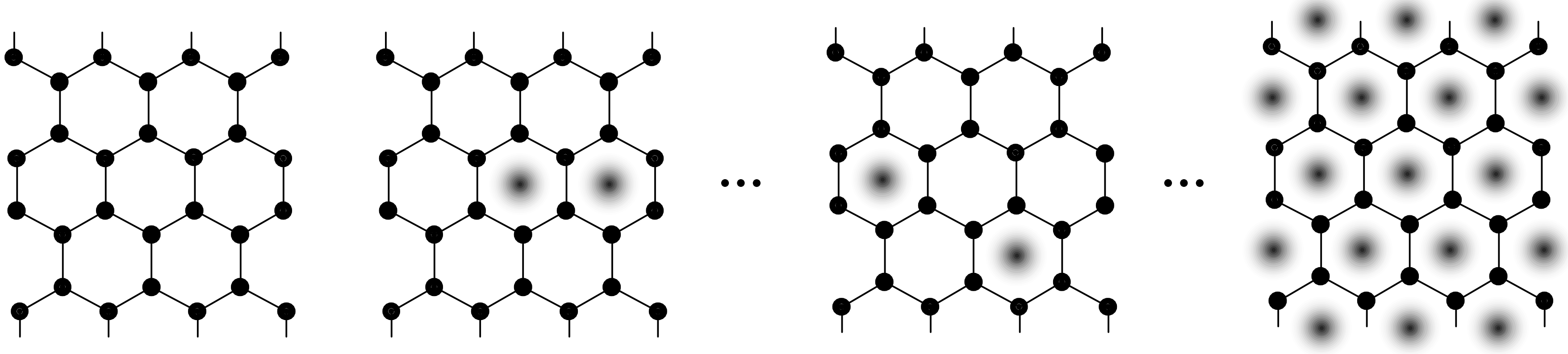
Each energy eigenstate |n> is characterized by some vortex configuration

$$\{w_p = \langle n|W_p|n\rangle = \pm 1\} \text{ for all plaquettes p}$$

also the vortices are always excited in pairs,
i.e. even-vortex configurations are relevant on closed surfaces or infinite plane,

the Hilbert space splits into vortex sectors, i.e. subspaces of the system with a particular configuration of vortices

$$L = \bigoplus_{w_1,\ldots\ldots,w_m} L_{w_1,\ldots,w_m}$$

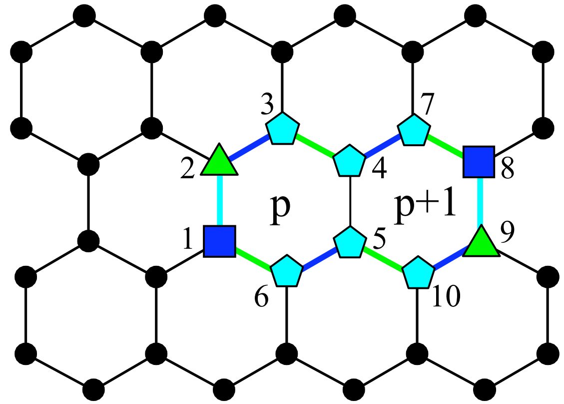


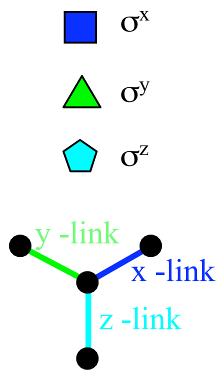
vortex free sector — examples from two-vortex sectors — full vortex sector

Products of vortex operators

$$W_p W_{p+1} = K^z_{1,2} K^x_{2,3} K^y_{3,4} K^x_{5,6} K^y_{6,1} K^x_{4,7} K^y_{7,8} K^z_{8,9} K^x_{9,10} K^y_{10,5}$$

(we used $(K^{\alpha}_{k,k+1})^2 = 1$)



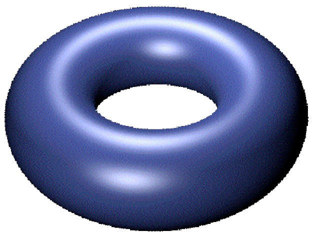


Products of vortex operators generate closed loops

$$K_{i,j}^{\alpha(1)} K_{j,k}^{\alpha(2)} \ldots K_{p,q}^{\alpha(M-1)} K_{q,i}^{\alpha(M)}$$

On torus, this gives the condition

$$\Pi_p W_p = 1$$



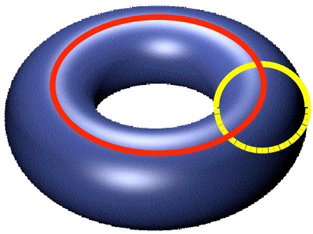
Loop symmetries on torus

For a system of N spins on torus (i.e. a system with N/2 plaquettes), $\Pi_p W_p = 1$ implies that there are N/2-1 independent vortex quantum numbers $\{w_1, \dots , w_{N/2-1}\}$.

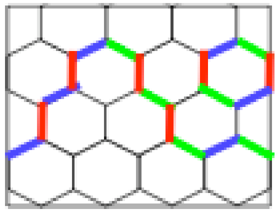
Loops on torus $K_{i,j}^{\alpha(1)} K_{j,k}^{\alpha(2)} \dots K_{p,q}^{\alpha(M-1)} K_{q,i}^{\alpha(M)}$

- all homologically trivial loops are generated by plaquette operators

- two distinct homologically nontrivial loops needed to generate the full loop symmetry group (the third nontrivial loop is a product of these two).



The full loop symmetry of the torus is the abelian group with N/2+1 independent generators of the order 2 ($loop^2=I$), i.e. $Z_2^{N/2+1}$.



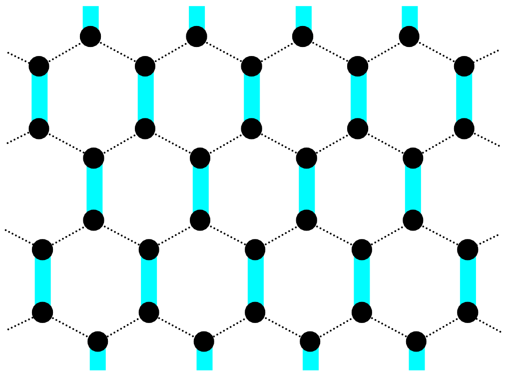
All loop symmetries can be written as

$$C_{(k,l)} = G_k F_l(W_1, W_2, \dots , W_{N-1})$$

where k is from {0,1,2,3} and $G_0 = I$, and G_1, G_2, G_3 are arbitrarily chosen symmetries from the three nontrivial homology classes, and F_l, with l from $\{1, \dots, 2^{N/2-1}\}$, run through all monomials in the W_p.

Effective (low energy) Hamiltonian

$J_z >> J_y, J_x$ $\quad H = H_D + U$



$$H_D = -J_z \sum_{z-links} \sigma_j^z \sigma_k^z,$$

$2^{N/2}$ degenerate ground state = "ground state manifold"

$$U = -J_x \sum_{x-links} \sigma_j^x \sigma_k^x - J_y \sum_{y-links} \sigma_j^y \sigma_k^y$$

Brillouin-Wigner perturbation theory

For any exact eigenstate |ψ> of the full Hamiltonian H, the projection onto ferromagnetic subspace $|\psi_0\rangle = \mathcal{P}|\psi\rangle$ satisfies

$$\left[E_0 + \sum_{n=1}^{\infty} H^{(n)}\right]|\psi_0\rangle = E|\psi_0\rangle = H_{\text{eff}}|\psi_0\rangle,$$

$H_D |\psi_0\rangle = E_0 |\psi_0\rangle$

where $\quad H^{(n)} = \mathcal{P}U\mathcal{G}^{n-1}\mathcal{P}$ $\quad \mathcal{G} = [1/(E - H_0)](1 - \mathcal{P})U$

$|\psi\rangle = (1 - \mathcal{G})^{-1} |\psi_0\rangle$

Calculating n-th order corrections is equivalent to finding the nonzero elements of the matrix $H^{(n)}$

Contributions to $H^{(n)}$ comes from the length n products $K_{ij}^{\alpha^{(1)}}, \ldots, K_{kl}^{\alpha^{(n)}}$ with $\alpha^{(m)} \in x, y$ that preserves the low energy subspace.

G. Kells, A. T. Bolukbasi, V. Lahtinen, J. K. Slingerland, J. K. Pachos and J. Vala,
Topological degeneracy and vortex manipulation in the Kitaev honeycomb model, Phys. Rev. Lett. **101**, 240404 (2008).

Effective (low energy) Hamiltonian

The resulting low-energy Hamiltonian can be written in terms of operators acting on the spins of the "dimers" using the transformation rules

$$\mathcal{P}[\sigma^x \otimes \sigma^y] \rightarrow +\sigma^y_e, \qquad \mathcal{P}[\sigma^x \otimes \sigma^x] \rightarrow +\sigma^x_e,$$
$$\mathcal{P}[\sigma^y \otimes \sigma^y] \rightarrow -\sigma^x_e, \qquad \mathcal{P}[\sigma^z \otimes I] \rightarrow +\sigma^z_e,$$
$$\mathcal{P}[\sigma^z \otimes \sigma^z] \rightarrow +I_e,$$

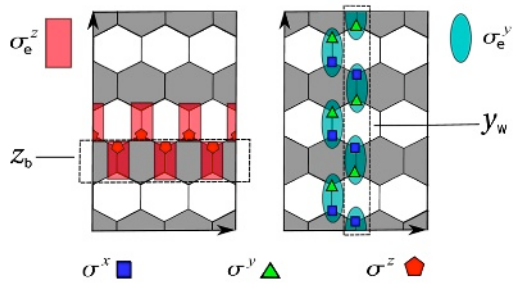
The lowest order non-constant contribution comes from the plaquette operators

$$\mathcal{P}[W_p] \rightarrow Q_p = \sigma^y_{e(l)}\sigma^y_{e(r)}\sigma^z_{e(u)}\sigma^z_{e(d)}$$

Expanding to all orders gives the contributions from all loop symmetries both homogically trivial and homologically nontrivial:

$$Z \equiv \prod_i \sigma^z_i$$

$$\mathcal{P}(Z) \rightarrow z$$



$$V \equiv \prod K^{x,y}_{jk} \prod K^{y,x}_{lm}$$

$$\mathcal{P}(V) \rightarrow y$$

$$H_{eff} = \sum_{i=0}^{3} \sum_{j=1}^{2^{N/2-2}} c_{i,j} G_i(z,y) F_j(Q_1, Q_2, \ldots\ldots Q_{N/2-2})$$

trivial
$W_p \longrightarrow Q_p$

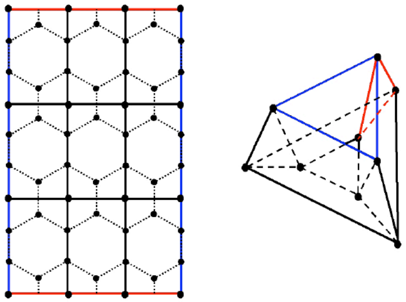
nontrivial
- reflects topology

G. Kells, A. T. Bolukbasi, V. Lahtinen, J. K. Slingerland, J. K. Pachos and J. Vala, ***Topological degeneracy and vortex manipulation in the Kitaev honeycomb model***, Phys. Rev. Lett. **101**, 240404 (2008).

Finite-size effects in small systems on torus

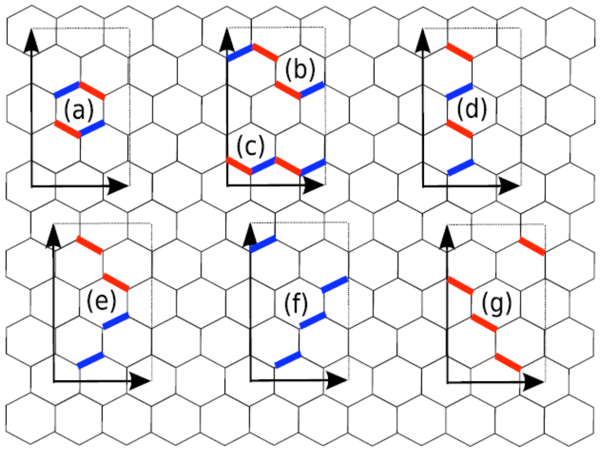
Toric code emerges on the 4th order of perturbation theory the low energy sector of H:

$$\sigma(H) - E_0 = \sigma(J_{eff} H_{TC}) + O(J^6) \qquad J_{eff} = \frac{J_x^2 J_y^2}{16|J_z|^3} = \frac{J^4}{16|J_z|^3} \qquad J = J_x = J_y \ll J_z$$



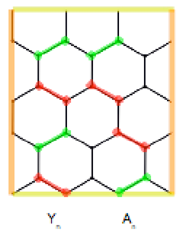
The minimal size of the lattice with no finite size terms on the 4th order is
N = 36
i.e. Toric code on the lattice of 3x3 square plaquettes which properly represents the torus

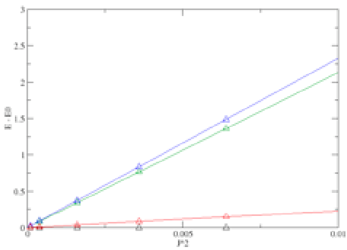
For smaller systems, the finite size effects are substantial on the 4th order, for example N=16:



$$\begin{aligned} H^{(4)} &= -\frac{J_x^2 J_y^2}{16|J_z|} \sum^{8} (Q_n + R_n - 5A_n) \\ &\quad - \frac{J_x^2 J_y^2}{16|J_z|} \sum^{4} (Z_n + 5Y_n) \\ &\quad - \frac{5}{16|J_z|} (J_x^4 \sum_{n=1}^{2} X_n + J_y^4 \sum_{n=3}^{4} X_n) \end{aligned}$$

The toric code spectrum can be reconstructed by extracting the finite size effects from the spectrum of the full model



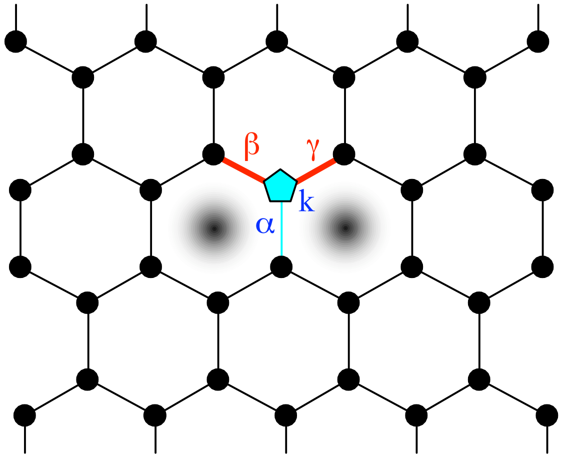


G. Kells, N. Moran and J. Vala,
Finite size effects in the Kitaev honeycomb lattice model on torus,
J. Stat. Mech. – Th. Exp., (2009) P03006

Creating and annihilating vortices

$$W_p = \sigma^z{}_1\sigma^y{}_2\,\sigma^x{}_3\sigma^z{}_4\,\sigma^y{}_5\sigma^x{}_6 = K^x{}_{1,2}K^z{}_{2,3}K^y{}_{3,4}K^x{}_{4,5}K^z{}_{5,6}K^y{}_{6,1}$$

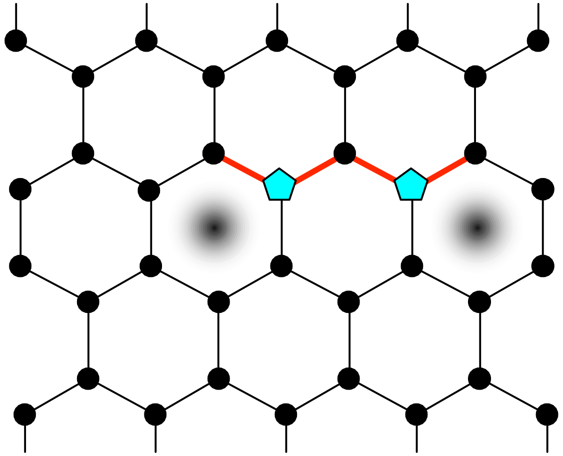
$$W_p|\psi\rangle = w_p|\psi\rangle \qquad \text{where } w_p = \pm 1$$

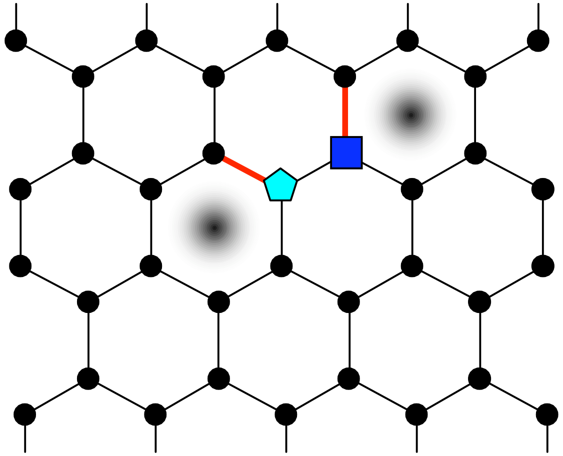


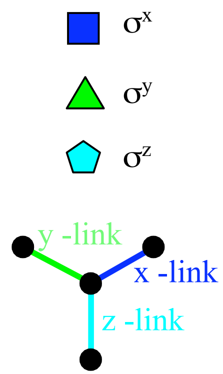
A Pauli operator $\sigma^\alpha{}_k$ at a vertex k
flips the vortex states of those plaquettes
which share the link $\alpha(k)$ originating at the vertex k

$$W_p\sigma^\alpha{}_k|\psi\rangle = -\,w_p|\psi\rangle$$

… and the same kind of operation can be used to move vortices between plaquettes …







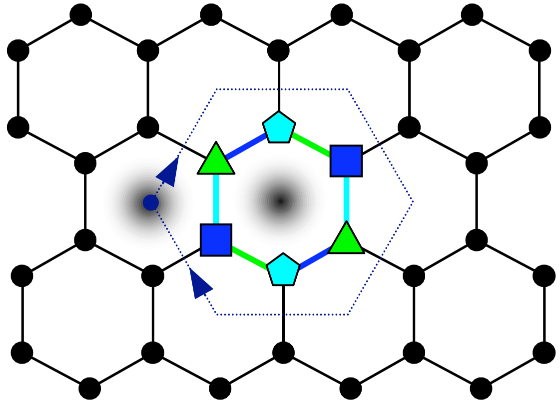
… however, moving vortices cost energy in general

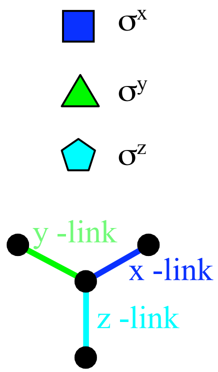
$$\langle\phi|\sigma^\alpha{}_k H \sigma^\alpha{}_k|\phi\rangle = \langle\phi|H|\phi\rangle + 2J_\beta\langle\phi|K^\beta{}_{\beta\,(k)}|\phi\rangle + 2J_\gamma\,\langle\phi|K^\gamma{}_{\gamma(k)}|\phi\rangle$$

and that in general spoils the statistical phase.

Statistics of vortices

Exploiting spin-statistics theorem



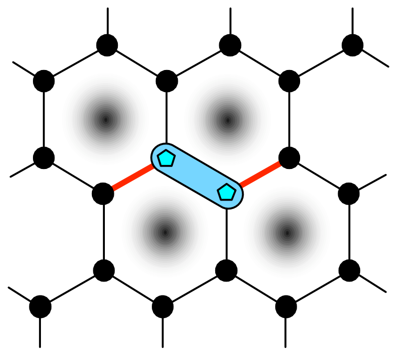


The statistical phase associated with 2π rotation of a pair of vortices is -1, that means

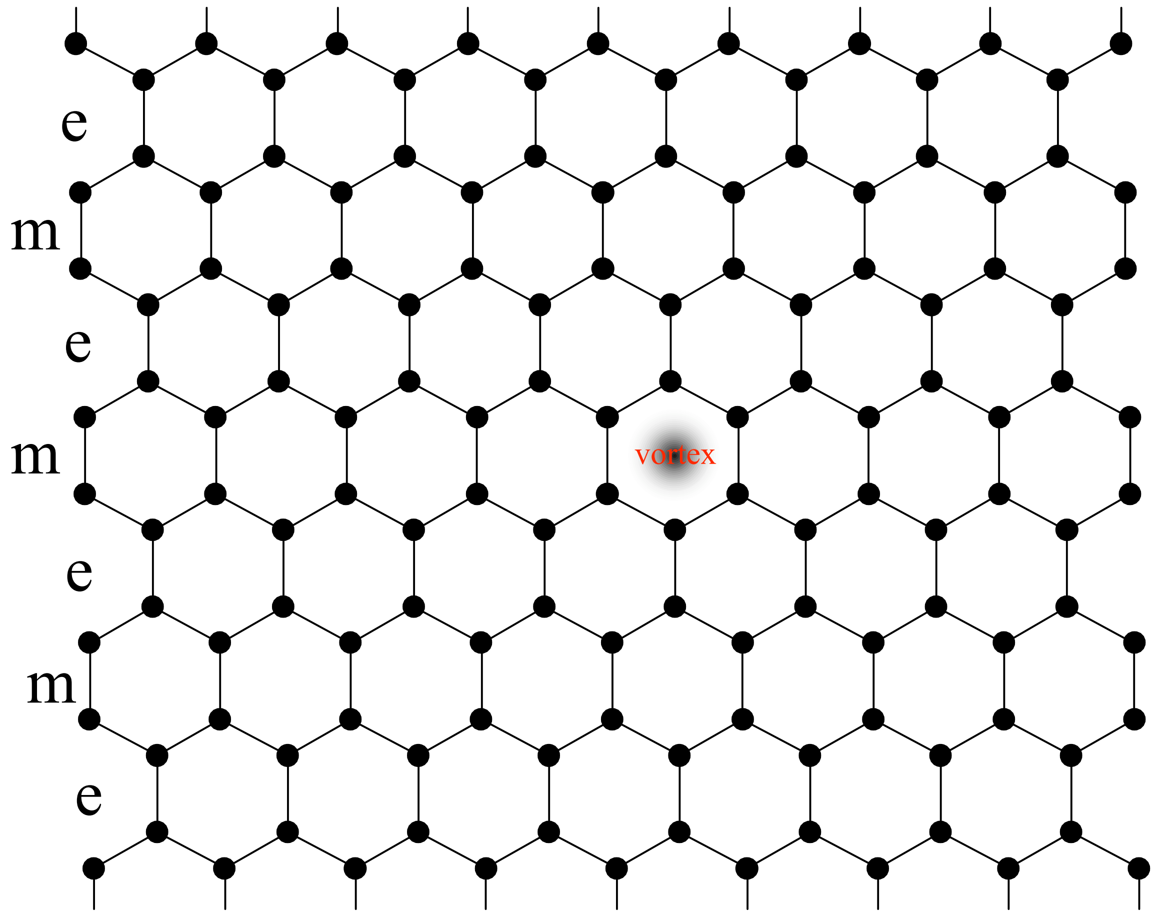
- the pair of vortices form a fermion (of a new kind)
- the vortices are semions, i.e. abelian anyons with the statistical phase $i = (-1)^{1/2}$

These fermions are created by applying a two-spin operator

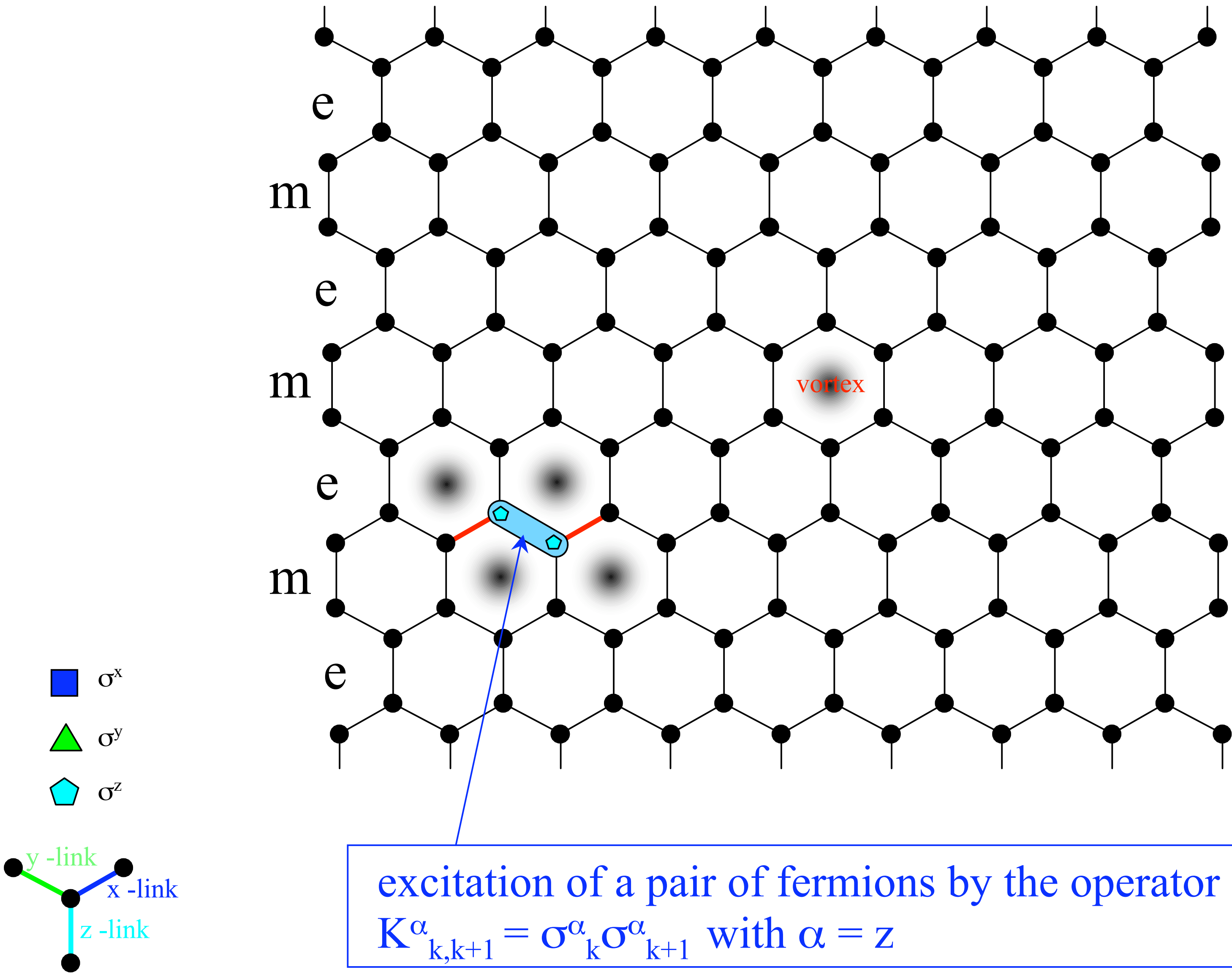
$$K^{\alpha}_{k,k+1} = \sigma^{\alpha}_{k}\sigma^{\alpha}_{k+1}$$



Mutual statistics of fermions and vortices

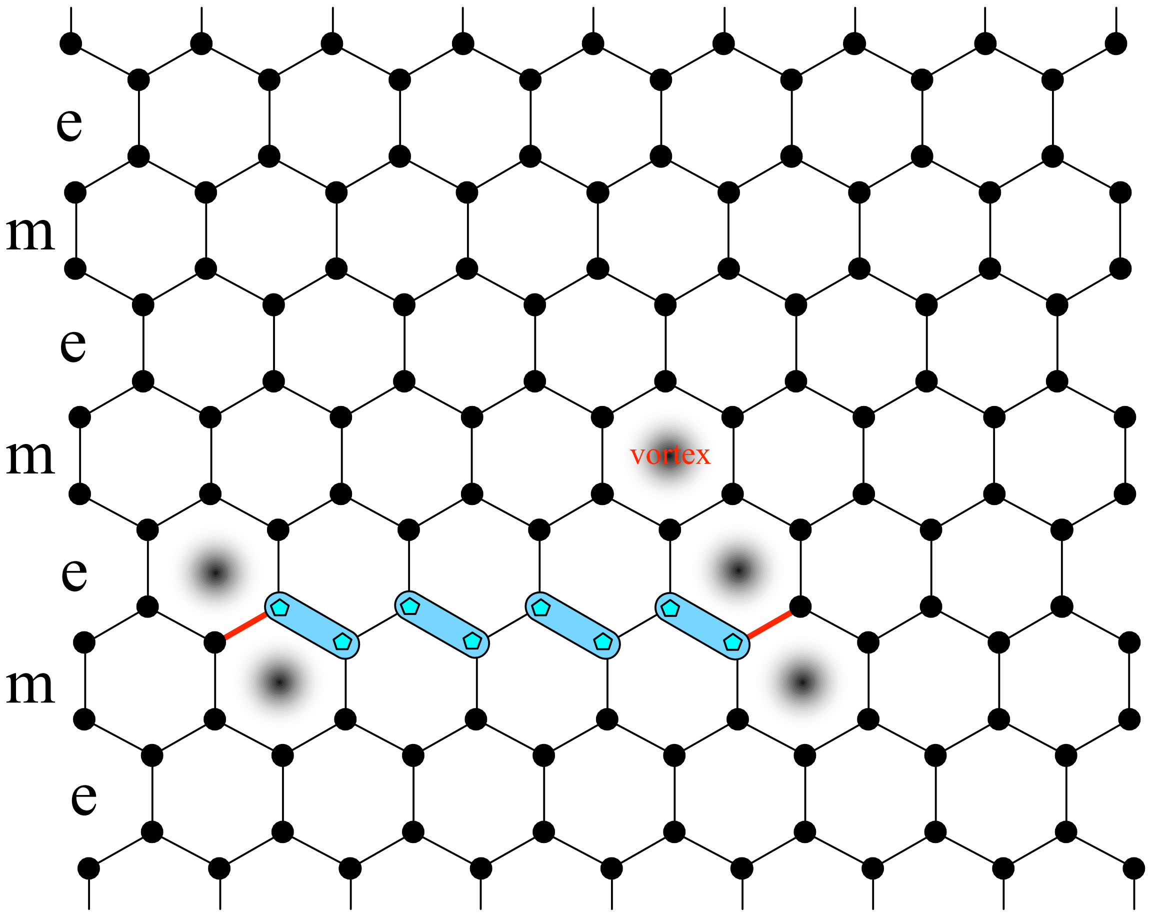


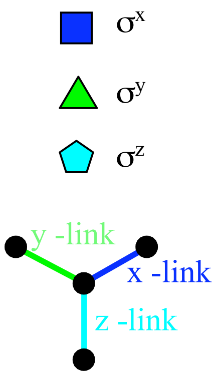
Mutual statistics of fermions and vortices



excitation of a pair of fermions by the operator
$K^{\alpha}_{k,k+1} = \sigma^{\alpha}_{k}\sigma^{\alpha}_{k+1}$ with $\alpha = z$

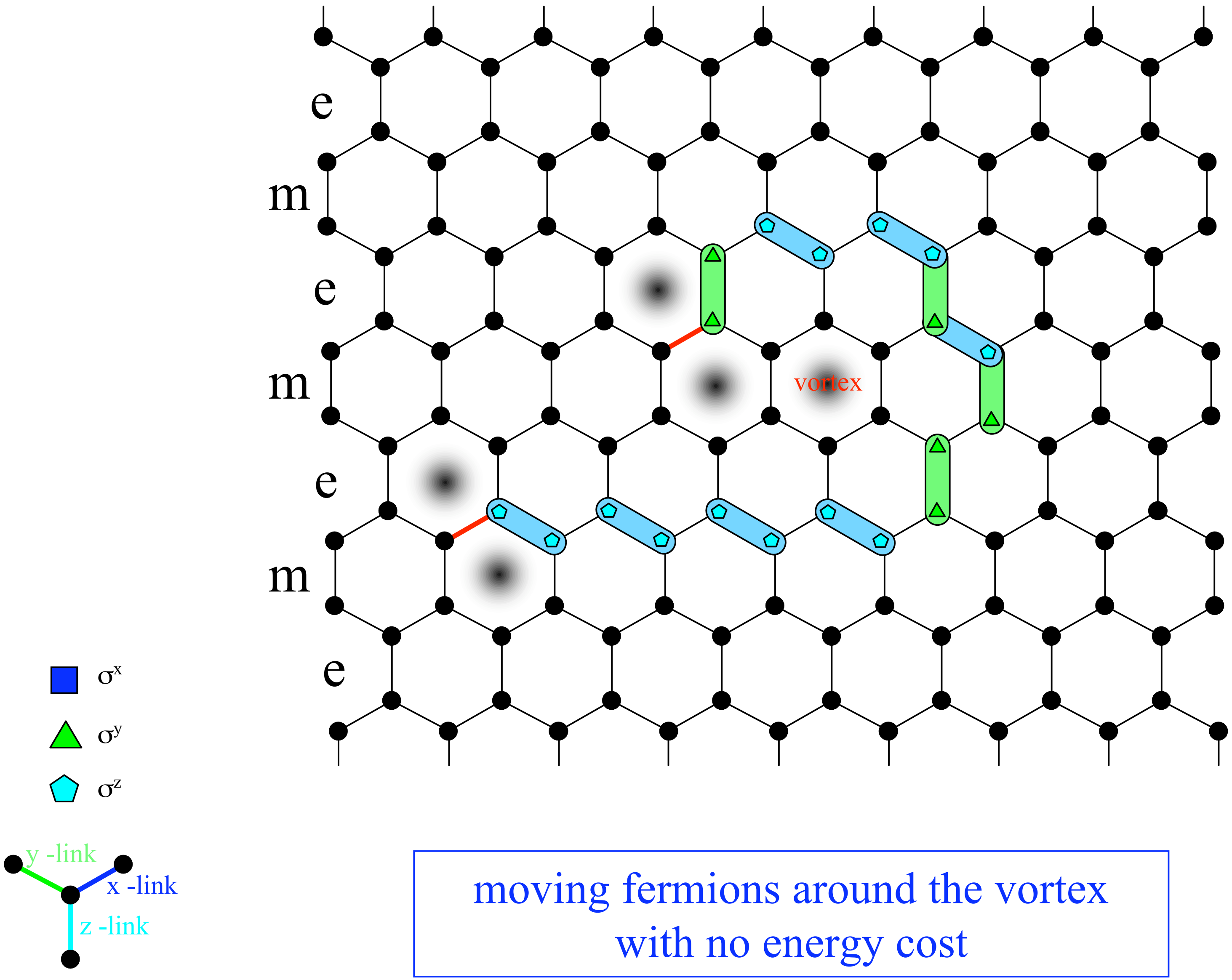
Mutual statistics of fermions and vortices





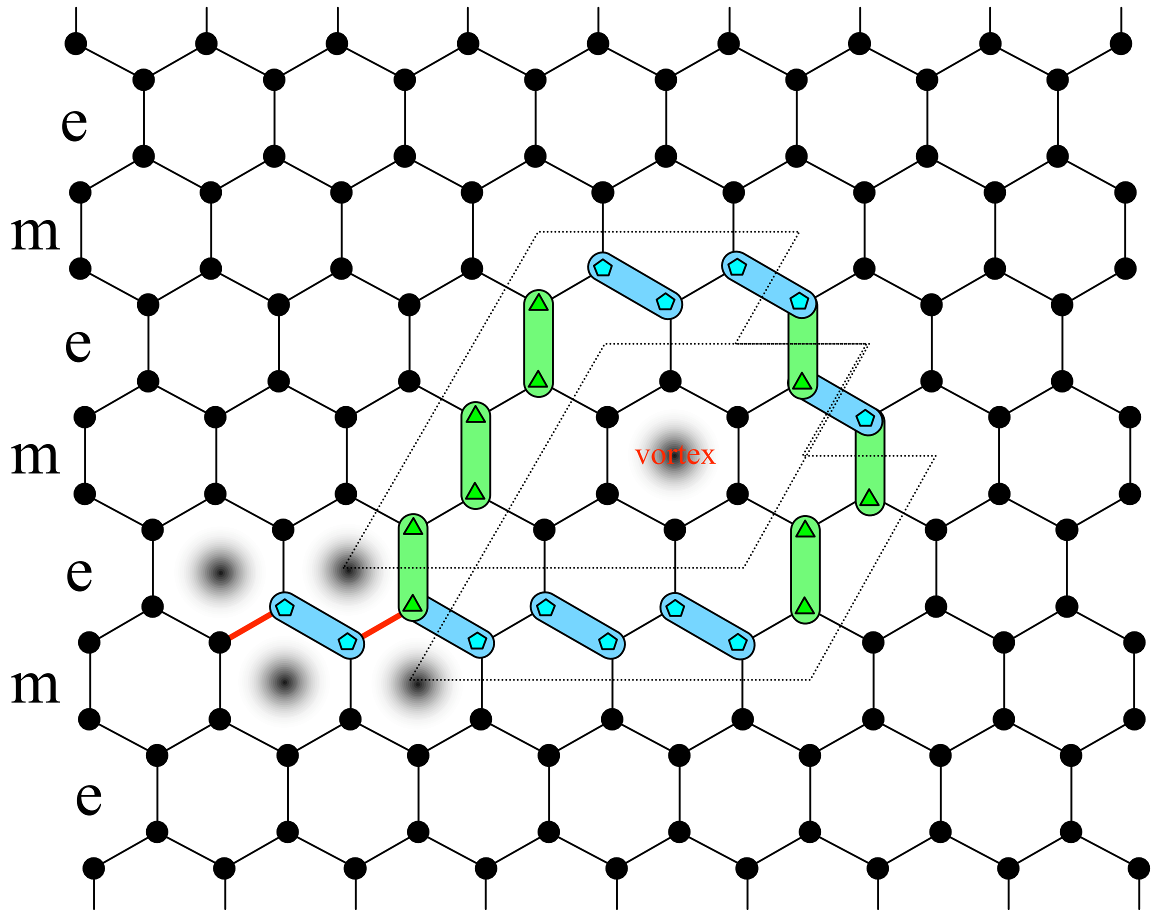
moving fermions costs no energy

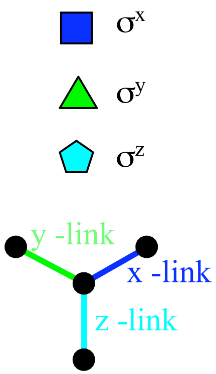
Mutual statistics of fermions and vortices



moving fermions around the vortex
with no energy cost

Mutual statistics of fermions and vortices





Phase associated with a closed loop operation is the vortex-vortex statistical phase, here between the “electric” vortex of the fermion and a “magnetic” vortex inside the loop.

As moving fermions costs no energy, the operation represents a clean realization of abelian statistics in the Kitaev honeycomb lattice model.

Conclusions

Symmetries

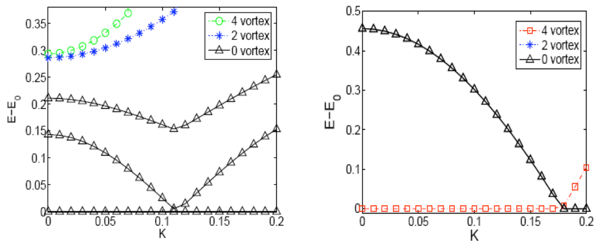
- classification of all closed loop symmetries of the Kitaev model valid for all parameter ranges and all possible lattice configurations
- exact perturbative derivation of the full effective Hamiltonian of the model on torus

Spectral properties

- complete classification of finite size effects on torus
- spectral properties of the this torus limit exhibits spectral features which are strikingly similar to the behavior of full model without and with magnetic field (in progress, numerics)
- perturbative magnetic field opens spectral gap in the nonabelian phase on torus (numerics)

G. Kells, N. Moran and J. Vala,
Finite size effects in the Kitaev honeycomb lattice model on torus,
J. Stat. Mech. – Theory Exp., (2009) P03006

V. Lahtinen, G. Kells, A. Carollo, T. Stitt, J. Vala, and J. Pachos,
Spectrum of the Non-Abelian Phase in Kitaev's Honeycomb Lattice Model,
Ann. Phys. 323, 2286 (2008).



Quasiparticles

- found a new kind of free fermions
- applied to realize anyonic statistics in the honeycomb model without relying on perturbative mapping of the toric code operations

G. Kells, A. T. Bolukbasi, V. Lahtinen, J. K. Slingerland, J. K. Pachos and J. Vala,
Topological degeneracy and vortex manipulation in the Kitaev honeycomb model, **Phys. Rev. Lett. 101**, 240404(2008).

G. Kells, A. T. J. K. Slingerland, and J. Vala,
A description of Kitaev's honeycomb model with toric code stabilizers, **arXiv:0903.5211 (2009).**

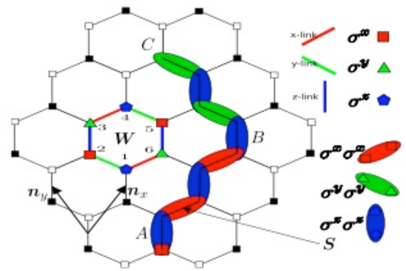
A decription of Kitaev's honeycomb model with toric code stabilizers

NEW & MORE DETAILED VERSION AT ARXIV TODAY G. Kells, A. T. J. K. Slingerland, and J. Vala, *A description of Kitaev's honeycomb model with toric code stabilizers,* **arXiv:0903.5211 (2009)**

$$H_0 = -\sum_{\alpha \in \{x,y,z\}} \sum_{i,j} J_\alpha K^\alpha_{i,j}$$

$$H_1 = -\kappa \sum_{\boldsymbol{q}} \sum_{l=1}^{6} P(\boldsymbol{q})^{(l)}$$

$$\sum_{l=1}^{6} P(\boldsymbol{q})^{(l)} = \sigma_1^x \sigma_6^y \sigma_5^z + \sigma_2^z \sigma_3^y \sigma_4^x + \sigma_1^y \sigma_2^x \sigma_3^z + \sigma_4^y \sigma_5^x \sigma_6^z + \sigma_3^x \sigma_4^z \sigma_5^y + \sigma_2^y \sigma_1^z \sigma_6^x .$$



Definition of hard-core bosons and effective spins of the "z-dimers"

$$|\uparrow_{\blacksquare}\uparrow_{\square}\rangle = |\Uparrow, 0\rangle, \quad |\downarrow_{\blacksquare}\downarrow_{\square}\rangle = |\Downarrow, 0\rangle,$$
$$|\uparrow_{\blacksquare}\downarrow_{\square}\rangle = |\Uparrow, 1\rangle, \quad |\downarrow_{\blacksquare}\uparrow_{\square}\rangle = |\Downarrow, 1\rangle.$$

$$\begin{aligned}
\sigma^x_{\boldsymbol{q},\blacksquare} &= \tau^x_{\boldsymbol{q}}(b^\dagger_{\boldsymbol{q}} + b_{\boldsymbol{q}}) \;, & \sigma^x_{\boldsymbol{q},\square} &= b^\dagger_{\boldsymbol{q}} + b_{\boldsymbol{q}}, \\
\sigma^y_{\boldsymbol{q},\blacksquare} &= \tau^y_{\boldsymbol{q}}(b^\dagger_{\boldsymbol{q}} + b_{\boldsymbol{q}}) \;, & \sigma^y_{\boldsymbol{q},\square} &= i\,\tau^z_{\boldsymbol{q}}(b^\dagger_{\boldsymbol{q}} - b_{\boldsymbol{q}}), \\
\sigma^z_{\boldsymbol{q},\blacksquare} &= \tau^z_{\boldsymbol{q}} \;, & \sigma^z_{\boldsymbol{q},\square} &= \tau^z_{\boldsymbol{q}}(I - 2b^\dagger_{\boldsymbol{q}} b_{\boldsymbol{q}}),
\end{aligned}$$

Hamiltonian

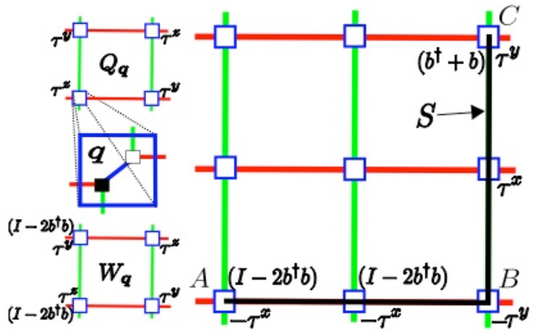
$$\begin{aligned}
H_0 = & -J_x \sum_{\boldsymbol{q}} (b^\dagger_{\boldsymbol{q}} + b_{\boldsymbol{q}}) \tau^x_{\boldsymbol{q}+\boldsymbol{n}_x} (b^\dagger_{\boldsymbol{q}+\boldsymbol{n}_x} + b_{\boldsymbol{q}+\boldsymbol{n}_x}) \\
& - J_y \sum_{\boldsymbol{q}} i\tau^z_{\boldsymbol{q}} (b^\dagger_{\boldsymbol{q}} - b_{\boldsymbol{q}}) \tau^y_{\boldsymbol{q}+\boldsymbol{n}_y} (b^\dagger_{\boldsymbol{q}+\boldsymbol{n}_y} + b_{\boldsymbol{q}+\boldsymbol{n}_y}) \\
& - J_z \sum_{\boldsymbol{q}} (I - 2b^\dagger_{\boldsymbol{q}} b_{\boldsymbol{q}}).
\end{aligned}$$

Vortex operators

$$\boldsymbol{W}_{\boldsymbol{q}} = (I - 2\boldsymbol{N}_{\boldsymbol{q}})(I - 2\boldsymbol{N}_{\boldsymbol{q}+\boldsymbol{n}_y})\boldsymbol{Q}_{\boldsymbol{q}}$$

$$\begin{aligned}
\boldsymbol{X}_{q_x,q_y} &\equiv \prod_{q'_y=0}^{q_y-1} \boldsymbol{W}_{q_x,q'_y} \\
&= (I - 2\boldsymbol{N}_{q_x,0})(I - 2\boldsymbol{N}_{q_x,q_y}) \prod_{q'_y=0}^{q_y-1} \boldsymbol{Q}_{q_x,q'_y}
\end{aligned}$$

A decription of Kitaev's honeycomb model with toric code stabilizers



Strings

$$\begin{aligned} S_{\boldsymbol{q}} \equiv\ & \sigma^y_{(q_x,q_y),\blacksquare}\sigma^y_{(q_x,q_y-1),\square}\sigma^z_{(q_x,q_y-1)\square} \\ & \ldots\sigma^y_{(q_x,1),\blacksquare}\sigma^y_{(q_x,0),\square}\sigma^z_{(q_x,0),\square}\sigma^z_{(q_x,0)\blacksquare}\sigma^x_{(q_x,0)\blacksquare} \\ & \ldots\sigma^x_{(1,0)\blacksquare}\sigma^x_{(0,0)\square}\sigma^z_{(0,0),\square}\sigma^z_{(0,0),\blacksquare}\sigma^x_{(0,0),\blacksquare}. \end{aligned}$$

Fermionic operators (Jordan-Wigner)

$$c^\dagger_{\boldsymbol{q}} = b^\dagger_{\boldsymbol{q}} S'_{\boldsymbol{q}}, \quad c_{\boldsymbol{q}} = b_{\boldsymbol{q}} S'_{\boldsymbol{q}}$$

$$\{c^\dagger_{\boldsymbol{q}}, c_{\boldsymbol{q}'}\} = \delta_{\boldsymbol{q}\boldsymbol{q}'}, \quad \{c^\dagger_{\boldsymbol{q}}, c^\dagger_{\boldsymbol{q}'}\} = 0, \quad \{c_{\boldsymbol{q}}, c_{\boldsymbol{q}'}\} = 0.$$

Hamiltonian

$$\begin{aligned} \boldsymbol{H} =\ & J_x \sum_{\boldsymbol{q}} \boldsymbol{X}_{\boldsymbol{q}}(c^\dagger_{\boldsymbol{q}} - c_{\boldsymbol{q}})(c^\dagger_{\boldsymbol{q}+\boldsymbol{n}_x} + c_{\boldsymbol{q}+\boldsymbol{n}_x}) \\ & + J_y \sum_{\boldsymbol{q}} \boldsymbol{Y}_{\boldsymbol{q}}(c^\dagger_{\boldsymbol{q}} - c_{\boldsymbol{q}})(c^\dagger_{\boldsymbol{q}+\boldsymbol{n}_y} + c_{\boldsymbol{q}+\boldsymbol{n}_y}) \\ & + J_z \sum_{\boldsymbol{q}} (2c^\dagger_{\boldsymbol{q}} c_{\boldsymbol{q}} - I), \end{aligned}$$

Momentum representation

$$c_{\boldsymbol{q}} = M^{-1/2} \sum c_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{q}}.$$

$$H = \sum_{\boldsymbol{k}} \left[\xi_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}} c_{\boldsymbol{k}} + \frac{1}{2}(\Delta c^\dagger_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}} + \Delta^* c_{-\boldsymbol{k}} c_{\boldsymbol{k}}) \right] - M J_z$$

$$\begin{aligned} \xi_{\boldsymbol{k}} &= \varepsilon_{\boldsymbol{k}} - \mu \\ \Delta_{\boldsymbol{k}} &= \alpha_{\boldsymbol{k}} + i\beta_{\boldsymbol{k}} \\ \mu &= -2J_z \\ \varepsilon_{\boldsymbol{k}} &= 2J_x \cos(k_x) + 2J_y \cos(k_y) \\ \alpha_{\boldsymbol{k}} &= 4\kappa(\sin(k_x) - \sin(k_y) - \sin(k_x - k_y)) \\ \beta_{\boldsymbol{k}} &= 2J_x \sin(k_x) + 2J_y \sin(k_y). \end{aligned}$$

Bogoliubov transformation

$$\gamma_{\boldsymbol{k}} = u_{\boldsymbol{k}} c_{\boldsymbol{k}} - v_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}},$$

$$H = \sum E_{\boldsymbol{k}}(\gamma^\dagger_{\boldsymbol{k}} \gamma_{\boldsymbol{k}} - 1/2)$$

$$\begin{aligned} E_{\boldsymbol{k}} &= \sqrt{\xi^2_{\boldsymbol{k}} + |\Delta_{\boldsymbol{k}}|^2} \\ u_{\boldsymbol{k}} &= \sqrt{1/2(1 + \xi_{\boldsymbol{k}}/E_{\boldsymbol{k}})} \\ v_{\boldsymbol{k}} &= i\sqrt{1/2(1 - \xi_{\boldsymbol{k}}/E_{\boldsymbol{k}})} \end{aligned}$$

BCS-type state

$$|\,\mathrm{gs}\rangle = \prod_{\boldsymbol{k}} (u_{\boldsymbol{k}} + v_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}})|\,\{W_{\boldsymbol{q}}\}, \{\emptyset\}\rangle.$$

In addition the paper includes also:

- **fermionization on torus**
- **effcetive magnetic field (non-abelian topological phase)**

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