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# Geometry of the SU(4) group


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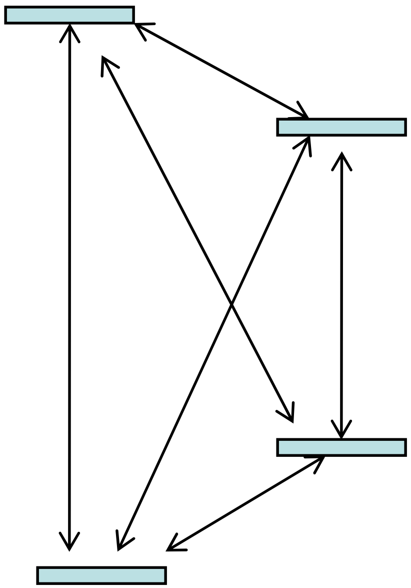
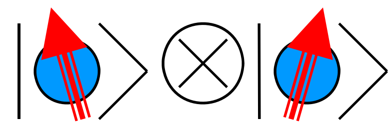
# 4-Level system and two spin ½ system





$$|1\rangle \longrightarrow |1\rangle \otimes |1\rangle$$

$$|2\rangle \longrightarrow |1\rangle \otimes |0\rangle$$

$$|3\rangle \longrightarrow |0\rangle \otimes |1\rangle$$

$$|4\rangle \longrightarrow |1\rangle \otimes |1\rangle$$

15 generators of the SU(4) group

$$Hamiltonian = \sum \hat{\chi}_n\, x_n(t), \quad \hat{\chi}_n \;\; are\ generators\ \ of\, SU(4)$$

$\sigma_x^{(1)}, \sigma_y^{(1)}, \sigma_z^{(1)}$ — $spin\ 1/2\ Local\, rotations\ \ SU(2)\, group$

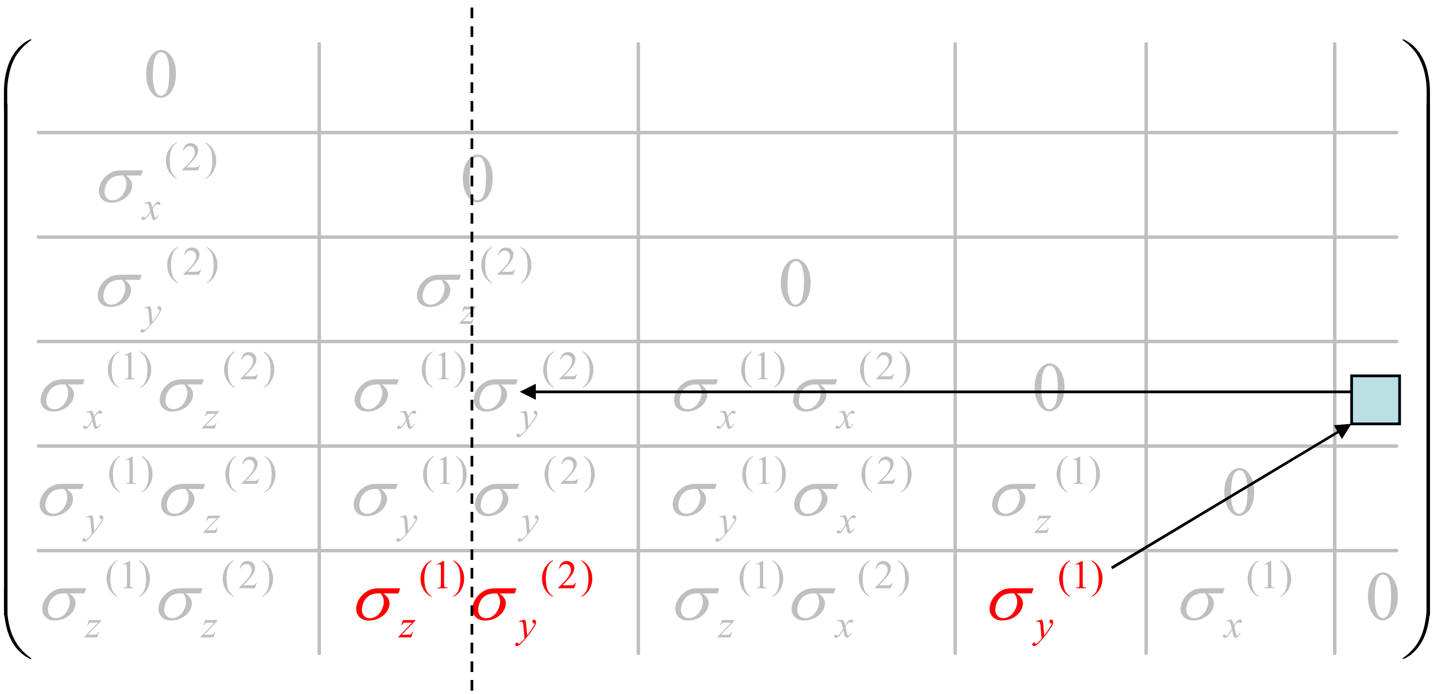
$\sigma_x^{(2)}, \sigma_y^{(2)}, \sigma_z^{(2)}$ — $spin\ 1/2\ Local\, rotations\ \ SU(2)\, group$

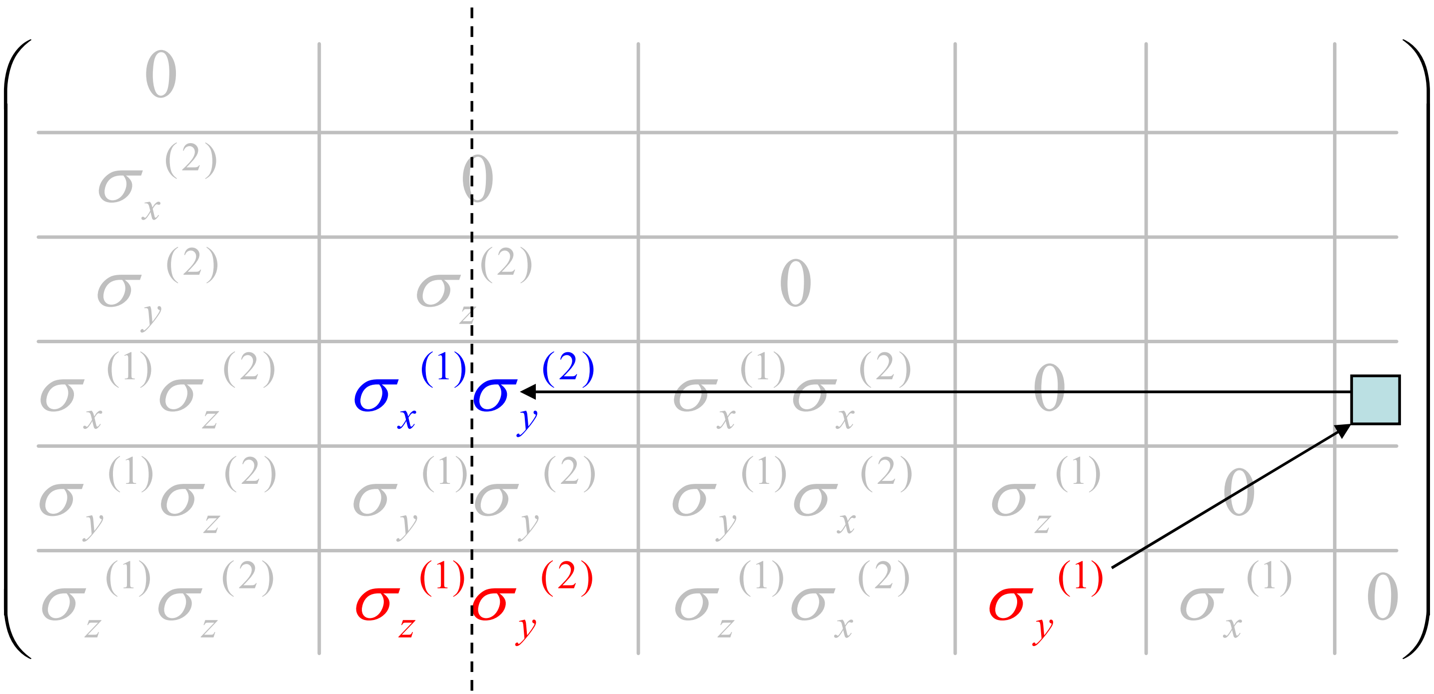
$$\left.\begin{matrix} \sigma_x^{(1)}\sigma_z^{(2)}, \sigma_x^{(1)}\sigma_y^{(2)}, \sigma_x^{(1)}\sigma_x^{(2)} \\ \sigma_y^{(1)}\sigma_z^{(2)}, \sigma_y^{(1)}\sigma_y^{(2)}, \sigma_y^{(1)}\sigma_x^{(2)} \\ \sigma_z^{(1)}\sigma_z^{(2)}, \sigma_z^{(1)}\sigma_y^{(2)}, \sigma_z^{(1)}\sigma_x^{(2)} \end{matrix}\right\} \; spin-spin\ coupling$$

| $[X_j;X_k]$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ | $X_{11}$ | $X_{12}$ | $X_{13}$ | $X_{14}$ | $X_{15}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X_1$ | 0 | $X_3$ | $X_2$ | 0 | 0 | 0 | 0 | 0 | 0 | $X_{13}$ | $X_{14}$ | $X_{15}$ | $X_{10}$ | $X_{11}$ | $X_{12}$ |
| $X_2$ | $X_3$ | 0 | $X_1$ | 0 | 0 | 0 | $X_{13}$ | $X_{14}$ | $X_{15}$ | 0 | 0 | 0 | $X_7$ | $X_8$ | $X_9$ |
| $X_3$ | $X_2$ | $X_1$ | 0 | 0 | 0 | 0 | $X_{10}$ | $X_{11}$ | $X_{12}$ | $X_7$ | $X_8$ | $X_9$ | 0 | 0 | 0 |
| $X_4$ | 0 | 0 | 0 | 0 | $X_6$ | $X_5$ | 0 | $X_9$ | $X_8$ | 0 | $X_{12}$ | $X_{11}$ | 0 | $X_{15}$ | $X_{14}$ |
| $X_5$ | 0 | 0 | 0 | $X_6$ | 0 | $X_4$ | $X_9$ | 0 | $X_7$ | $X_{12}$ | 0 | $X_{10}$ | $X_{15}$ | 0 | $X_{13}$ |
| $X_6$ | 0 | 0 | 0 | $X_5$ | $X_4$ | 0 | $X_8$ | $X_7$ | 0 | $X_{11}$ | $X_{10}$ | 0 | $X_{14}$ | $X_{13}$ | 0 |
| $X_7$ | 0 | $X_{13}$ | $X_{10}$ | 0 | $X_9$ | $X_8$ | 0 | $X_6$ | $X_5$ | $X_3$ | 0 | 0 | $X_2$ | 0 | 0 |
| $X_8$ | 0 | $X_{14}$ | $X_{11}$ | $X_9$ | 0 | $X_7$ | $X_6$ | 0 | $X_4$ | 0 | $X_3$ | 0 | 0 | $X_2$ | 0 |
| $X_9$ | 0 | $X_{15}$ | $X_{12}$ | $X_8$ | $X_7$ | 0 | $X_5$ | $X_4$ | 0 | 0 | 0 | $X_3$ | 0 | 0 | $X_2$ |
| $X_{10}$ | $X_{13}$ | 0 | $X_7$ | 0 | $X_{12}$ | $X_{11}$ | $X_3$ | 0 | 0 | 0 | $X_6$ | $X_5$ | $X_1$ | 0 | 0 |
| $X_{11}$ | $X_{14}$ | 0 | $X_8$ | $X_{12}$ | 0 | $X_{10}$ | 0 | $X_3$ | 0 | $X_6$ | 0 | $X_4$ | 0 | $X_1$ | 0 |
| $X_{12}$ | $X_{15}$ | 0 | $X_9$ | $X_{11}$ | $X_{10}$ | 0 | 0 | 0 | $X_3$ | $X_5$ | $X_4$ | 0 | 0 | 0 | $X_1$ |
| $X_{13}$ | $X_{10}$ | $X_7$ | 0 | 0 | $X_{15}$ | $X_{14}$ | $X_2$ | 0 | 0 | $X_1$ | 0 | 0 | 0 | $X_6$ | $X_5$ |
| $X_{14}$ | $X_{11}$ | $X_8$ | 0 | $X_{15}$ | 0 | $X_{13}$ | 0 | $X_2$ | 0 | 0 | $X_1$ | 0 | $X_6$ | 0 | $X_4$ |
| $X_{15}$ | $X_{12}$ | $X_9$ | 0 | $X_{14}$ | $X_{13}$ | 0 | 0 | 0 | $X_2$ | 0 | 0 | $X_1$ | $X_5$ | $X_4$ | 0 |

$$
\left(\begin{array}{c|c|c|c|c|c}
0 & & & & & \\
\hline
\sigma_x^{(2)} & 0 & & & & \\
\hline
\sigma_y^{(2)} & \sigma_z^{(2)} & 0 & & & \\
\hline
\sigma_x^{(1)}\sigma_z^{(2)} & \sigma_x^{(1)}\sigma_y^{(2)} & \sigma_x^{(1)}\sigma_x^{(2)} & 0 & & \\
\hline
\sigma_y^{(1)}\sigma_z^{(2)} & \sigma_y^{(1)}\sigma_y^{(2)} & \sigma_y^{(1)}\sigma_x^{(2)} & \sigma_z^{(1)} & 0 & \\
\hline
\sigma_z^{(1)}\sigma_z^{(2)} & \sigma_z^{(1)}\sigma_y^{(2)} & \sigma_z^{(1)}\sigma_x^{(2)} & \sigma_y^{(1)} & \sigma_x^{(1)} & 0
\end{array}\right)
$$

$$
\begin{pmatrix}
0 & & & & & \\
\sigma_x^{(2)} & 0 & & & & \\
\sigma_y^{(2)} & \sigma_z^{(2)} & 0 & & & \\
\sigma_x^{(1)}\sigma_z^{(2)} & \sigma_x^{(1)}\sigma_y^{(2)} & \sigma_x^{(1)}\sigma_x^{(2)} & 0 & & \\
\sigma_y^{(1)}\sigma_z^{(2)} & \sigma_y^{(1)}\sigma_y^{(2)} & \sigma_y^{(1)}\sigma_x^{(2)} & \sigma_z^{(1)} & 0 & \\
\sigma_z^{(1)}\sigma_z^{(2)} & \sigma_z^{(1)}\sigma_y^{(2)} & \sigma_z^{(1)}\sigma_x^{(2)} & \sigma_y^{(1)} & \sigma_x^{(1)} & 0
\end{pmatrix}
$$


0
σ_x^(2) 0
σ_y^(2) σ_z^(2) 0
σ_x^(1)σ_z^(2) σ_x^(1)σ_y^(2) σ_x^(1)σ_x^(2) 0
σ_y^(1)σ_z^(2) σ_y^(1)σ_y^(2) σ_y^(1)σ_x^(2) σ_z^(1) 0
σ_z^(1)σ_z^(2) σ_z^(1)σ_y^(2) σ_z^(1)σ_x^(2) σ_y^(1) σ_x^(1) 0



$$\left[\sigma_z^{(1)}\sigma_y^{(2)},\sigma_y^{(1)}\right]=\sigma_z^{(1)}\sigma_y^{(2)}\sigma_y^{(1)}-\sigma_y^{(1)}\sigma_z^{(1)}\sigma_y^{(2)}=\sigma_x^{(1)}\sigma_y^{(2)}$$

$$
\left(\begin{array}{c|c|c|c|c|c}
0 & & & & & \\
\hline
\sigma_x^{(2)} & 0 & & & & \\
\hline
\sigma_y^{(2)} & \sigma_z^{(2)} & 0 & & & \\
\hline
\sigma_x^{(1)}\sigma_z^{(2)} & \sigma_x^{(1)}\sigma_y^{(2)} & \sigma_x^{(1)}\sigma_x^{(2)} & 0 & & \\
\hline
\sigma_y^{(1)}\sigma_z^{(2)} & \sigma_y^{(1)}\sigma_y^{(2)} & \sigma_y^{(1)}\sigma_x^{(2)} & \sigma_z^{(1)} & 0 & \\
\hline
\sigma_z^{(1)}\sigma_z^{(2)} & \sigma_z^{(1)}\sigma_y^{(2)} & \sigma_z^{(1)}\sigma_x^{(2)} & \sigma_y^{(1)} & \sigma_x^{(1)} & 0
\end{array}\right)
$$

$$SU(2)\otimes SU(2)\simeq Spin(3)\otimes Spin(3)$$

$$
\left(\begin{array}{c|c|c|c|c|c}
0 & & & & & \\
\hline
\sigma_x^{(2)} & 0 & & & & \\
\hline
\sigma_y^{(2)} & \sigma_z^{(2)} & 0 & & & \\
\hline
\sigma_x^{(1)}\sigma_z^{(2)} & \sigma_x^{(1)}\sigma_y^{(2)} & \sigma_x^{(1)}\sigma_x^{(2)} & 0 & & \\
\hline
\sigma_y^{(1)}\sigma_z^{(2)} & \sigma_y^{(1)}\sigma_y^{(2)} & \sigma_y^{(1)}\sigma_x^{(2)} & \sigma_z^{(1)} & 0 & \\
\hline
\sigma_z^{(1)}\sigma_z^{(2)} & \sigma_z^{(1)}\sigma_y^{(2)} & \sigma_z^{(1)}\sigma_x^{(2)} & \sigma_y^{(1)} & \sigma_x^{(1)} & 0
\end{array}\right)
$$

$$Spin(4)$$

$$
\left(\begin{array}{c|c|c|c|c|c}
0 & & & & & \\ \hline
\sigma_x^{(2)} & 0 & & & & \\ \hline
\sigma_y^{(2)} & \sigma_z^{(2)} & 0 & & & \\ \hline
\sigma_x^{(1)}\sigma_z^{(2)} & \sigma_x^{(1)}\sigma_y^{(2)} & \sigma_x^{(1)}\sigma_x^{(2)} & 0 & & \\ \hline
\sigma_y^{(1)}\sigma_z^{(2)} & \sigma_y^{(1)}\sigma_y^{(2)} & \sigma_y^{(1)}\sigma_x^{(2)} & \sigma_z^{(1)} & 0 & \\ \hline
\sigma_z^{(1)}\sigma_z^{(2)} & \sigma_z^{(1)}\sigma_y^{(2)} & \sigma_z^{(1)}\sigma_x^{(2)} & \sigma_y^{(1)} & \sigma_x^{(1)} & 0
\end{array}\right)
$$

$$Spin(5)$$

$$
\left(\begin{array}{c|c|c|c|c|c}
0 & & & & & \\
\hline
\sigma_x^{(2)} & 0 & & & & \\
\hline
\sigma_y^{(2)} & \sigma_z^{(2)} & 0 & & & \\
\hline
\sigma_x^{(1)}\sigma_z^{(2)} & \sigma_x^{(1)}\sigma_y^{(2)} & \sigma_x^{(1)}\sigma_x^{(2)} & 0 & & \\
\hline
\sigma_y^{(1)}\sigma_z^{(2)} & \sigma_y^{(1)}\sigma_y^{(2)} & \sigma_y^{(1)}\sigma_x^{(2)} & \sigma_z^{(1)} & 0 & \\
\hline
\sigma_z^{(1)}\sigma_z^{(2)} & \sigma_z^{(1)}\sigma_y^{(2)} & \sigma_z^{(1)}\sigma_x^{(2)} & \sigma_y^{(1)} & \sigma_x^{(1)} & 0
\end{array}\right)
$$

$$SU(4) \cong Spin(6)$$

$$SU(4)/Z_2 \cong SO(6)$$

# What are Spin groups and why SU(4) is isomorphic to Spin(6)

An observation: generators of su(4) map one-one onto generators of so(6)=spin(6) as follows

$$\begin{pmatrix} 0 & \hat{p}_{12} & \hat{p}_{13} & \hat{p}_{14} & \hat{p}_{15} & \hat{p}_{16} \\ \hat{p}_{21} & 0 & \hat{p}_{23} & \hat{p}_{24} & \hat{p}_{25} & \hat{p}_{26} \\ \hat{p}_{31} & \hat{p}_{32} & 0 & \hat{p}_{34} & \hat{p}_{35} & \hat{p}_{36} \\ \hat{p}_{41} & \hat{p}_{42} & \hat{p}_{43} & 0 & \hat{p}_{45} & \hat{p}_{46} \\ \hat{p}_{51} & \hat{p}_{52} & \hat{p}_{53} & \hat{p}_{54} & 0 & \hat{p}_{56} \\ \hat{p}_{61} & \hat{p}_{62} & \hat{p}_{63} & \hat{p}_{64} & \hat{p}_{65} & 0 \end{pmatrix} = i \begin{pmatrix} 0 & -\sigma_x^{(2)} & +\sigma_y^{(2)} & -\sigma_x^{(1)}\sigma_z^{(2)} & -\sigma_y^{(1)}z^{(2)} & -\sigma_z^{(1)}\sigma_z^{(2)} \\ \sigma_x^{(2)} & 0 & -\sigma_z^{(2)} & -\sigma_x^{(1)}y^{(2)} & -\sigma_y^{(1)}\sigma_y^{(2)} & -\sigma_z^{(1)}y^{(2)} \\ -\sigma_y^{(2)} & \sigma_z^{(2)} & 0 & -\sigma_x^{(1)}\sigma_x^{(2)} & -\sigma_z^{(1)}\sigma_y^{(2)} & -\sigma_z^{(1)}\sigma_x^{(2)} \\ \sigma_z^{(1)}\sigma_z^{(2)} & \sigma_z^{(1)}y^{(2)} & \sigma_z^{(1)}\sigma_x^{(2)} & 0 & -\sigma_z^{(1)} & \sigma_y^{(1)} \\ \sigma_y^{(1)}z^{(2)} & \sigma_y^{(1)}\sigma_y^{(2)} & \sigma_y^{(1)}\sigma_x^{(2)} & -\sigma_x^{(1)} & 0 & -\sigma_x^{(1)} \\ \sigma_x^{(1)}\sigma_z^{(2)} & \sigma_x^{(1)}\sigma_y^{(2)} & \sigma_x^{(1)}\sigma_x^{(2)} & \sigma_y^{(1)} & -\sigma_z^{(1)} & 0 \end{pmatrix}$$

It is easy to check that so(6) generators $(p_{n,m})_{p,k} = \delta_{n,p}\,\delta_{m,k}$ have identical commutators as second-grade elements $\epsilon_i \epsilon_j$ of algebra *Clifford(6)*, generating a group *Spin(6)*

$$\epsilon_i \epsilon_j + \epsilon_j \epsilon_i = -2\delta_{i,j}$$

# Basic facts about Spin groups are:

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*Spin(6)* group is constructed as a group generated by second-grade elements (bivectors) of *Clifford(6)* .

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15(+1) second-grade elements of *Clifford(5)* are scew – Hermitian generators of U(4).

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Full *Clifford(6)* has 63(+1) elements, of which a subset of 31(+1) even-graded elements is isomorphic to *Clifford(5)*=(Algebra of 4×4 Complex Matrices).

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*SU(2)=Spin(3)* is the double cover of *SO(3)=SU(2)/Z2*
*SU(4) =Spin(6)* is the double cover of *SO(6)=SU(4)/Z2*

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Coset spaces *Spin(n+1)/ Spin(n)=*$S^n$ are n-D Spheres

$$\text{Hadamard}(1) \times \text{Hadamard}(2) = \left(\begin{array}{c|c|c|c} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \hline \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \hline -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \hline -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

$$\text{SO-form} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\sqrt{\text{SWAP}} = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \hline 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

SO(6) form

$$\left(\begin{array}{ccc|ccc} \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \end{array}\right)$$

# Two qubits are useful for description of a 4-level system. Whether 4-level system can be useful for encoding ONE qubit ?

Error-correction schemes by redundant two-qubit encoding.
P. Zoller and J. Cirac 1996

“Time-optimal synthesis of unitary transformations
in coupled fast and slow qubit system”
Navin Khaneja, et al RPA (2008)

# Encoding of information into two – dimensional **subspaces**

Standard Error Correction by Redundant encoding

$$\alpha\left|\uparrow\right\rangle+\beta\left|\downarrow\right\rangle\rightarrow\ \alpha\left|1\right\rangle\otimes\left|1\right\rangle...\otimes\left|1\right\rangle+\alpha\left|0\right\rangle\otimes\left|0\right\rangle...\otimes\left|0\right\rangle$$

Information Encoding into Linear subspaces

$$\alpha\left|\uparrow\right\rangle+\beta\left|\downarrow\right\rangle\rightarrow\ \alpha\left|\Psi_1\right\rangle+\beta\left|\Psi_2\right\rangle;$$

$$\left|\Psi_1\right\rangle\in Span\{\left|1\right\rangle,\left|2\right\rangle\};\quad \left|\Psi_2\right\rangle\in Span\{\left|3\right\rangle,\left|4\right\rangle\}$$

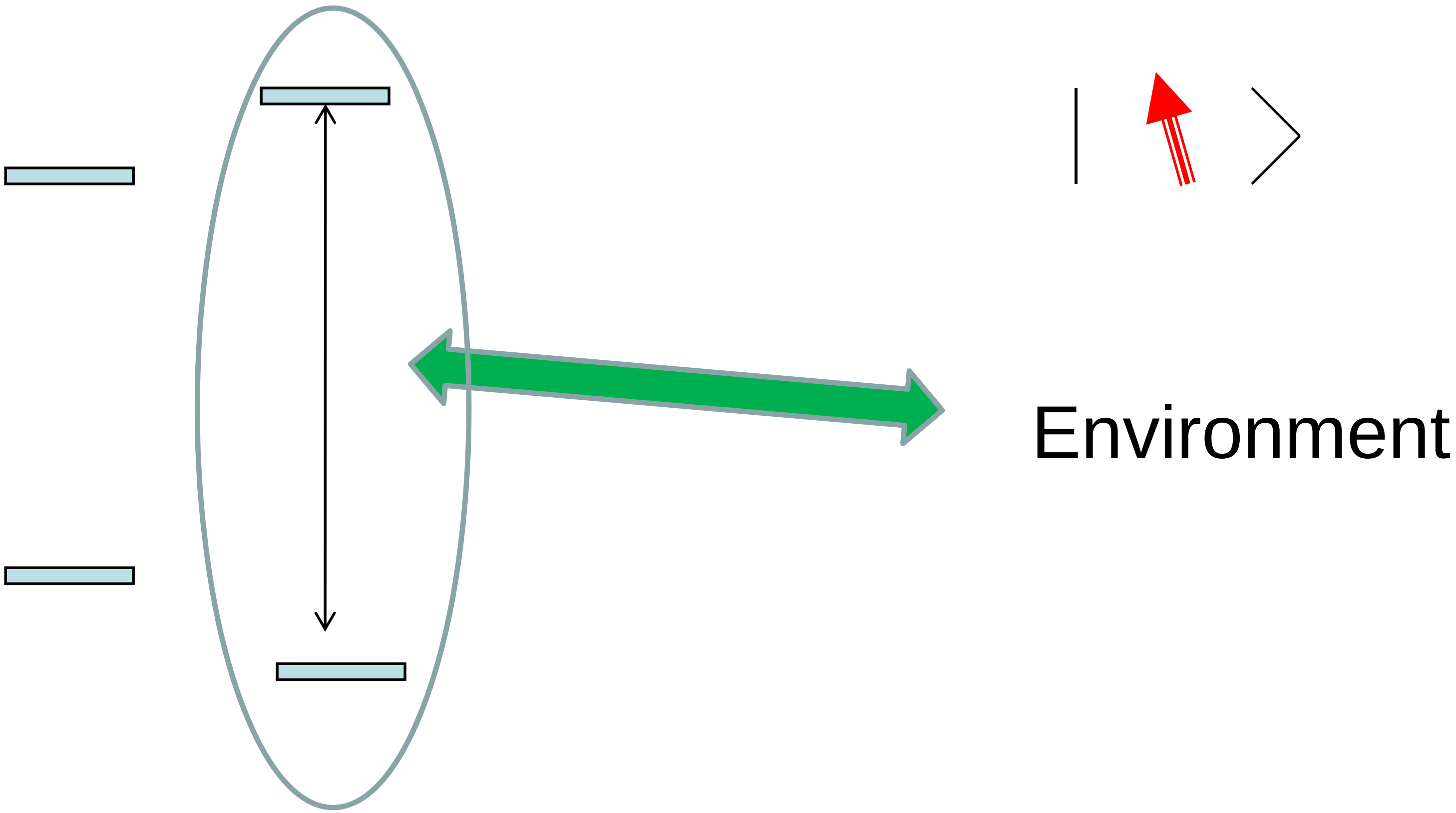
$$Span\{\left|3\right\rangle,\left|4\right\rangle\}=\xi_1\left|3\right\rangle+\xi_2\left|4\right\rangle;\quad \xi_{1,2}\in C^2$$

Information readout by Projective operators

$$P_1=\left|1\right\rangle\left\langle 1\right|+\left|2\right\rangle\left\langle 2\right|;\qquad P_2=\left|3\right\rangle\left\langle 3\right|+\left|4\right\rangle\left\langle 4\right|;$$

$$P_1+P_2=I;\ \ P_{1,2}\cdot P_{1,2}=P_{1,2};\ \ P_1\cdot P_2=0$$

# 4-Level system



Are there dynamic equations for subspaces used to encode the information?

$$\alpha\left|\uparrow\right\rangle + \beta\left|\downarrow\right\rangle \rightarrow \alpha\left|\Psi_1\right\rangle + \beta\left|\Psi_2\right\rangle;$$

$$Span\left\{\begin{pmatrix} u_{11} \\ u_{11} \\ u_{11} \\ u_{11} \end{pmatrix}, \begin{pmatrix} u_{12} \\ u_{22} \\ u_{32} \\ u_{42} \end{pmatrix}\right\} = Span\left\{\begin{pmatrix} u_{11}\, u_{12} \\ u_{11}\, u_{22} \\ u_{11}\, u_{32} \\ u_{11}\, u_{42} \end{pmatrix} \times \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right\}$$

$$\left|\Psi_1\right\rangle \in Span\{\left|1\right\rangle, \left|2\right\rangle\} \in Grassmannian(2,4,C)$$

$$Grassmannian(2,4,C) = U(4)/(SU(2)\times SU(2)\times U(1))$$

We suggest an algebraic descriptions of dynamics of 2-dimentional subspaces of 4-leves systems in a form quite similar to the Bloch-vector description of 2-level density matrices, using transformed Plucker Coordinates on Grassmanian manifolds G(2,4,C)

Six Plucker Coordinates are 6 minors of 2×4 matrix

$$P_{1,2..6} = \left\{ \mathcal{P}_{1,2}, -\mathcal{P}_{1,3}, \mathcal{P}_{1,4}, \mathcal{P}_{2,3}, \mathcal{P}_{2,4}, \mathcal{P}_{3,4} \right\}$$

$$\begin{pmatrix} u_{11} \, u_{12} \\ u_{11} \, u_{22} \\ u_{11} \, u_{32} \\ u_{11} \, u_{42} \end{pmatrix}$$

*for example,*

$$\mathcal{P}_{1,2} = \det \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$$i\dot{\mathbf{P}} = \boldsymbol{H}_p \boldsymbol{P}$$

$$\boldsymbol{H}_p = \begin{pmatrix} (H_{11}+H_{22}) & -H_{23} & H_{24} & -H_{13} & -H_{14} & 0 \\ -H_{32} & (H_{11}+H_{33}) & -H_{34} & -H_{12} & 0 & H_{14} \\ H_{42} & -H_{43} & (H_{11}+H_{44}) & 0 & H_{12} & H_{13} \\ -H_{31} & -H_{21} & 0 & (H_{22}+H_{33}) & H_{34} & -H_{24} \\ -H_{41} & 0 & H_{21} & H_{43} & (H_{22}+H_{44}) & H_{23} \\ 0 & H_{41} & H_{31} & -H_{42} & H_{32} & (H_{33}+H_{44}) \end{pmatrix}$$

$$\begin{pmatrix} P_{1,2} \\ P_{1,3} \\ P_{1,4} \\ P_{2,3} \\ P_{2,4} \\ P_{3,4} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} q_6 + iq_5 \\ -q_1 + iq_2 \\ q_3 + iq_4 \\ q_3 - iq_4 \\ q_1 + iq_2 \\ q_6 - iq_5 \end{pmatrix}$$

$$\dot{\boldsymbol{q}}_{\boldsymbol{i}} = 2\boldsymbol{F}_{i,j}\boldsymbol{q}_{\boldsymbol{j}},$$

$\boldsymbol{F}_{i,j}$ 6-dimensional antisymmetric tensor

*Bloch vector* $\dot{m}_i = -2e_{i,j,k}B_j m_k$

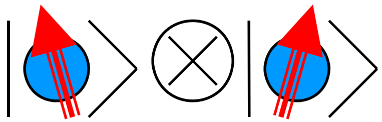
$$\dot{\boldsymbol{q}}_{\boldsymbol{i}} = 2\boldsymbol{F}_{i,j}\boldsymbol{q}_{\boldsymbol{j}}, \qquad \boldsymbol{F}_{i,j}$$

*Bloch vector* $\dot{m}_i = -2e_{i,j,k}B_jm_k$

$$\boldsymbol{H} = F_{2,1}\sigma_3^{(2)} - F_{3,1}\,\sigma_2^{(2)} + F_{3,2}\,\sigma_1^{(2)} - F_{4,i}\,\sigma_3^{(1)}\sigma_i^{(2)} + F_{5,i}\,\sigma_1^{(1)}\sigma_i^{(2)}$$
$$-F_{5,4}\,\sigma_2^{(1)} + F_{6,i}\sigma_2^{(1)}\sigma_i^{(1)} + F_{6,4}\sigma_1^{(1)} + F_{6,5}\sigma_3^{(1)}$$

| | | | | | |
|---|---|---|---|---|---|
| 0 | | | | | |
| $\sigma_x^{(2)}$ | 0 | | | | |
| $\sigma_y^{(2)}$ | $\sigma_z^{(2)}$ | 0 | | | |
| $\sigma_x^{(1)}\sigma_z^{(2)}$ | $\sigma_x^{(1)}\sigma_y^{(2)}$ | $\sigma_x^{(1)}\sigma_x^{(2)}$ | 0 | | |
| $\sigma_y^{(1)}\sigma_z^{(2)}$ | $\sigma_y^{(1)}\sigma_y^{(2)}$ | $\sigma_y^{(1)}\sigma_x^{(2)}$ | $\sigma_z^{(1)}$ | 0 | |
| $\sigma_z^{(1)}\sigma_z^{(2)}$ | $\sigma_z^{(1)}\sigma_y^{(2)}$ | $\sigma_z^{(1)}\sigma_x^{(2)}$ | $\sigma_y^{(1)}$ | $\sigma_x^{(1)}$ | 0 |

To illustrate how the method may lead to some physical insights in a complicated non-stationary quantum problem consider two interacting qubits – the cornerstone problem in quantum information theory.



The full dynamic group is 15-dimentional *SU(4)* group of unitary transformations. Present method allows to obtain semi-analytic solution for the *Spin(5)* 10-dimentional subgroup of the *SU(4).*

The Hamiltonian of the problem has the form of a linear combination of ten generators with time-dependent coefficients. We group the latter in a form of 5×5 antisymmetric real matrix (for the purpose which will be clear shortly). Using common representation of *su(4)* generators as tensor products of standard Pauli matrices we write the Hamiltonian as

$$\boldsymbol{H} = F_{2,1}\sigma_3^{(2)} - F_{3,1}\,\sigma_2^{(2)} + F_{3,2}\,\sigma_1^{(2)} - F_{4,i}\,\sigma_3^{(1)}\sigma_i^{(2)} + F_{5,i}\,\sigma_1^{(1)}\sigma_i^{(2)} - F_{5,4}\,\sigma_2^{(1)}$$

Ten generators in this expression span Lie algebra .
According to our method we partition into 2×2 blocks

$$H = \left(\begin{array}{c|c} H^{(1)} & \mathbf{V} \\ \hline \mathbf{V}^{\dagger} & H^{(2)} \end{array}\right)$$

$$\tilde{\mathbf{H}}^{(1),(2)} = \left(F_{32} \pm F_{41}\right)\sigma_1 - \left(F_{31} \pm F_{42}\right)\sigma_2 + \left(F_{21} \mp F_{43}\right)\sigma_3$$

$$\mathbf{V} = iF_{54} + F_{5i}\,\sigma_i$$

$$U = \frac{1}{\sqrt{1 + z_n z_n}} \left(\begin{array}{c|c} I & -\mathbf{z}^{\dagger} \\ \hline -\mathbf{z}^{\dagger} & I \end{array}\right) \left(\begin{array}{c|c} U^{(1)} & 0 \\ \hline 0 & U^{(2)} \end{array}\right)$$

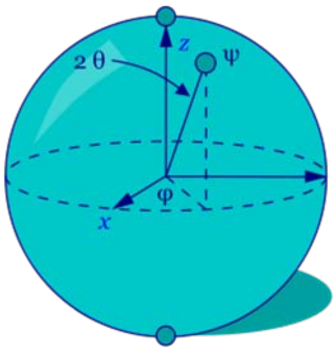
$$\mathbf{z} = z_4 + i z_i \sigma_i$$

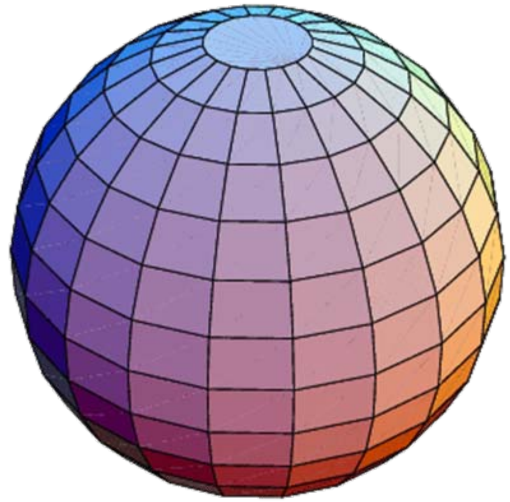
Stereographic projection from $\left(z_1, z_2, z_3, z_4\right)$ to $\mathbf{S}^4$

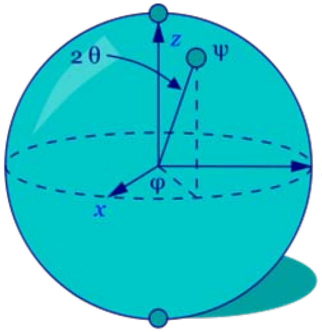
$$m_i = -2 z_i \Big/ \left(1 + \sum_{n=1}^{4} z_n^{\,2}\right); \; m_5 = \left(1 - \sum_{n=1}^{4} z_n^{\,2}\right) \Big/ \left(1 + \sum_{n=1}^{4} z_n^{\,2}\right), \quad \vec{m} \subset \mathbf{S}^4$$

# $S^4$ plays the role of a multidimensional Bloch Sphere

$$\dot{m}_n = 2F_{nk} m_k$$



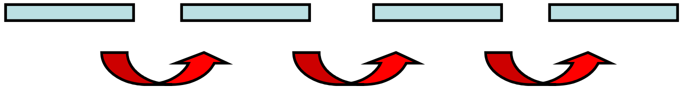



$$\dot{m}_i^{(1)} = -2e_{i,j,k} B_j^{(1)(eff)} m^{(2)}{}_k$$

$$\dot{m}_i^{(2)} = -2e_{i,j,k} B_j^{(2)(eff)} m^{(2)}{}_k$$

$$B_i^{(1,2)(eff)} = B^{(1,2)} + \left(\varepsilon_{n,m,i} F_{5,n} z_{\mathbf{m}} \pm F_{5,i} z_{\mathbf{4}} \pm F_{5,4} z_{\mathbf{i}}\right)$$

# Introduction



4-level coupling scheme

$$i\frac{d}{dt}\begin{pmatrix}a_1\\a_2\\a_3\\a_4\end{pmatrix}=\begin{pmatrix}0&\alpha(t)&0&0\\\alpha(t)&0&\beta(t)&0\\0&\beta(t)&0&\gamma(t)\\0&0&\gamma(t)&0\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\\a_4\end{pmatrix}$$

## Complete Population Transfer

at $t=t_0$ $\begin{pmatrix}a_1\\a_2\\a_3\\a_4\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $\Rightarrow unitary\ evolution \Rightarrow$ $\begin{pmatrix}a_1\\a_2\\a_3\\a_4\end{pmatrix}=\begin{pmatrix}0\\0\\e^{i\varphi}\\0\end{pmatrix}$ at $t=t_1$

## The 4-level coupling matrix ⇨ two commuting su(2) algebras

$$\begin{pmatrix} 0 & \alpha & 0 & 0 \\ \alpha & 0 & \beta & 0 \\ 0 & \beta & 0 & \gamma \\ 0 & 0 & \gamma & 0 \end{pmatrix} = \frac{\alpha+\gamma}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \frac{\beta}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \frac{\alpha-\gamma}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} + \frac{\beta}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$W = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

$4 \times 4 \quad matrices$

$\sigma_x^{(2)}$ $\sigma_z^{(2)}$ → $\sigma_y^{(2)}$

Cliff(1)

su(2)

$\sigma_x^{(1)}$ $\sigma_z^{(1)}$ → $\sigma_y^{(1)}$

Cliff(1)

su(2)

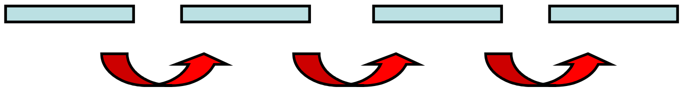
$SU(4) \qquad SO(4) \qquad SU(2)\otimes SU(2) \qquad SU(4)$

Complete population transfer occurs if

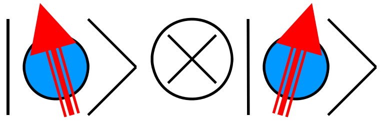
$$\alpha : \beta : \gamma \Leftrightarrow 5:3:4$$

This “magic” relation is an indication of a geometric character underlying evolution dynamics.

## **Part II.** Search for two qubits embedded in a 4-state system: a geometric (Clifford) algebra approach



?



Using quaternion algebra the problem is reduced to an algebraic relation between two unitary quaternions

$$U^{(1)} \Leftrightarrow n_0^{(1)} + n_1^{(1)} i + n_2^{(1)} j + n_3^{(1)} k \qquad \mathbf{n}^{(1)} \in S^3 \;\; in \;\; \mathbb{R}^4$$

$$U^{(2)} \Leftrightarrow n_0^{(2)} + n_1^{(2)} i + n_2^{(2)} j + n_3^{(2)} k \qquad \mathbf{n}^{(2)} \in S^3 \;\; in \;\; \mathbb{R}^4$$

$$\left(n_0^1 + n_1^1 i + n_2^1 j + n_3^1 k\right) = j\left(n_0^2 + n_1^2 i + n_2^2 j + n_3^2 k\right)^*$$

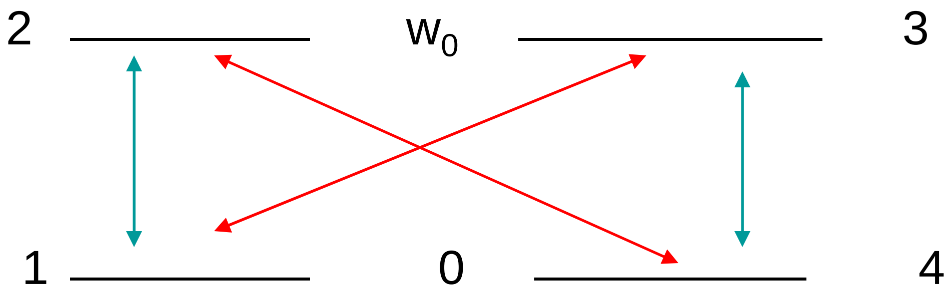
For the 4-level coupling scheme this relation has the form

$n^{(1)} \cdot n^{(2)} = 0,$ both $n^{(1)}$ and $n^{(2)}$ are in the $zy$ plane.

The general solution for complete population transfer

$$\alpha : \beta : \gamma \Leftrightarrow \frac{(2n+1)^2 + 1}{2} : (2n+1) : \frac{(2n+1)^2 - 1}{2}$$

$$A_x = \begin{pmatrix} 0 & \Omega_y & \Omega_z & 0 \\ \Omega_y & \omega_0 & 0 & \Omega_z \\ \Omega_z & 0 & \omega_0 & \Omega_y \\ 0 & \Omega_z & \Omega_y & 0 \end{pmatrix}$$




Combination of $O_1$,$O_4$,$O_5$,$O_8$,$O_9$,$O_{12}$,$O_{14}$; $O_{13}$ commutes with all.
Same Hamiltonian with Josephson junctions as two qubits occurs in Yu. A. Pashkin etal, Nature **421**, 823 and **425**, 941 (2003).

only seven operators: su(2) X su(2) X u(1)

## Construction for general SU(N)

Partition N-dim H as (N-n)- and n-dim blocks:

$$\mathbf{H}^{(N)} = \begin{pmatrix} \tilde{\mathbf{H}}^{(N-n)} & \mathbf{V} \\ \mathbf{V}^\dagger & \tilde{\mathbf{H}}^{(n)} \end{pmatrix}.$$

Write U again as three factors, first two nilpotent structure

$$\mathbf{U}^{(N)}(t) = \tilde{U}_1 \tilde{U}_2, \;\; \tilde{U}_1 = e^{\mathbf{z}(t)A_+} e^{\mathbf{w}^\dagger(t)A_-},$$

$$\tilde{U}_2 = \begin{pmatrix} \tilde{\mathbf{U}}^{(N-n)}(t) & \mathbf{0} \\ \mathbf{0}^\dagger & \tilde{\mathbf{U}}^{(n)}(t) \end{pmatrix}$$

$$i\dot{\tilde{U}}_2 = H_{\text{eff}} \tilde{U}_2, \;\; H_{\text{eff}} = \tilde{U}_1^{-1} H \tilde{U}_1 - i \tilde{U}_1^{-1} \dot{\tilde{U}}_1$$

$$i\dot{\mathbf{z}} = \tilde{\mathbf{H}}^{(N-n)}\mathbf{z} + \mathbf{V} - \mathbf{z}(\mathbf{V}^\dagger \mathbf{z} + \tilde{\mathbf{H}}^{(n)})$$

Matrix Riccati equation, **z**(0)=0. For n=1, **z**: (N-1) vector

Hamiltonian for (N-1) residual problem:

$$\mathbf{H}^{(N-1)} = \tilde{\mathbf{H}}^{(N-1)} - \frac{\mathbf{z}\mathbf{V}^\dagger + \mathbf{V}\mathbf{z}^\dagger}{\sqrt{\gamma}+1} - \frac{\mathbf{z}(\mathbf{z}^\dagger\mathbf{V} + \mathbf{V}^\dagger\mathbf{z})\mathbf{z}^\dagger}{2(\sqrt{\gamma}+1)^2}$$

explicitly Hermitian.

D.Uskov and ARPR: arXiv quant-ph/0511192

four-level/two-qubit SU(4): ($z_1$,$z_2$,$z_3$) Riccati, next ($z_1$,$z_2$), final z, three su-phases in all and six complex z's

vs

two-level/qubit SU(2): one phase and one complex z or $S^2$

SU(N): [SU(N)/(SU(N-1) X U(1))] X ((SU(N-1) X U(1))

base manifold fiber

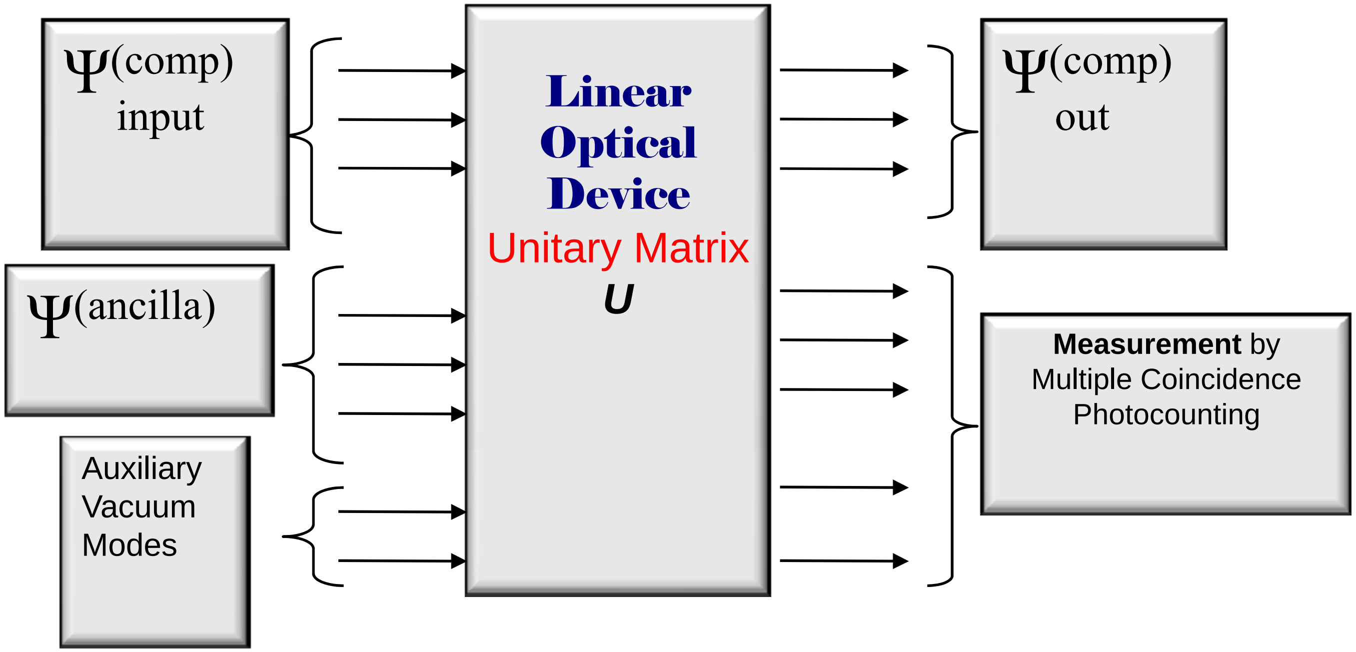
[SU(N)/(U(1) X U(1) …. U(1))] X (U(1) X U(1) … U(1))

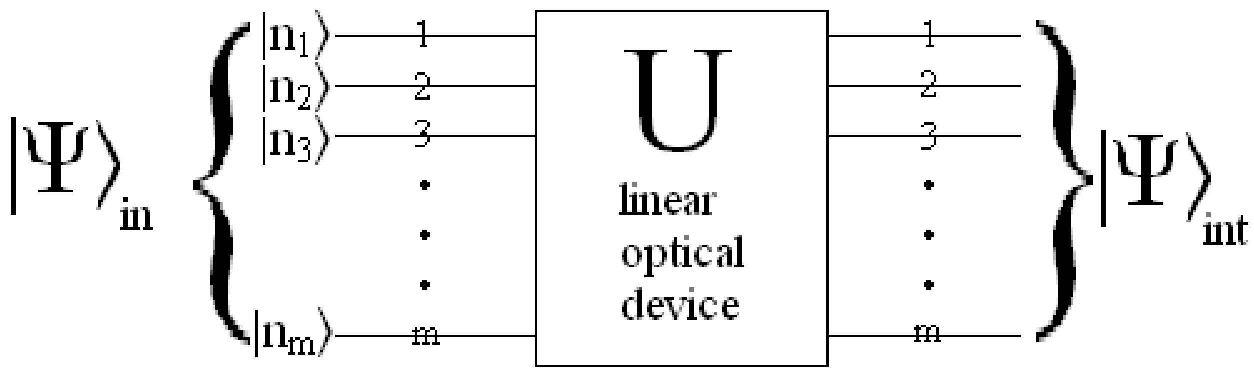
Schwinger philosophy

(N-1) su-phases

# Designing Optimal States and Transformations for Quantum Metrology
# And Communication

# Optimization of Success Probabilities of Quantum Photonic Gates





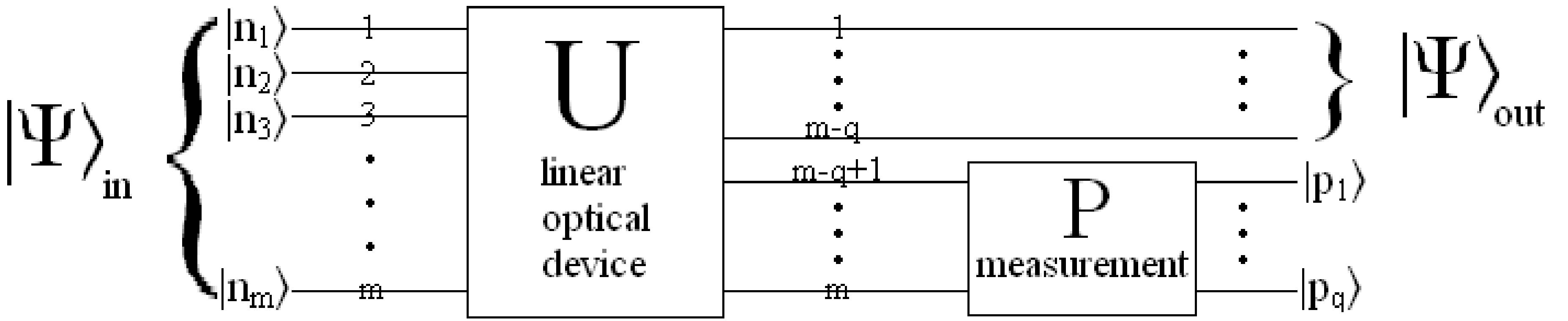

*m*-mode disentangled optical input state:

$$|\psi\rangle_{in} = |n_1\rangle_1 \otimes \ldots \otimes |n_m\rangle_m = \prod_{i=1}^{m} \frac{1}{\sqrt{n_i\,!}} \left(a_i^+\right)^{n_i} |0\rangle^{\otimes m}$$

Linear optical device (unitary transformation of the modes) is applied:

$$|\psi\rangle_{\text{int}} = \prod_{i=1}^{m} \frac{1}{\sqrt{n_i\,!}} \left( \sum_{j=1}^{m} \boldsymbol{u}_{i,j} \cdot a_{j,out}^+ \right)^{n_i} |0\rangle^{\otimes m}$$

# LOQSG




Output of "1" $\Longrightarrow$ successful state generation with:

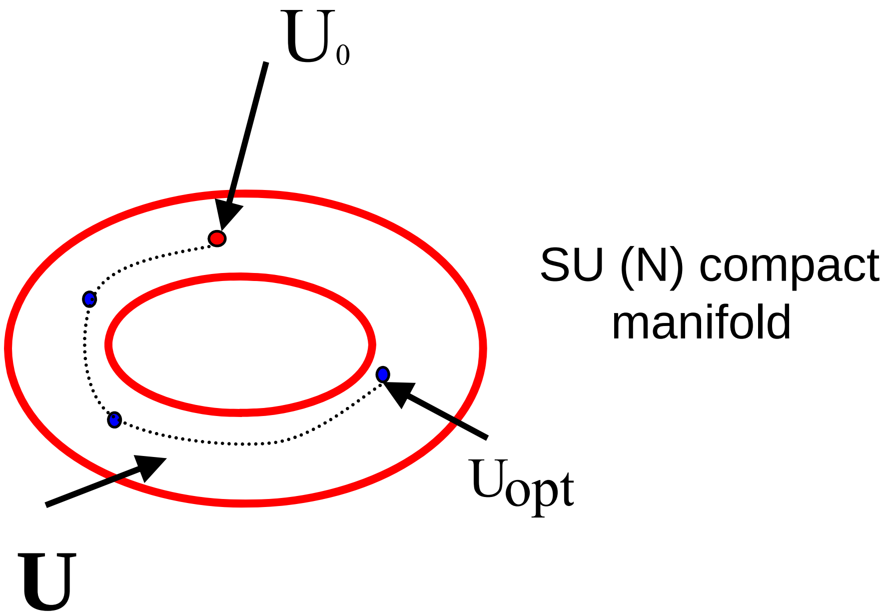
$$|\psi\rangle_{out} = \hat{P} \cdot \tilde{\hat{U}} |\psi\rangle_{in} \qquad \tilde{\hat{U}} = \{\tilde{u}_{i,j}\}, \quad \tilde{\hat{U}}^{+} \tilde{\hat{U}} = I$$

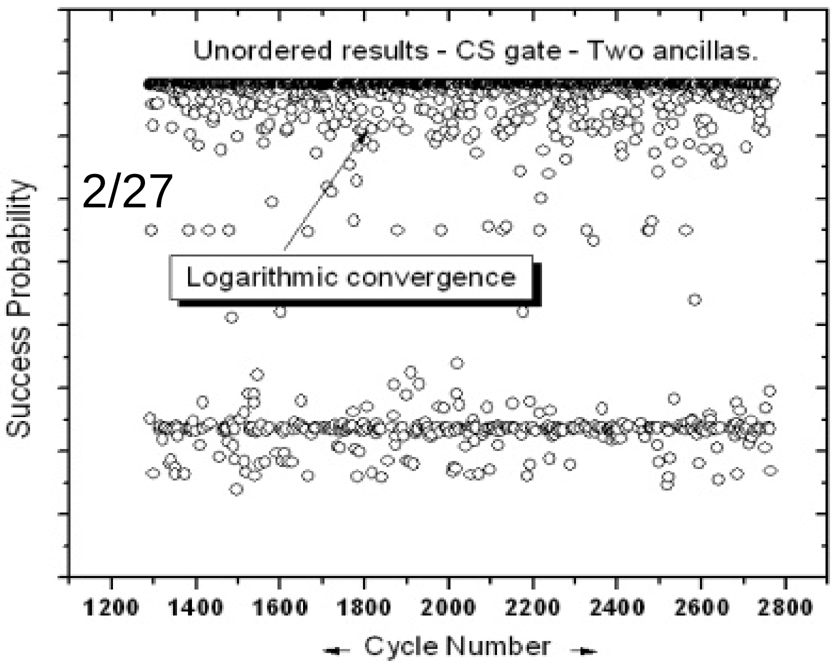
$$\hat{P} = I_1 \otimes I_2 ... \otimes I_{m-q} \otimes |p_1\rangle\langle p_1|_{m-q+1} \otimes ... \otimes |p_m\rangle\langle p_m|_m$$

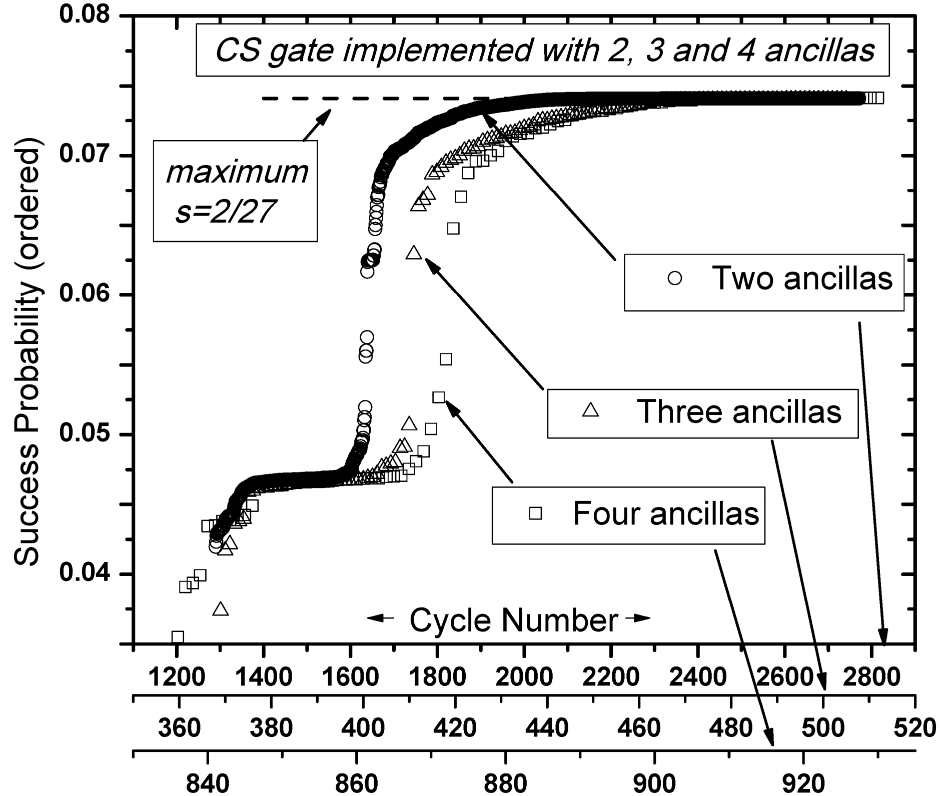
- Optimization of fidelity function or success probability F(U):

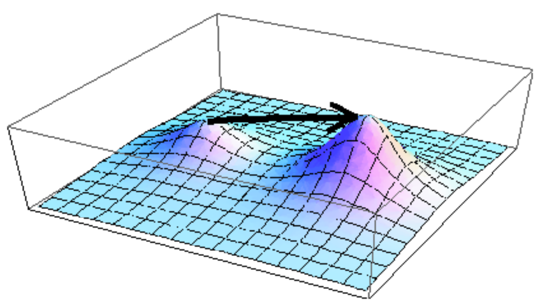
1. chose a random starting point $U_0$
2. calculate the next point as

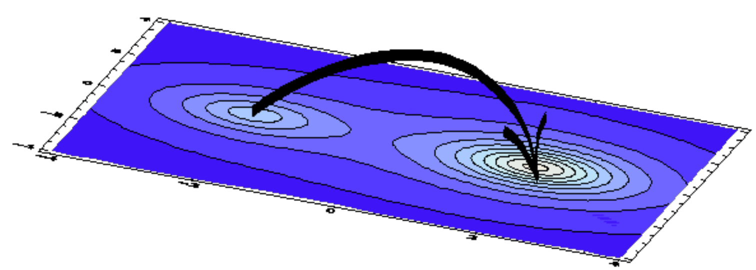
$$U_0^{\text{next}} = U_0 e^{\sum_{i=1}^{8} \nabla F_i \, \hat{\chi}_i} , \quad \hat{\chi}_i \text{ span the } su(\text{N}) \text{ Lie Algebra}$$

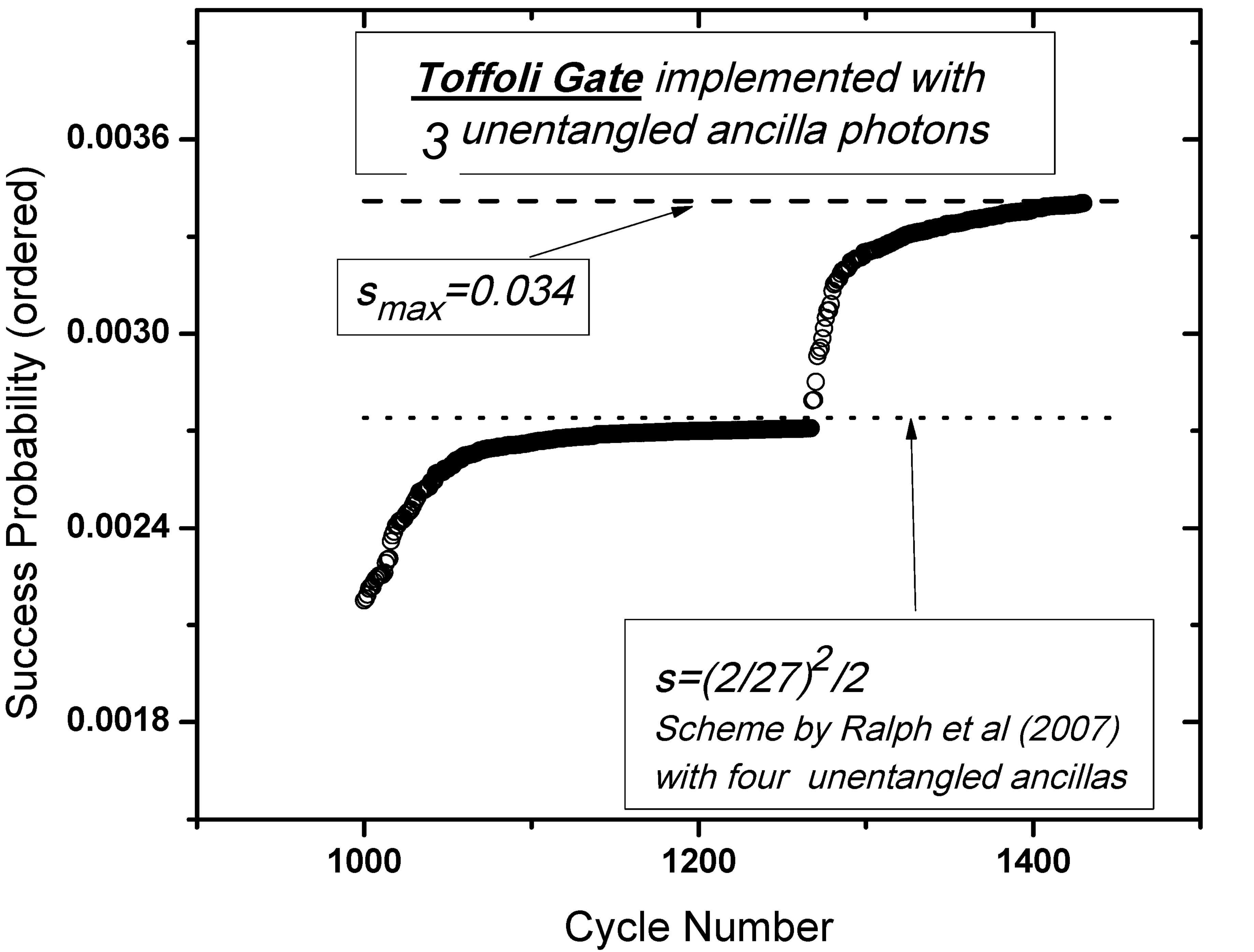



Unordered results - CS gate - Two ancillas.
2/27
Logarithmic convergence
Success Probability
Cycle Number
1200
1400
1600
1800
2000
2200
2400
2600
2800


CS gate implemented with 2, 3 and 4 ancillas
maximum
s=2/27
Two ancillas
Three ancillas
Four ancillas
Success Probability (ordered)
Cycle Number
0.08
0.07
0.06
0.05
0.04
1200
1400
1600
1800
2000
2200
2400
2600
2800
360
380
400
420
440
460
480
500
520
840
860
880
900
920

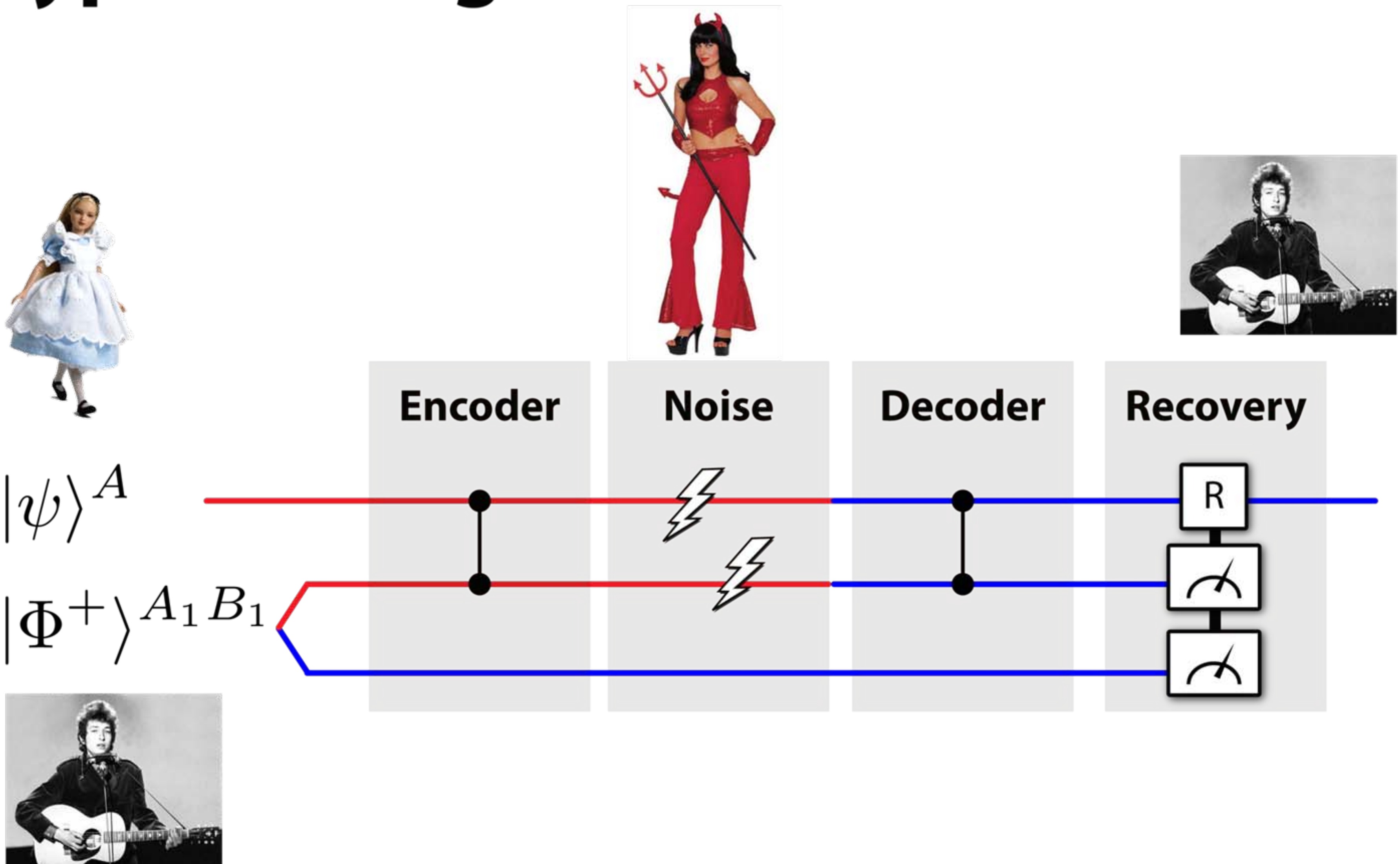






Toffoli Gate implemented with
3 unentangled ancilla photons
s_max=0.034
s=(2/27)^2/2
Scheme by Ralph et al (2007)
with four unentangled ancillas
Success Probability (ordered)
0.0036
0.0030
0.0024
0.0018
1000
1200
1400
Cycle Number

# Operation of the Hyperentanglement–Assisted Code




*M. M. Wilde and D. B. Uskov, arXiv:0807.4906 (2008).*

# Error Syndrome Table

| Error | Recovery | Syndrome |
|---|---|---|
| $I$ | $I$ | $\Phi^+$ |
| $X^A$ | $X$ | $\Phi^-$ |
| $X^{A_1}$ | $Z$ | $\Psi^+$ |
| $X^A X^{A_1}$ | $ZX$ | $\Psi^-$ |

**Syndrome table** reduces to **superdense coding**!

*M. M. Wilde and D. B. Uskov, arXiv:0807.4906 (2008).*

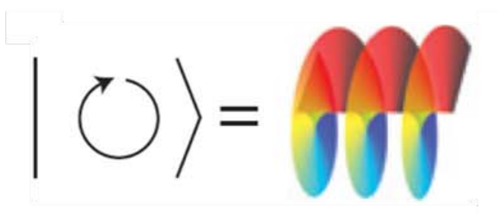
# Hyperentanglement

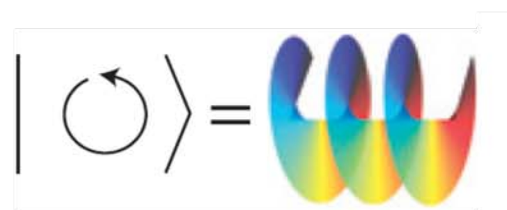
A **hyperentangled** state is simultaneously entangled in

**polarization** and **orbital angular momentum**

$$\frac{1}{2}\left(|HH\rangle^{AB} + |VV\rangle^{AB}\right) \otimes \left(|\circlearrowleft\circlearrowright\rangle^{AB} + |\circlearrowright\circlearrowleft\rangle^{AB}\right)$$

where





*J. T. Barreiro, T.-C. Wei, and P. G. Kwiat, Nature Physics* ***4****, 282 (2008).*

# Encoding and Decoding Circuit

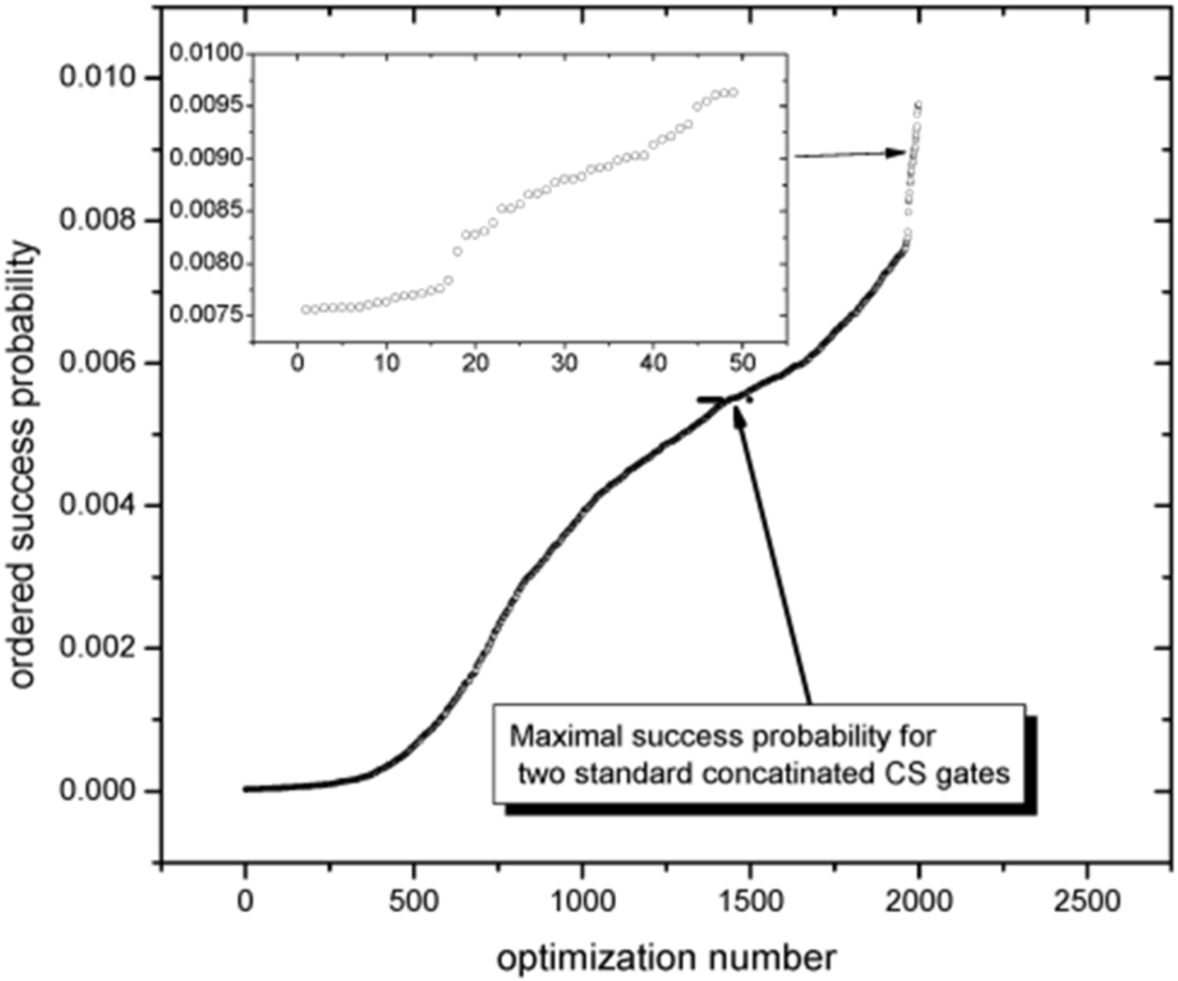
**Transform** the following basis states

$$|V\rangle^{A}|V \circlearrowleft\rangle^{A_1} \rightarrow -|V\rangle^{A}|V \circlearrowleft\rangle^{A_1},$$
$$|V\rangle^{A}|V \circlearrowright\rangle^{A_1} \rightarrow -|V\rangle^{A}|V \circlearrowright\rangle^{A_1}.$$

and leave the others **alone**!

*M. M. Wilde and D. B. Uskov,* PRA 2008

# Gate Optimization

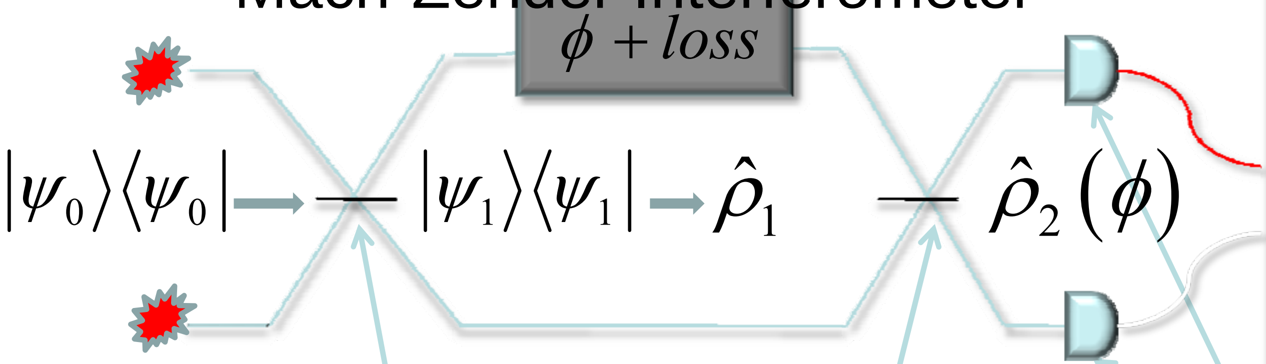



Gate requires only **3 ancilla modes** and has success probability of **0.0096**

*M. M. Wilde and D. B. Uskov, arXiv:0807.4906 (2008).*

# Phase Estimation

Mach-Zender Interferometer




## Stochastic Process

**(N+1)(N+2)/2 possible outcomes**

$$p_m(\phi) = Tr\left(A_m \hat{\rho}_2 A_m^{\dagger}\right)$$

$$m = \{n_1, n_2\}$$

## Beam splitters and phase shifters

$$\hat{U} = e^{\alpha \hat{J}_z} e^{\beta \hat{J}_y} e^{\gamma \hat{J}_z}$$

$$\hat{J}_z = i(a_1^{\dagger} a_1 - a_2^{\dagger} a_2), \; \hat{J}_y = (a_1^{\dagger} a_2 - a_1 a_2^{\dagger})$$

## Number resolving detectors

ideal number-resolving detectors implement POVM with Kraus operators

$$A_{\{n_1,n_2\}} = |vac\rangle\langle n_1, n_2|$$

## Kramer-Rao Bound

$$(\delta\phi_e)^2 = 1/\Im$$

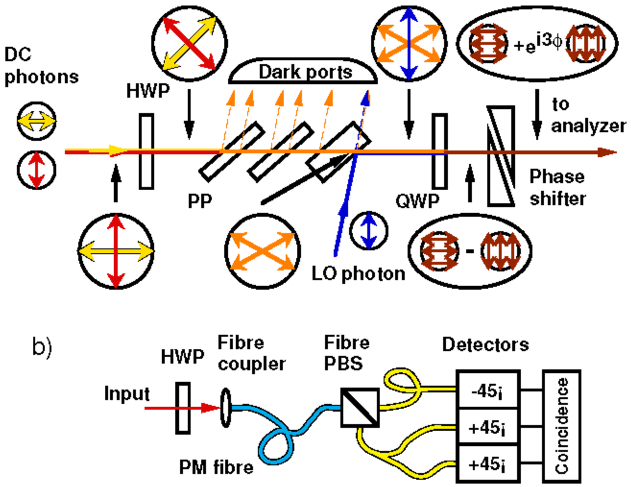
Classical Limit $\delta\phi_e = 1/\sqrt{N}$

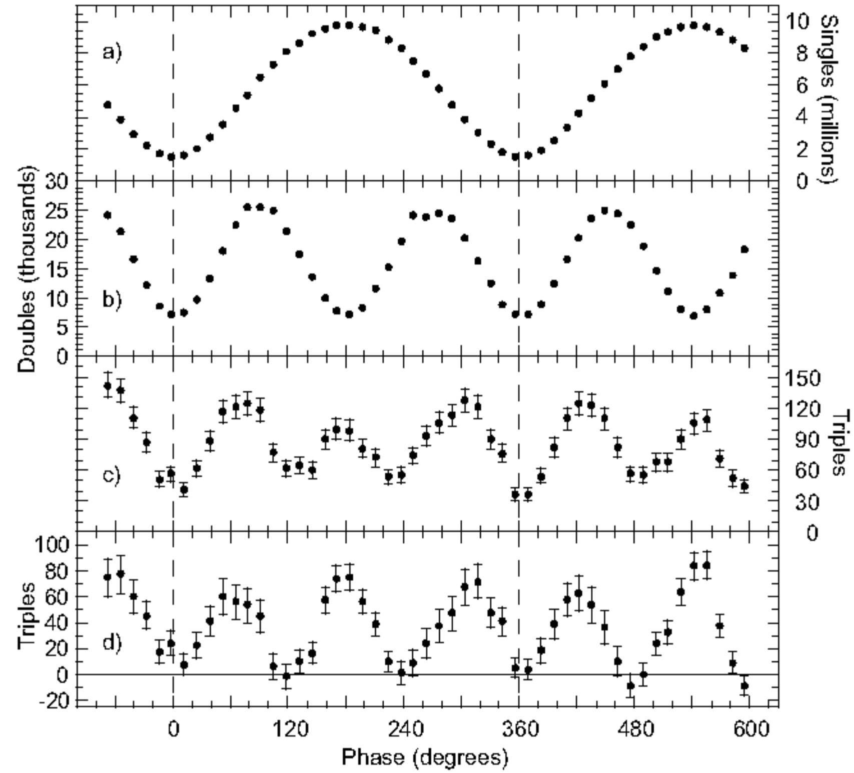
Heisenberg Limit $\delta\phi_e = 1/N$

## *We optimize Fisher Information*

$$\Im(\psi_1) = \sum_{\{n_1,n_2\}} \frac{1}{p_{\{n_1,n_2\}}} \left(\partial_{\phi}\, p_{\{n_1,n_2\}}\right)^2$$

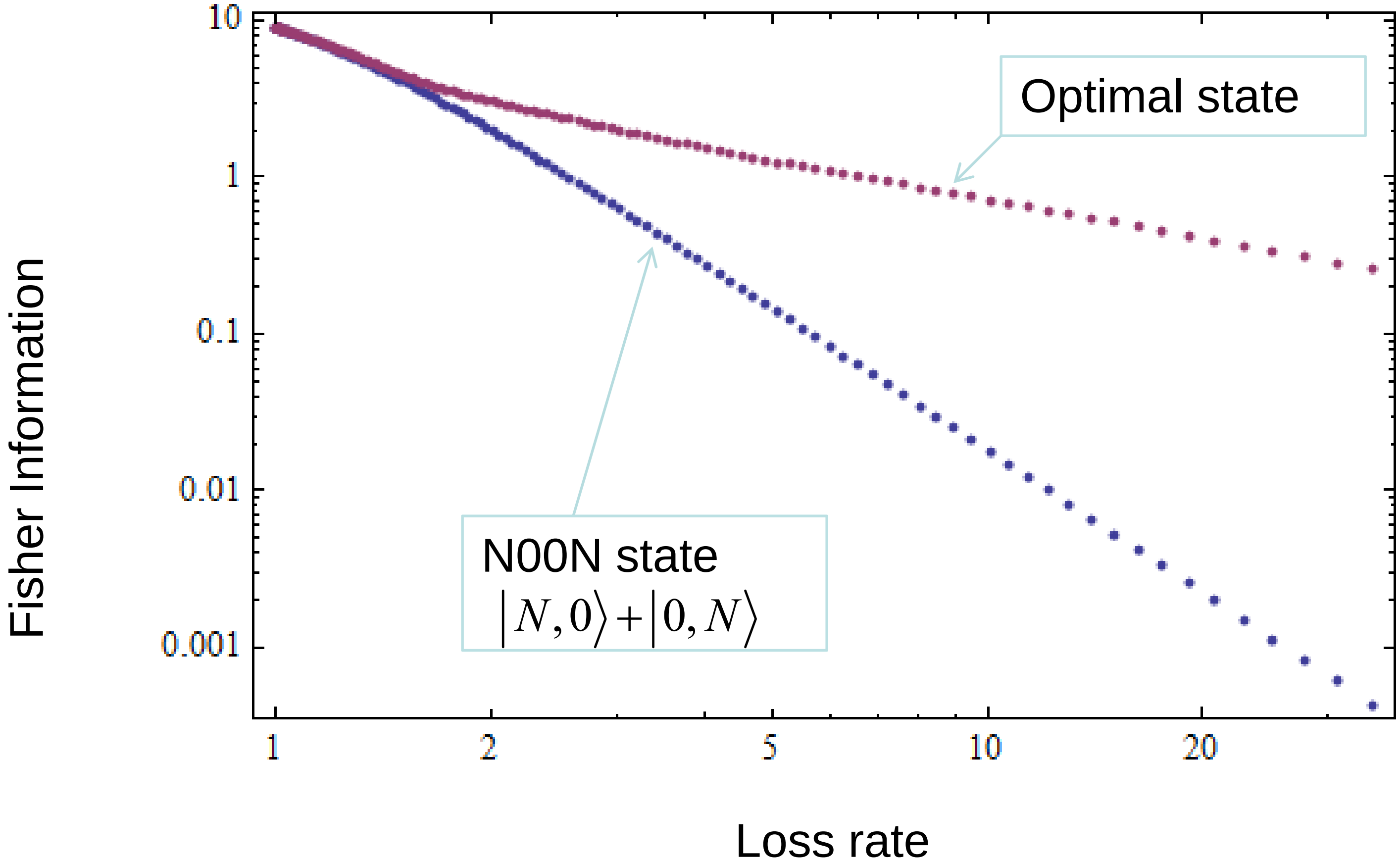
# Mitchell, J. S. Lundeen and A. M. Steinberg 2003



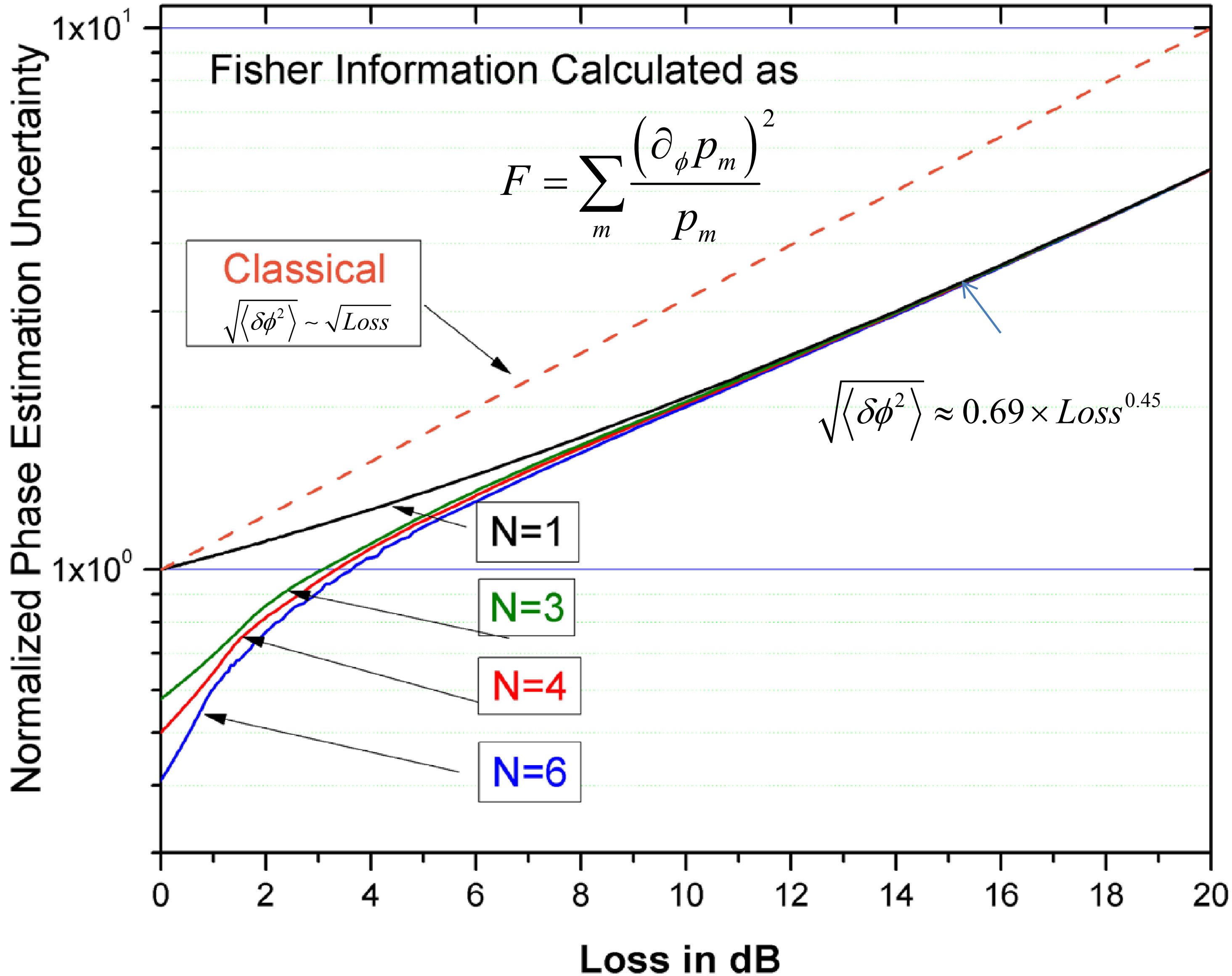
# What is the Optimal State $\psi_1$ ?

Fisher Information



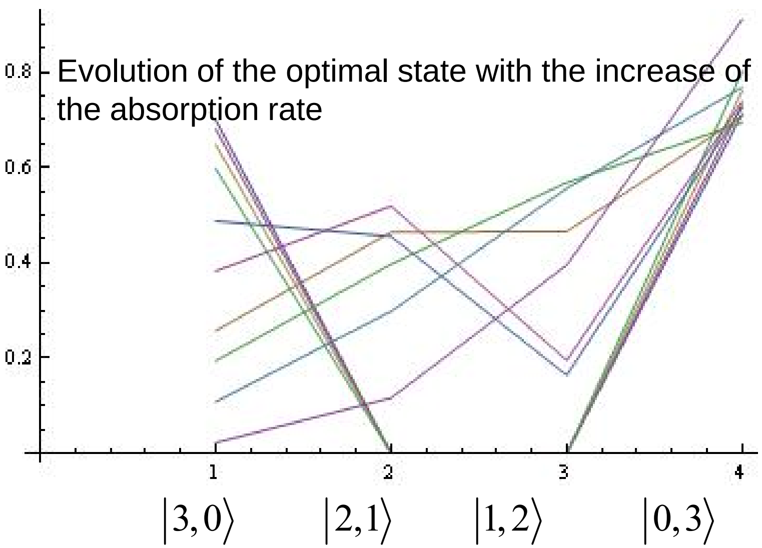
$$\sqrt{N_{shots} \times N_{photons} \times \langle \delta\phi^2 \rangle} \sim 1 / \sqrt{(\text{Fisher Information}/N_{photons})}$$

**Minimal Phase Uncertainty Normalized per One Photon per One Shot for N=1, 3, 4, 6 Photons in the Input State**

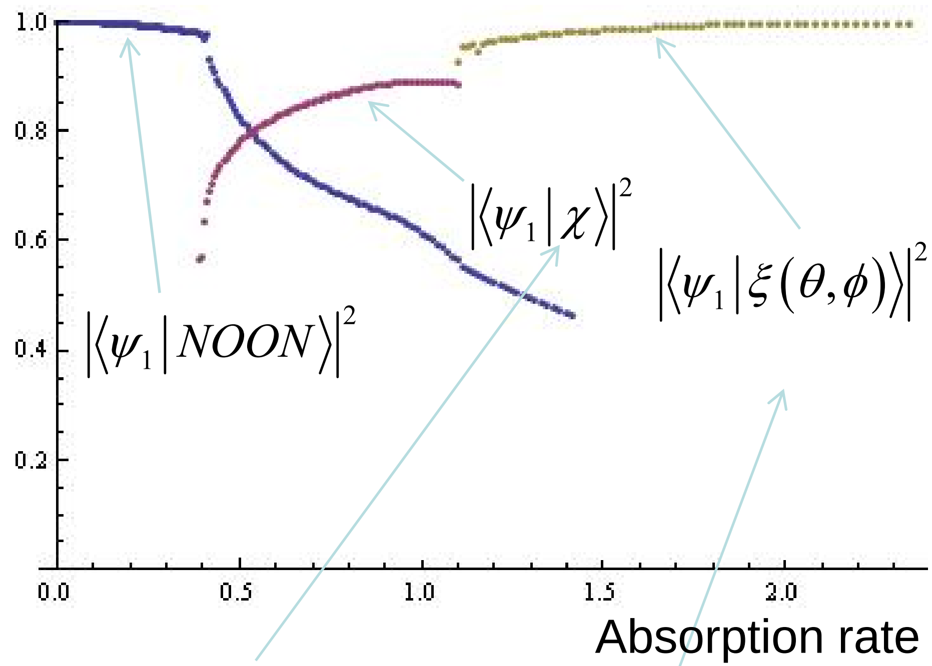


# What is the Optimal State $\psi_1$ ?

Absorption rate




Structure of the optimal state




$$\chi = \frac{1}{\sqrt{2}}\left(|N-1,1\rangle + |0,N\rangle\right)$$

$$|\xi(\theta,\phi)\rangle = \textit{Generalized Coherent State}$$

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