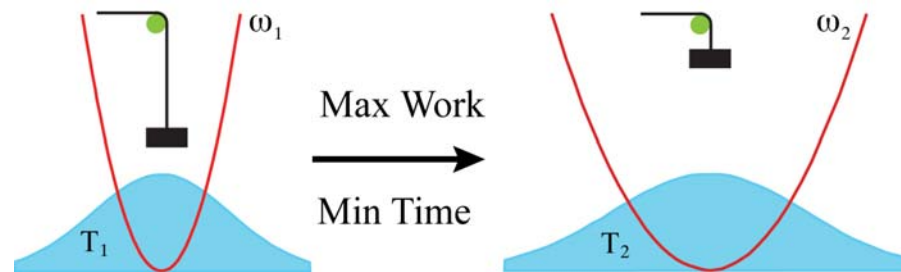


Fast Adiabatic Control of a Harmonic Oscillator's Frequency

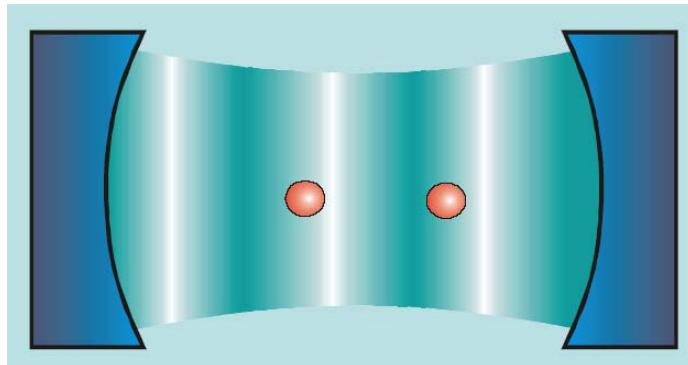
Peter Salamon
Dept. of Math & Stats
San Diego State University



Ensemble of Independent Harmonic Oscillators Sharing a Controlled Frequency ω

$$H = \frac{1}{2}(P^2 + \omega^2 Q^2)$$

Cool atoms in an optical lattice. Lattice created by lasers and having an easily controlled ω .



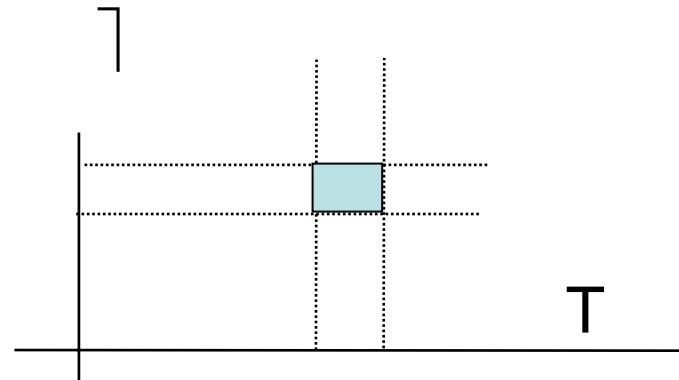
The Heat Engine

Rate
Limiting
Step

- Contact with $T=T_H$ at $\omega = \omega_1$.
- Adiabatic change from $\omega = \omega_1$ to $\omega = \omega_2$.
- Contact with $T=T_C$ at $\omega = \omega_2$.
- Adiabatic change from $\omega = \omega_2$ to $\omega = \omega_1$.

Controls: It's all in the **timing**.

Time for thermal contacts and rate at which ω changes on adiabats.



Dynamic Third Law

- The second law limits the rate of cooling

For a cycle operating between T_c and T_h and exchanging heat, the net entropy production rate is

$$\sigma = -\dot{Q}_c/T_c + \dot{Q}_h/T_h > 0.$$

For bounded $|\dot{Q}_h| < C$ this rearranges to give

$$\left(\frac{C}{T_h}\right) T_c > \dot{Q}_c \propto T^\delta$$

Definition

- A **prelude process** is a **reversible** process performed as a prelude to a thermal process. □



Heisenberg Representation

$$\frac{dA}{dt} = i[H, A] + \frac{\partial A}{\partial t} \quad \text{adiabats}$$

$$\frac{dA}{dt} = i[H, A] + \frac{\partial A}{\partial t} + \mathcal{L}_D^*(A) \quad \text{thermal contacts}$$

with Lindblad operator

$$\mathcal{L}_D(\rho) = k_{\downarrow}(a^{\dagger}\rho a - \frac{1}{2}\{aa^{\dagger}, \rho\}) + k_{\uparrow}(a\rho a^{\dagger} - \frac{1}{2}\{a^{\dagger}a, \rho\})$$

The working fluid: Quantum Harmonic Oscillator

$$\begin{aligned} H &= \frac{1}{2}(P^2 + \omega^2 Q^2) && \text{Hamiltonian} \\ L &= \frac{1}{2}(P^2 - \omega^2 Q^2) && \text{Lagrangian} \\ C &= \frac{\omega}{2}(PQ + QP) && \text{Correlation} \end{aligned}$$

C is needed to close the Lie algebra

Dynamics on Adiabats

$$\dot{E} = \frac{\dot{\omega}}{\omega}(E - L)$$

$$\omega_{\min} \leq \omega \leq \omega_{\max}$$

$$\dot{L} = -\frac{\dot{\omega}}{\omega}(E - L) - 2\omega C$$

$$-\infty \leq \dot{\omega} \leq \infty$$

$$\dot{C} = 2\omega L + \frac{\dot{\omega}}{\omega}C$$

or, using sudden jumps $\omega_i \rightarrow \omega_f$

$$\begin{pmatrix} E \\ L \\ C \end{pmatrix}_{\omega_f} = \frac{1}{2} \begin{pmatrix} 1 + \left(\frac{\omega_f}{\omega_i}\right)^2 & 1 - \left(\frac{\omega_f}{\omega_i}\right)^2 & 0 \\ 1 - \left(\frac{\omega_f}{\omega_i}\right)^2 & 1 + \left(\frac{\omega_f}{\omega_i}\right)^2 & 0 \\ 0 & 0 & \frac{2\omega_f}{\omega_i} \end{pmatrix} \begin{pmatrix} E \\ L \\ C \end{pmatrix}_{\omega_i}$$

Sudden adiabats **not optimal** due to quantum friction.

Fixed omega dynamics

$$\omega(t) = \omega(0)$$

$$E(t) = E(0)$$

$$L(t) = \cos(2\omega t)L(0) - \sin(2\omega t)C(0)$$

$$C(t) = \sin(2\omega t)L(0) + \cos(2\omega t)C(0)$$

Dynamics for Heat Exchange

- Lindblad dynamics

$$E(t) = e^{-\Gamma t} (E(0) - E_{eq}(T)) + E_{eq}(T)$$

$$\begin{pmatrix} L(t) \\ C(t) \end{pmatrix} = e^{-\Gamma t} \begin{pmatrix} \cos(2\omega t) & -\sin(2\omega t) \\ \sin(2\omega t) & \cos(2\omega t) \end{pmatrix} \begin{pmatrix} L(0) \\ C(0) \end{pmatrix}$$

where $\Gamma = k_{\downarrow} - k_{\uparrow}$

Heat bath = coherence decay

PHYSICAL REVIEW E **68**, 016101 (2003)

Quantum four-stroke heat engine: Thermodynamic observables in a model with intrinsic friction

Tova Feldmann and Ronnie Kosloff*

Department of Physical Chemistry, The Hebrew University, Jerusalem 91904, Israel



QuickTime™ and a
decompressor
are needed to see this picture.

Irreversible performance of a quantum harmonic heat engine

Yair Rezek¹ and Ronnie Kosloff

Department of Physical Chemistry, the Hebrew University,
Jerusalem 91904, Israel

New Journal of Physics **8** (2006) 83

Quantum Entropy

The Von Neumann entropy

$$S_{VN} = \text{Tr} (\rho \log(\rho))$$

is conserved.

Effective entropy in contact with the heat bath is the **energy entropy**

$$S_E = \sum_n P_n \log(P_n)$$

where $P = \text{diag}(\rho_E)$ and ρ_E is the density matrix in an energy basis.

$$S_{VN} \leq S_E$$

Quantum Friction

Problem: Changing ω at a finite rate or jumping from

$$\omega_i \rightarrow \omega_f$$

creates “extra” entropy by increasing S_E .

During a heat exchange, the energy in the LC oscillation becomes heat. This is Feldmann & Kosloff’s quantum friction.

Adiabatic Switching

If we change ω infinitely slowly, we can keep S_{VN} constant.

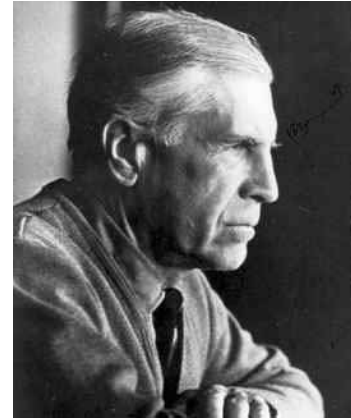
$$S_{\text{VN}} = \ln \left(\sqrt{X - \frac{1}{4}} \right) + \sqrt{X} \operatorname{asinh} \left(\frac{\sqrt{X}}{X - \frac{1}{4}} \right)$$

$$X = \frac{E^2 - L^2 - C^2}{\hbar^2 \omega^2}.$$

- Sets energy minimum
- Thermal equilibrium at $L=C=0$

Optimal Control

Easier and more powerful calculus of variations.



The Problem:

$$\frac{dx}{dt} = f(x, u); \quad \int f_0(x, u) dt \rightarrow \text{Min}$$

The Tool:

$$H(x, \lambda, u) = \sum_{i=0}^n \lambda_i f_i \quad \left\{ \begin{array}{l} \frac{dx}{dt} = \frac{\partial H}{\partial \lambda} \\ \frac{d\lambda}{dt} = -\frac{\partial H}{\partial x} \end{array} \right.$$

The Optimality Conditions:

H constant in time
 H maximum in u at each t

Optimal Adiabats

Problem: How to choose $\omega(t)$?

$$\dot{E} = \frac{\dot{\omega}}{\omega}(E - L)$$

$$\dot{L} = -\frac{\dot{\omega}}{\omega}(E - L) - 2\omega C$$

$$\dot{C} = 2\omega L + \frac{\dot{\omega}}{\omega}C$$

augment with

$$\dot{\omega} = u\omega$$

Optimal control Hamiltonian

$$\begin{aligned} H &= \lambda_1 u\omega + \lambda_2 u(E - L) - \lambda_3(u(E - L) + 2\omega C) + \lambda_4(2\omega L + uC) \\ &= (\lambda_1\omega + (\lambda_2 - \lambda_3)(E - L))u + 2\omega(\lambda_4 L - \lambda_3 C) \\ &= \sigma u + \alpha \end{aligned}$$

Linear in $u!!!$

Singular Control Problems

$$H = \sigma(x, \lambda)u + \alpha(x, \lambda)$$

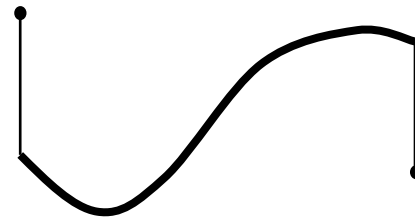
σ = switching
function

$\sigma > 0$; $u = u_{\text{Max}}$

$\sigma < 0$; $u = u_{\text{Min}}$

$\sigma \equiv 0$; $u = ?$

This structure usually leads to **turnpike theorems**.



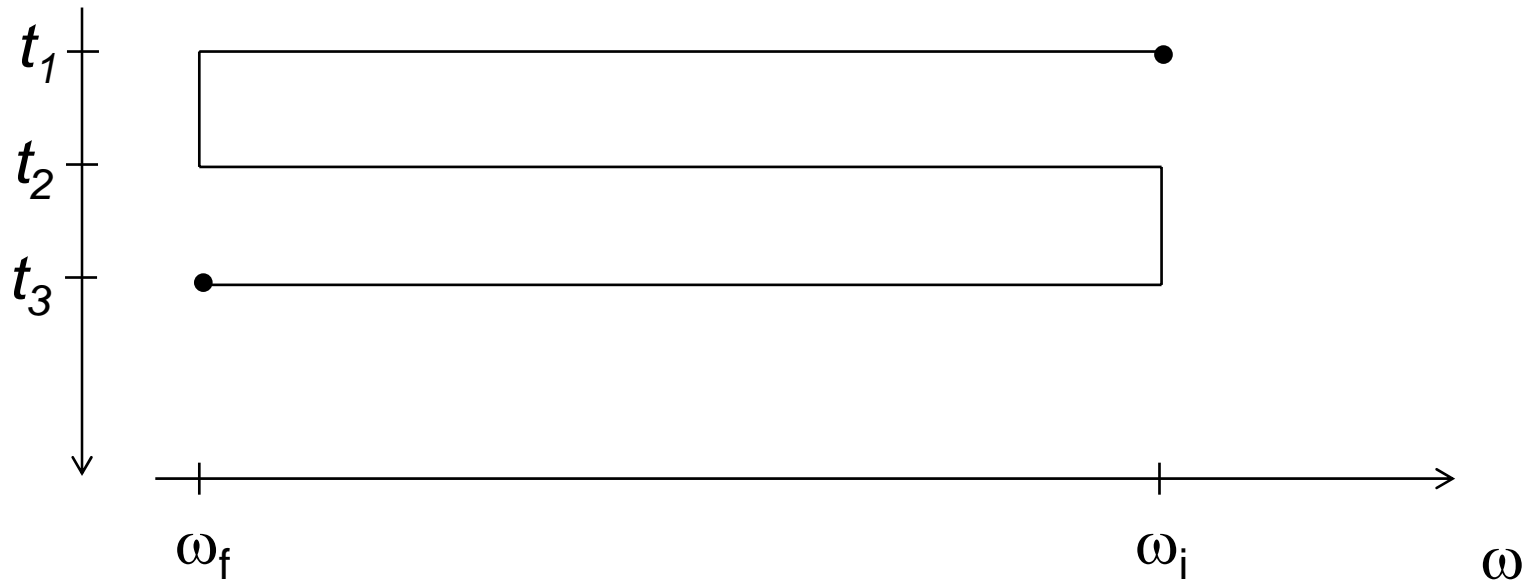
Theorem: Optimal control of the harmonic oscillator is bang-bang.

Singular branches are never used.

Best Adiabats

$$\omega_f \leq \omega_1, \omega_2 \leq \omega_i \quad \Longrightarrow \quad \begin{aligned} \omega_1^{\text{opt}} &= \omega_f \\ \omega_2^{\text{opt}} &= \omega_i \end{aligned}$$

$$\tau_{\min} = \frac{1}{2} \left(\frac{1}{\omega_i} + \frac{1}{\omega_f} \right) \text{Arccos} \left(\frac{\omega_i^2 + \omega_f^2}{(\omega_i + \omega_f)^2} \right)$$



Best Adiabatic

When ω_1 and ω_2 are forced to lie within $[\omega_i, \omega_f]$, The best we can do is to take $\omega_1 = \omega_f$ and $\omega_2 = \omega_i$. For this case, the two cosines are equal; the minimum time to reach $(E_0 \frac{\omega_f}{\omega_i}, 0, 0)$ given by

$$t_3 - t_2 = \frac{1}{2\omega_2} \text{Arccos} \left(\frac{\omega_i^2 + \omega_f^2}{(\omega_i + \omega_f)^2} \right)$$

and

$$t_2 - t_1 = \frac{1}{2\omega_1} \text{Arccos} \left(\frac{\omega_i^2 + \omega_f^2}{(\omega_i + \omega_f)^2} \right)$$

Total time on the order of one oscillation !!!

The Magic

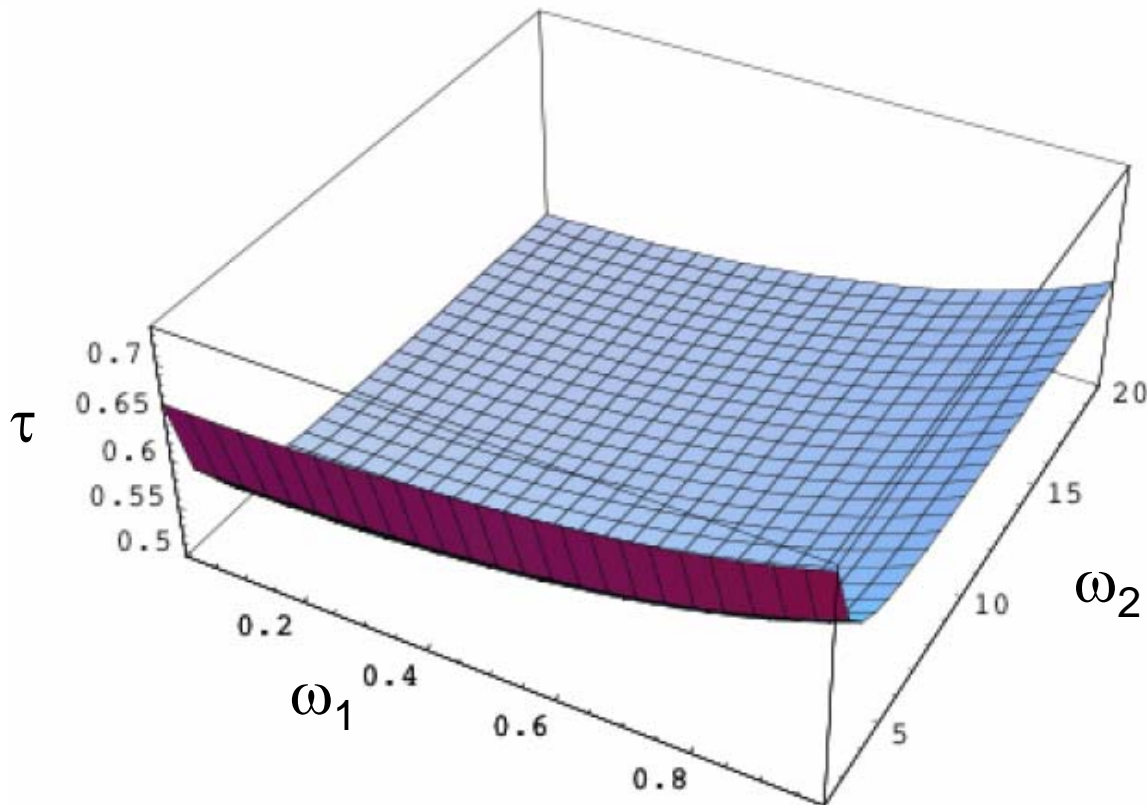
- Fast(est) adiabatic switching.
- Can only extract the full maximum work available from the change if
time > min time
else must create parasitic oscillations.
-- New type of finite-time Availability
- Time limiting branch in a heat cycle to cool system toward $T=0$.
 - Implies

$$\dot{Q}_c \propto T^{\frac{3}{2}}$$

Without constraining the intermediate frequencies

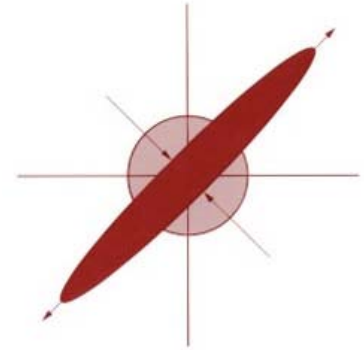
The minimum time to go from a thermal ensemble with ω_i to $\omega_f < \omega_i$ is with $\omega_1 = 0$ and $\omega_2 = \infty$. The resulting time is

$$\tau_{\min} = \frac{\sqrt{1 - \omega_f/\omega_i}}{\sqrt{\omega_f\omega_i}}$$





Squeezed States

- What is the optimal control problem?
- Can calculate the effect of the fast adiabatic switch, but what are we to expect out of it?





Proof of Optimality

- Consider a small step with $u = \frac{\dot{\omega}}{\omega} \neq 0, \pm\infty$
- Next consider replacing this step with a small move between the same initial and final states using only $u \in \{0, \infty\}$
- For $C u < 0$, use  For $C u > 0$, use 
- With the result that

$$\tau_{u \in \{0, \infty\}} - \tau_{u \text{ finite}} = \left\{ \begin{array}{ll} \frac{5}{12} \frac{\omega(E-L)u}{C} t^3, & C u < 0 \\ -\frac{1}{12} \frac{\omega(E-L)u}{C} t^3, & C u > 0 \end{array} \right\} < 0$$

Questions of Finite Time Thermodynamics

- What is the **maximum power** that can be delivered by a heat engine in finite time?

$$\eta_{\text{Max}P} = 1 - \sqrt{\frac{T_C}{T_H}}$$

- Given A_{init} , A_{final} , and τ , what is the **minimum entropy** that must be **produced** in changing the state of system A from A_{init} to A_{final} in time τ ?
 - Time as an observable dual to energy
 - Elapsed time dual to entropy production
 - Probe of open system quantum dynamics
- Given A_{init} , A_{final} , what is the **minimum time** for changing the state of system A from A_{init} to A_{final} ?
- How fast can we **approach $T=0$** ?