Optimal control of rare events

QControl09, UCSB, July 15, 2009

Michael Khasin
khasinmi@pa.msu.edu

In collaboration with
Mark Dykman Michigan State University
Baruch Meerson Hebrew University of Jerusalem

Outlook

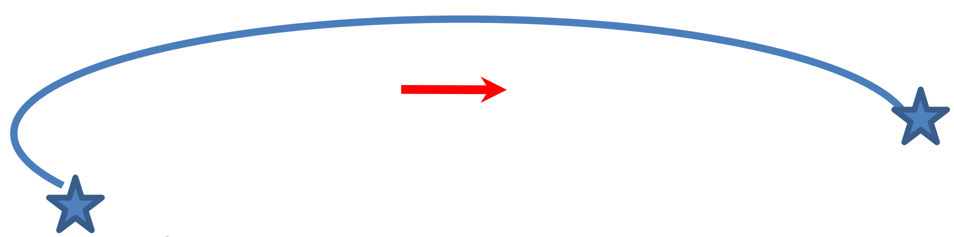
1. Rare events. Optimal fluctuation path
2. Controlling rare events
3. Optimal control with weak periodic force

Optimal fluctuation path

Optimal control problem

$$\dot{x} = K(x, u)$$

Cost functional $S[u]$



$$x_i \;,\; t_i \qquad\qquad x_f \;,\; t_f$$

Optimal fluctuation path

Assume $$\int e^{-\Omega S[u]} \mathcal{D}u < \infty$$ $$0 < \Omega < \infty$$

Then $$P[u] = e^{-\Omega S[u]}$$ Probability functional of noise u

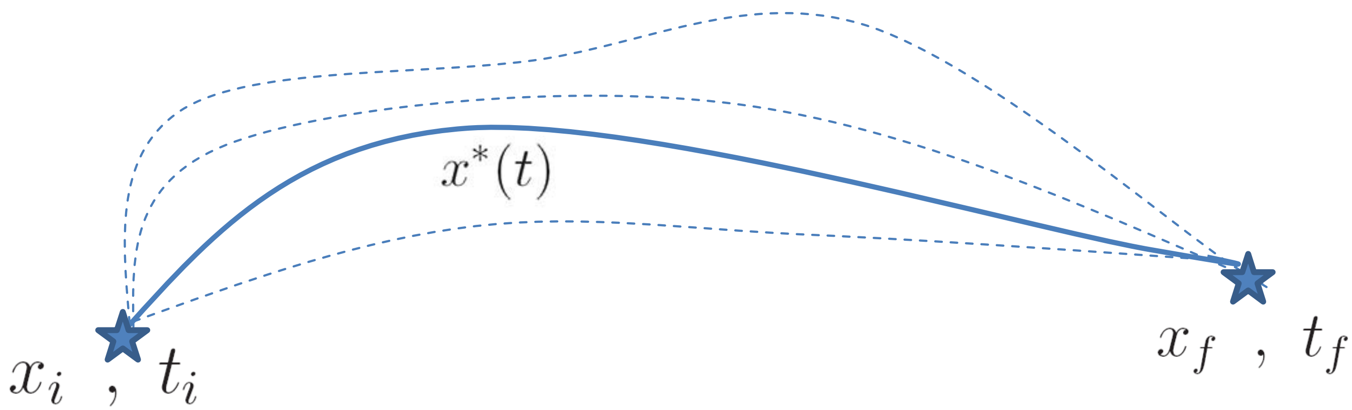
$$\dot{x} = K(x, u)$$ Stochastic dynamics driven by noise u

$$S[u^*] = \min_u S[u]$$

$$u^*(t)$$ Most probable realization of noise taking system from x_i , t_i to x_f , t_f

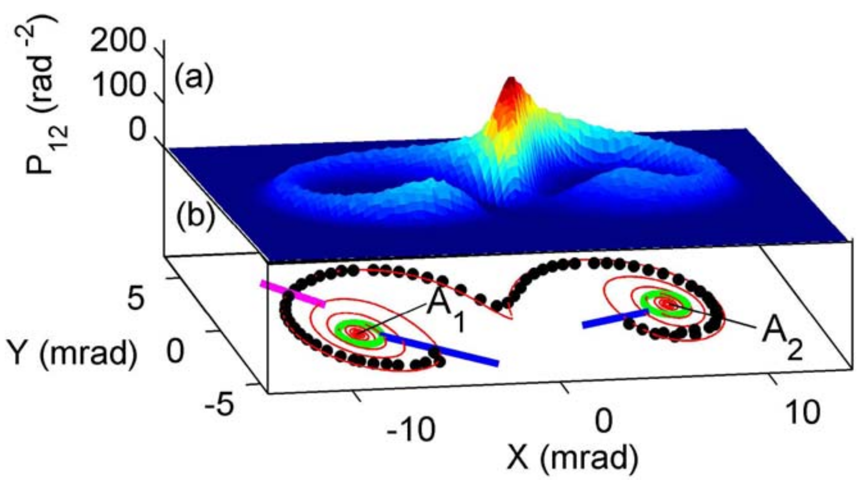
$$x^*(t)$$ *Optimal fluctuation path*

Optimal fluctuation path



$\Omega^{-1} \ll 1$ ➡ $P[u^*] = e^{-\Omega S[u^*]}$ has exponentially large contribution to the switching

"noise strength",
noise is weak on average



Switching trajectories in a parametrically excited micromechanical oscillator, Chan et al. (2008)

Optimal fluctuation path. Example

White Gaussian noise, nonlinear systems: Wentzell & Freidlin (1970) ; Dykman & Krivoglaz (1979)

$$\dot{x} = K(x) + u$$

$$\langle u(t)u(t')\rangle = D\delta(t - t') \quad \Rightarrow \quad P[u] = e^{-\Omega \int u^2 dt} \qquad \Omega = \frac{1}{2D}$$

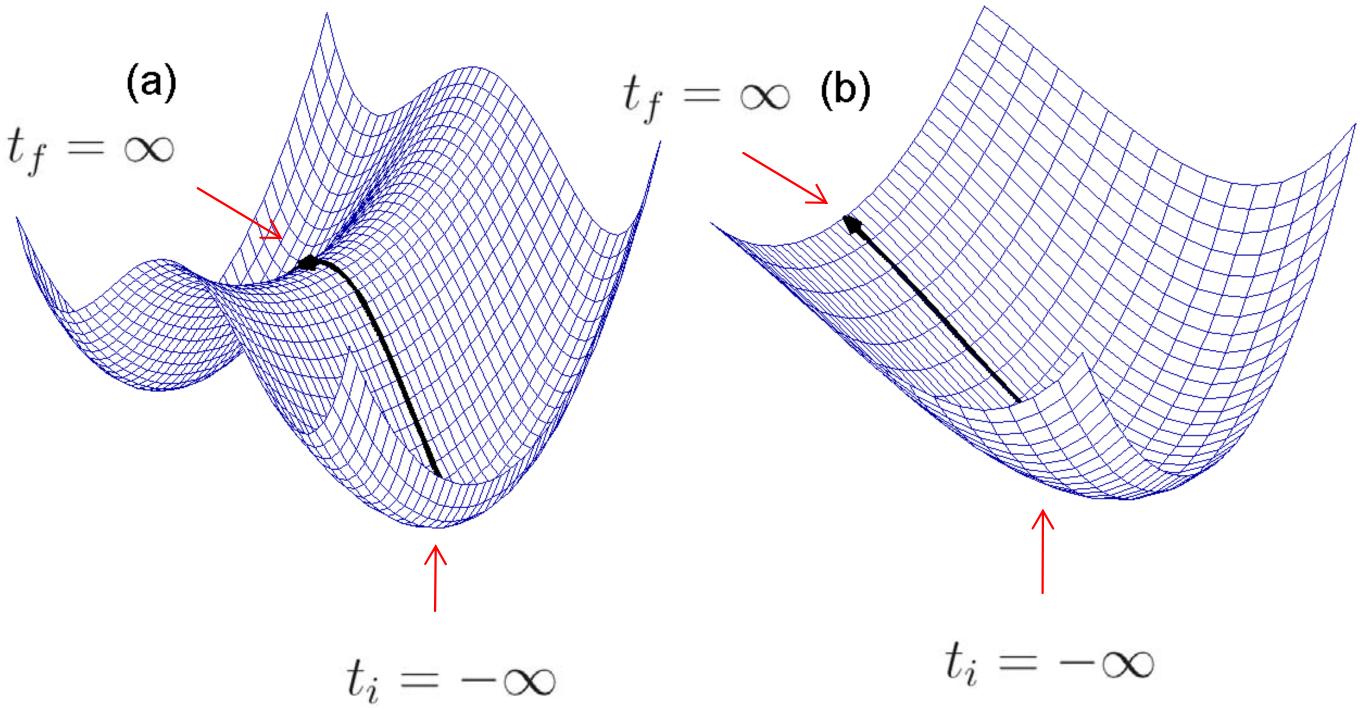
Minimizing $$\mathcal{R}[u,p,x] = \frac{1}{2}\int u^2 dt + \int \lambda(\dot{x} - K(x) - u)\, dt$$

$$\left.\begin{aligned} p &= u \\ \dot{x} &= K(x) + \lambda \\ \dot{\lambda} &= -\lambda K(x)' \end{aligned}\right\} \quad H(x,\lambda) = K\lambda + \frac{1}{2}\lambda^2$$

Activation barrier $$S[u^*] = \frac{1}{2}\int (\dot{x} - K(x))^2\, dt$$

Optimal fluctuation path. Boundary conditions

Switching **Extinction**



The probability of the *rare event* – following the optimal fluctuation path – defines the rate of switching or extinction of the system.

Example. Fluctuational disease extinction

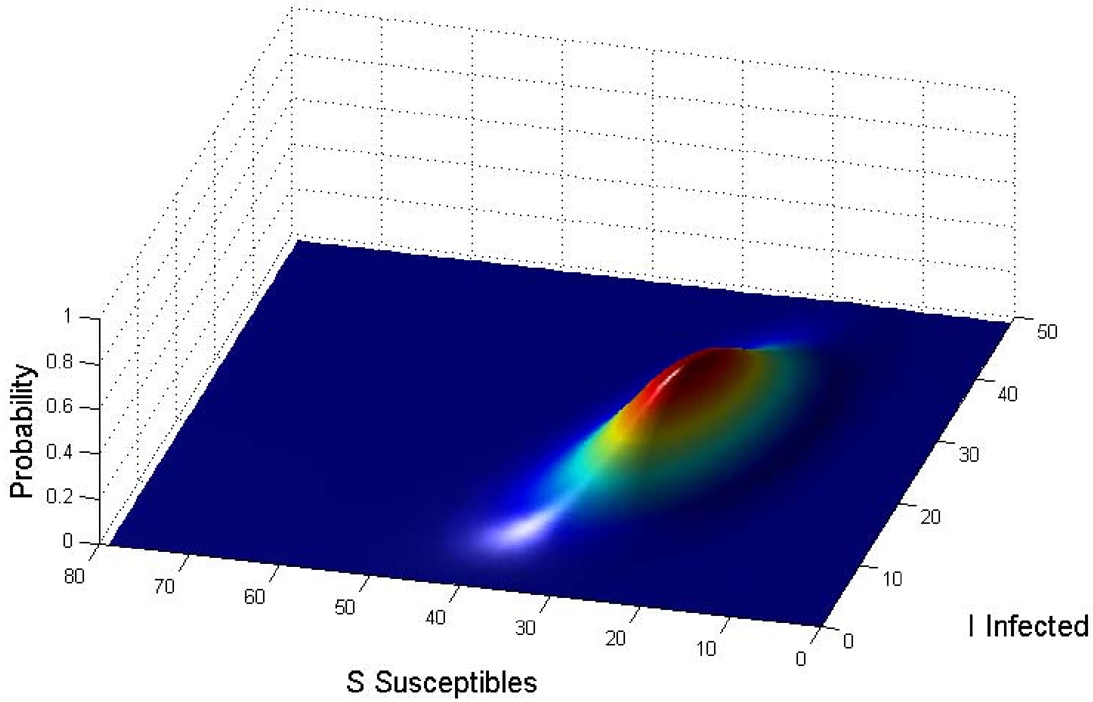
Endemic state $\mathbf{X}_{end}$ = (**I**, **S**, ...) – metastable

State of disease extinction $\mathbf{X}_{ext}$ = (**I=0**, **S**, ...) – stable

The population evolves from the endemic state to the state of disease extinction at **the** rate **W** exponentially small in the size of the system

$$W \propto e^{-\Omega S[x^*]}$$

$$\Omega = N$$



Intermediate summary I

Finding the probability of a rare event driven by weak noise can be formulated as an optimal control problem:

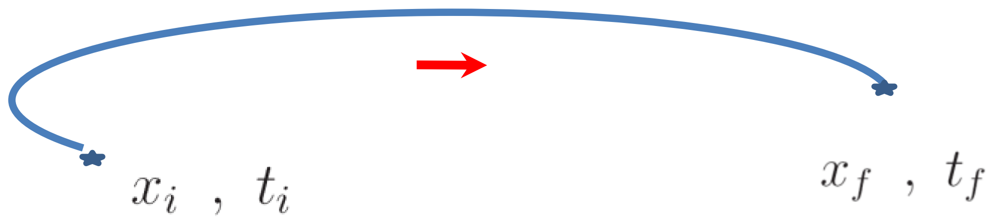
Control field – the noise

Cost functional – the probability functional of the noise

Optimal control of rare events

$$\dot{x} = K(x, u, \xi)$$

noise **control field**



Cost functional

$$P[u] = e^{-\Omega S[u]}$$

Double optimization: **Minimization of P[u] with respect to noise u**

Optimization of P[u] with respect to control field ξ

V.N. Smelyanskiy and M.I. Dykman, Phys. Rev. E 55, 2516 (1997).
B. E. Vugmeister and H. Rabitz, Phys. Rev. E 55, 2522 (1997).

Optimal control of rare events

No constraint on the control field:

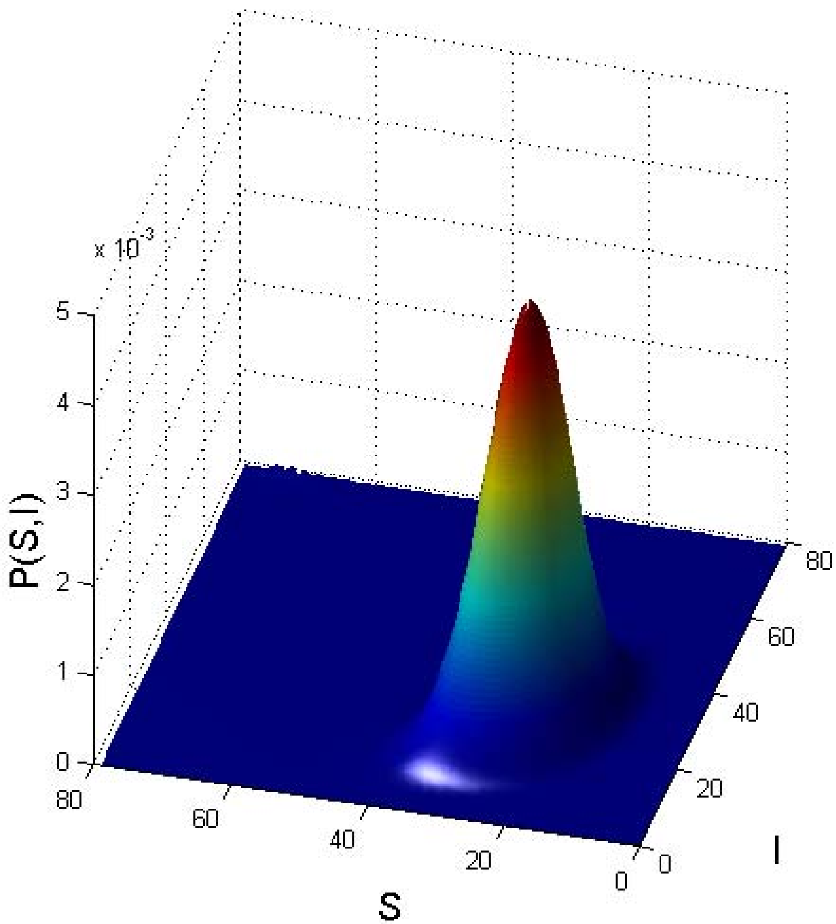
Optimal solution – to ignore the weak noise and bring the system to the target with probability 1

Weak control field (optimal fluctuation path changes slightly)

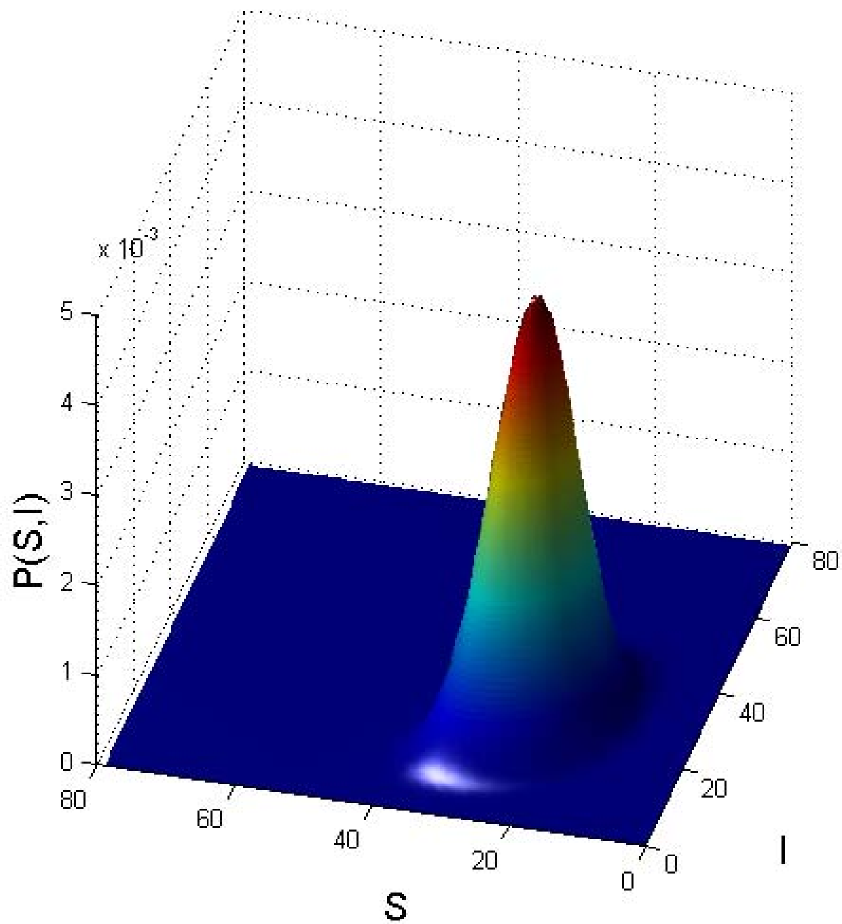
Switching/extinction is still a rare event. It is necessary to cooperate with the optimal fluctuation.

Optimal control of rare events. Numerical example

No vaccination



Vaccination



Intermediate summary II

Optimal control of rare events:

Double optimization – with respect to both noise and control field

Weak control field - cooperation with the fluctuation

Weak control field leads to an exponentially strong response in the inverse strength of the noise Ω

Part II. Controlling rare events with weak periodic control field.

Constraints on the control field:

1. Weak field (interesting)
2. Periodic field (due to the boundary conditions)
3. Some additional constraints (see below)

Problem: to minimize/maximize the probability of a rare event (switching or extinction)

Solution is universal!

Weak control field. Linear response.

Probability of the rare event

$$P[\xi] = e^{-\Omega S[u^*,\xi]}$$

Linear response:

$$S[\xi] = \int_{-\infty}^{\infty} L(\dot{x}^*, x^*, \xi)\ dt = S^{(0)} + S^{(1)}[\xi]$$

$$S^{(1)}[\xi] \ = \ \int_{-\infty}^{\infty} \frac{\partial L(\dot{x}, x, \xi)}{\partial \xi}|_{x=x^{*(0)},\xi=0}\ \xi(t)\ dt$$

$$\equiv \ \int_{-\infty}^{\infty} \chi(t)\xi(t)\ dt$$

Logarithmic susceptibility to the control field

Weak control field. Linear response.

Probability of the rare event

$$P[\xi] = e^{-\Omega S[u^*,\xi]}$$

Linear response:

$$S[\xi] = \int_{-\infty}^{\infty} L(\dot{x}^*, x^*, \xi)\ dt = S^{(0)} + S^{(1)}[\xi]$$

$$S^{(1)}[\xi] \ = \int_{-\infty}^{\infty} \frac{\partial L(\dot{x}, x, \xi)}{\partial \xi}|_{x=x^{*(0)},\xi=0}\ \xi(t)\ dt$$

$$\equiv \int_{-\infty}^{\infty} \chi(t)\xi(t)\ dt$$

Exponentially strong effect:

$$P[\xi] = P^{(0)} e^{-\Omega S^{(1)}[\xi]}$$

$\Omega^{-1} \ll 1$ **weak noise**

Synchronization of the optimal fluctuation path with the control field.

$$S^{(1)}[\xi] = \min_{-\infty<t_0<\infty} \underbrace{\int_{-\infty}^{\infty} \chi(t-t_0)\xi(t)\, dt}_{S_{t_0}^{(1)}[\xi]}$$

Synchronization of the optimal fluctuation path is a consequence of the double optimization principle

Minimizing the probability of a rare event with weak periodic control field.

Periodic control field: $$\xi(t) = \sum_{n=-\infty}^{\infty} \xi_n e^{\frac{i2\pi nt}{T}}$$

$$\begin{aligned} S^{(1)}[\xi] &= \min_{-\infty<t_0<\infty} \int_{-\infty}^{\infty} \chi(t-t_0)\xi(t)\, dt \\ &= \min_{-\infty<t_0<\infty} \left\{ 2\pi\xi_0\tilde{\chi}(0) + 2\pi \sum_{n\neq 0} \xi_n e^{i\frac{2\pi n t_0}{T}} \tilde{\chi}\left(\frac{2\pi n}{T}\right) \right\} \end{aligned}$$

$$\max_{\xi} S^{(1)}[\xi] = S^{(1)}[\xi_0]$$

Constant field is the optimal solution!

Maximizing the probability of a rare event with weak periodic control field.

Additional constraints $$\frac{1}{T}\int_0^T \xi(t)dt_0 = \nu \qquad \xi(t) \geq 0$$

Motivation:

$$\dot{\rho}(\mathbf{X}) = \sum_r [W(\mathbf{X}-\mathbf{r};\mathbf{r})\rho(\mathbf{X}-\mathbf{r}) - W(\mathbf{X};\mathbf{r})\rho(\mathbf{X})]$$

Transition probability $$W(\mathbf{X};\mathbf{r}): \quad \mathbf{X} \to \mathbf{X}+\mathbf{r}$$

Inflow rate $$W(\mathbf{X};\mathbf{r}') = \mu$$

Maximizing the probability of a rare event with weak periodic control field.

Additional constraints $$\frac{1}{T}\int_0^T \xi(t)dt_0 = \nu \qquad \xi(t) \geq 0$$

Motivation:

$$\dot{\rho}(\mathbf{X}) = \sum_r [W(\mathbf{X}-\mathbf{r};\mathbf{r})\rho(\mathbf{X}-\mathbf{r}) - W(\mathbf{X};\mathbf{r})\rho(\mathbf{X})]$$

Transition probability $$W(\mathbf{X};\mathbf{r}): \quad \mathbf{X} \to \mathbf{X}+\mathbf{r}$$

Inflow rate $$W(\mathbf{X};\mathbf{r}') = \mu \pm \xi(t)$$

Maximizing the probability of a rare event with weak periodic control field.

$$S_{opt}^{(1)} = \min_{\substack{-\infty < t_0 < \infty \\ 0 \le \xi(t) \\ \xi(t+T) = \xi(t)}} \left\{ S_{t_0}^{(1)}[\xi] + \lambda \left(\frac{1}{T} \int_0^T \xi(t)dt - \nu \right) \right\}$$

$$S_{t_0}^{(1)}[\xi] = \int_{-\infty}^{\infty} \chi(t - t_0)\xi(t) \; dt = \int_0^T f(t - t_0) \; \xi(t)dt$$

$$f(t) \equiv \sum_{n=-\infty}^{\infty} \chi(t + nT)$$

Maximizing the probability of a rare event with weak periodic control field.

$$\begin{aligned}
\int_0^T f(t-t_0)\;\xi(t)dt \;&\leq\; \int_0^T \xi(t)dt \min_{0\leq t\leq T} f(t-t_0) \\
&=\; \nu T f(\tau - t_0) \\
&=\; \int_0^T f(t-t_0)\;\nu T \delta[t-\tau]dt
\end{aligned}$$

Maximizing the probability of a rare event with weak periodic control field.

$$
\begin{aligned}
\int_0^T f(t-t_0)\,\xi(t)dt \;&\le\; \int_0^T \xi(t)dt \min_{0\le t\le T} f(t-t_0) \\
&= \nu T f(\tau - t_0) \\
&= \int_0^T f(t-t_0)\,\nu T\delta[t-\tau]dt
\end{aligned}
$$

$$
\boxed{\xi(t)_{opt} = \nu T \sum_n \delta(t - nT)}
$$

Maximizing the probability of a rare event with weak periodic control field.

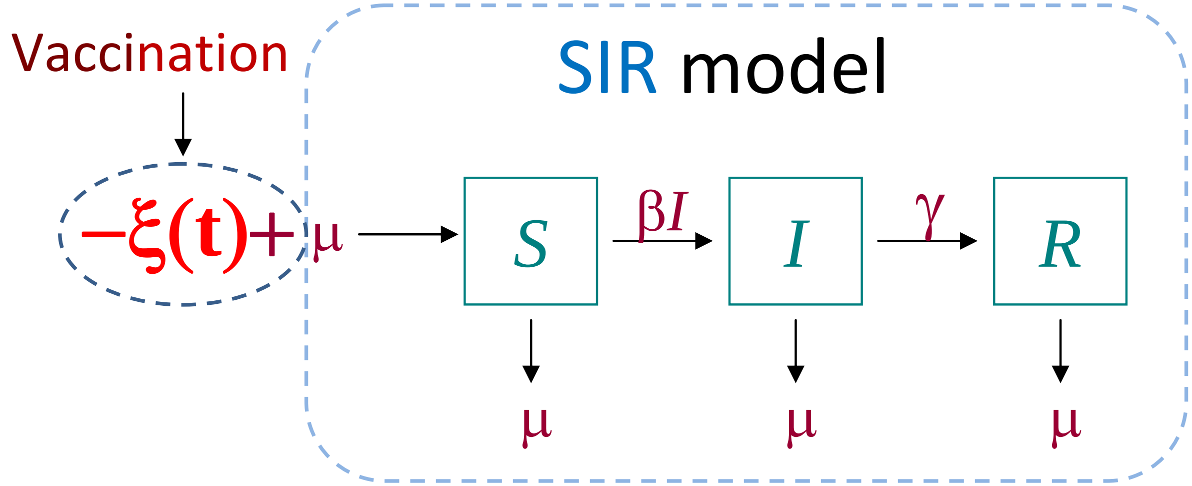
$$S^{(1)}_{opt} = \nu T \min_{-\infty \le t_0 \le \infty} \sum_{n=-\infty}^{\infty} \chi(nT - t_0)$$

$$= \nu \min_{-\infty \le t_0 \le \infty} \left\{ \sum_{n=-\infty}^{\infty} \tilde{\chi}\left(\frac{2\pi n}{T}\right) e^{i\frac{2\pi n}{T} t_0} \right\}$$

T-dependence. Limiting cases:

- $T \ll \tau_{rel}$ $\quad S^{(1)} = -\nu\tilde{\chi}(0)$
- $T \gg \tau_{rel}$ $\quad S^{(1)} = \nu T \min_{-\infty < t < \infty} \chi(t)$
- $T \sim \tau_{rel}$ $\quad$ Resonances on the subharmonics of the characteristic frequency : $\omega^* = 2\pi n / T$

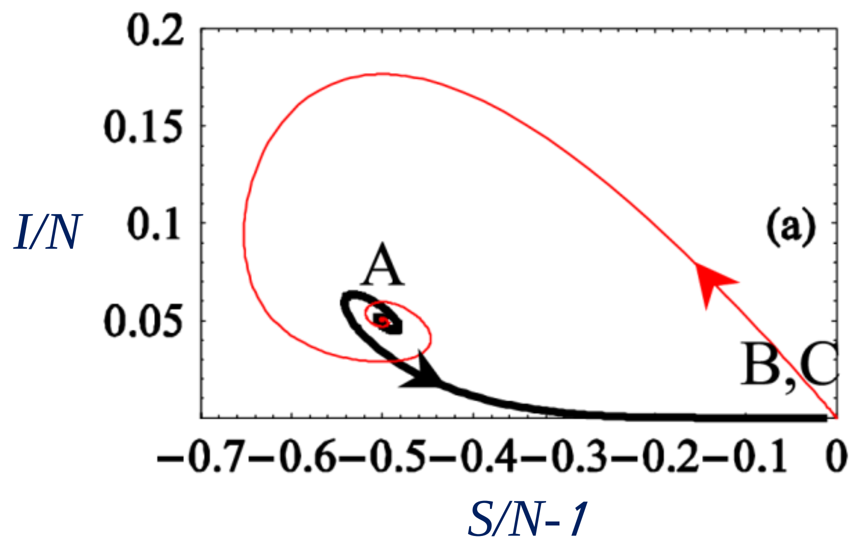
Application: optimal vaccination in SIR model



Oscillatory behavior:

- Epidemic outburst

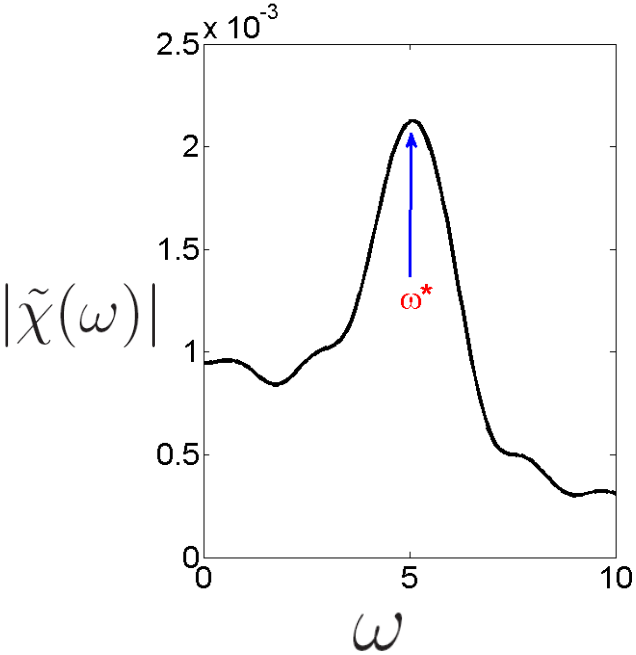
-Optimal path to disease extinction



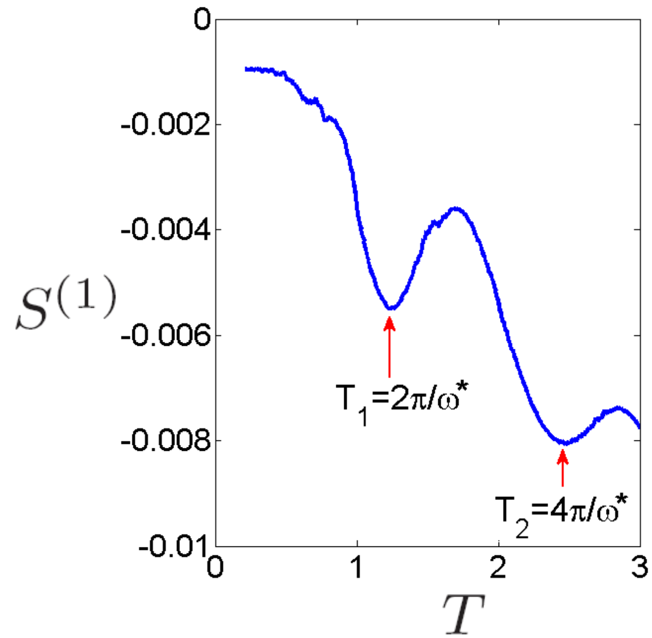
A. Kamenev and B. Meerson, PRE (2008)

Application: optimal vaccination in SIR model

Fourier spectrum of the log susceptibility $|\tilde{\chi}(\omega)|$



Decrease of the entropic barrier $S^{(1)}$ as a function of the vaccination period T



Exponential increase of the extinction rate:

$$W_{extinction} = W^{(0)}_{extinction} e^{N|S^{(1)}|}$$

Practical advice: tune vaccination in resonance for exponential gain!

Summary

- *Finding the probability* of a rare event is formulated as an optimal control problem
- *Controlling rare events* is exponentially efficient
- Optimal weak periodic control of a rare event (switching or extinction) is model-independent.
- Switching or extinction rate is *minimized* by a constant control field.
- Switching or extinction rate is *maximized* by a sequence of δ-like pulses.