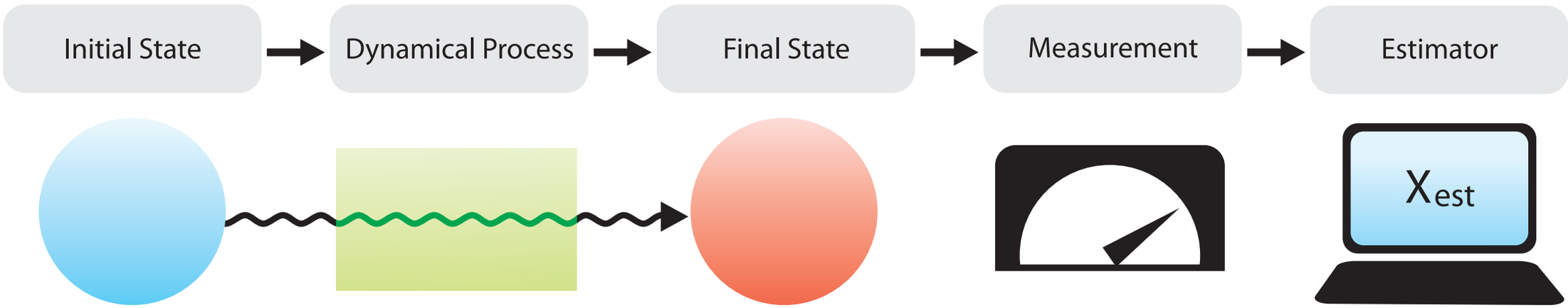
Quantum metrology of open dynamical systems: Precision limits through environment control

Application to optical interferometry and quantum speed limit

Luiz Davidovich
Instituto de Física - Universidade Federal do Rio de Janeiro

Based on work with Bruno M. Escher, Camille L. Latune, Marcio M.Taddei, Nicim Zagury, and Ruynet L. de Matos Filho

Parameter estimation in classical and quantum physics



1. Prepare probe in suitable initial state
2. Send probe through process to be investigated
3. Choose suitable measurement
4. Associate each experimental result *j* with estimation

$$\delta X \equiv \sqrt{\langle [X_{\text{est}}(j) - X]^2 \rangle_j \Big|_{X = X_{\text{true}}}} \rightarrow \text{Merit quantifier}$$

$$\langle X_{\text{est}} \rangle = X_{\text{true}}, \; d\langle X_{\text{est}} \rangle / dX = 1 \rightarrow \text{Unbiased estimator}$$

Cramér, Rao, and Fisher

H. Cramér

C. R. Rao

R. A. Fisher

Fisher information

Cramér-Rao bound for unbiased estimators:

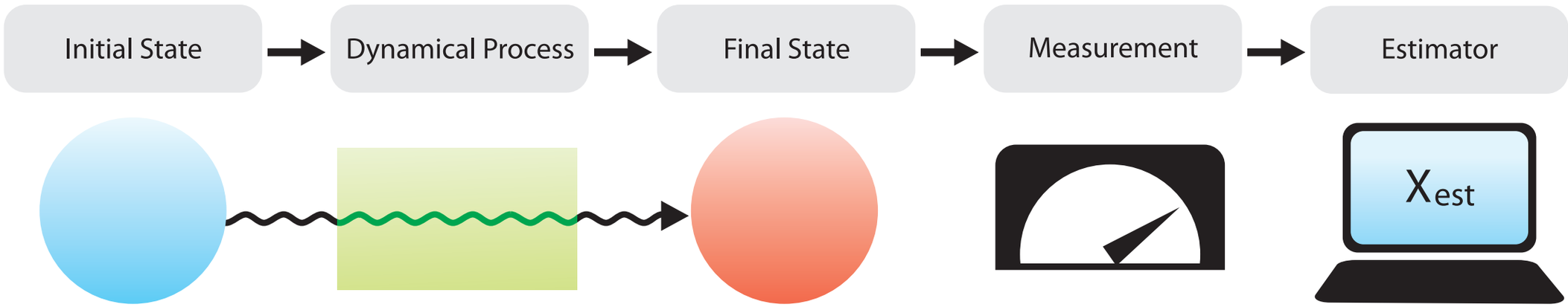
$$\delta X \geq 1/\sqrt{\nu F(X_{\text{true}})}, \quad F(X) \equiv \int d\xi \; p(\xi \mid X) \left(\frac{d \ln\left[p(\xi \mid X) \right]}{dX} \right)^2$$

$\nu \rightarrow$ Number of repetitions of the experiment

$p(\xi \mid X) \rightarrow$ probability density of getting an experimental result ξ

Fisher's theorem: Inequality can be saturated (i.e., it is possible to make it an equality) when $\nu \rightarrow \infty$, by choosing an appropriate estimator X_{est}.

Quantum parameter estimation

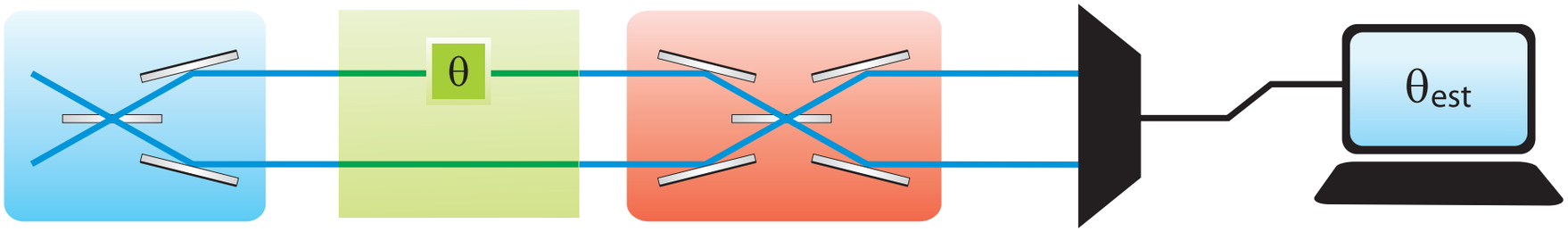


NEW POINTS TO CONSIDER

1. Precision in determination of parameter depends on the distinguishability between quantum states corresponding to nearby values of the parameter.
2. Measurement matters!
3. Is it possible to get better precision (for the same amount of resources) by using special quantum states?

RESOURCES: Number of atoms in atomic spectroscopy, number of photons in optical interferometry, average energy of a harmonic oscillator...

Example: Optical interferometry



Standard limit: $\delta\theta \approx \frac{1}{\sqrt{\langle n \rangle}}$

(Ignoring repetitions of the experiment)

$$\left|\langle \alpha | \alpha e^{i\delta\theta} \rangle\right|^2 = \exp\left(-\left|\alpha\left(1-e^{i\delta\theta}\right)\right|^2\right)$$
$$\approx \exp\left[-\langle n \rangle (\delta\theta)^2\right] \Rightarrow \delta\theta \approx 1/\sqrt{\langle n \rangle}$$

Possible method to increase precision for the same average number of photons: Use NOON states [D. Wineland et al., J. P. Dowling]

$$|\psi(N)\rangle = \left(|N,0\rangle + |0,N\rangle\right)/\sqrt{2} \rightarrow |\psi(N,\theta)\rangle = \left(|N,0\rangle + e^{iN\theta}|0,N\rangle\right)/\sqrt{2}, \quad \left(\langle n \rangle = N\right)$$

$$\left|\langle \psi(N) | \psi(N,\delta\theta) \rangle\right|^2 = \cos^2\left(N\delta\theta/2\right) \Rightarrow \delta\theta \approx 1/N$$

Heisenberg limit

Precision is better, for the same amount of resources (average number of photons)!

Quantum Fisher Information

$$F\left(X;\{\hat{E}_\xi\}\right) \equiv \int d\xi\, p(\xi \mid X)\left(\frac{d\,\ln\left[p(\xi \mid X)\right]}{dX}\right)^2$$

$$p(\xi \mid X) = \mathrm{Tr}\left[\hat{\rho}(X)\hat{E}_\xi\right]$$

$$\int d\xi \hat{E}_\xi = \hat{1}$$

POVM

This corresponds to a given quantum measurement. Ultimate lower bound for $\langle(\Delta X_{\mathrm{est}})^2\rangle$: optimize over all quantum measurements so that

$$\mathcal{F}_Q(X) = \max_{\{E_\xi\}} F\left(X;\{E_\xi\}\right)$$

Quantum Fisher Information

$$\delta X \equiv \sqrt{\langle(\Delta X_{\mathrm{est}})^2\rangle} \geq 1/\sqrt{\nu \mathcal{F}_Q(X)}$$

Ultimate precision limit

Asymptotically attainable when $\nu \to \infty$

Bures' Fidelity: $\Phi\left(\hat{\rho}_1,\hat{\rho}_2\right) \equiv \left(\mathrm{Tr}\sqrt{\hat{\rho}_1^{1/2}\hat{\rho}_2\hat{\rho}_1^{1/2}}\right)^2$

$$\Rightarrow \Phi\left[\hat{\rho}(X_{\mathrm{true}}),\hat{\rho}(X)\right] = 1-(\delta X/2)^2\mathcal{F}_Q\left[\hat{\rho}(X_{\mathrm{true}})\right]+O\left[(\delta X)^4\right]$$

Related to distance between states!

Quantum Fisher information for pure states

Initial state of the probe: $|\psi(0)\rangle$

Final X-dependent state: $|\psi(X)\rangle = \hat{U}(X)|\psi(0)\rangle$, $\hat{U}(X)$ unitary operator.

Then (Helstrom 1976):

$$\mathcal{F}_Q(X) = 4\langle(\Delta\hat{H})^2\rangle_0\,, \quad \langle(\Delta\hat{H})^2\rangle_0 \equiv \langle\psi(0)|\left[\hat{H}(X) - \langle\hat{H}(X)\rangle_0\right]^2|\psi(0)\rangle$$

where

$$\hat{H}(X) \equiv i\frac{d\hat{U}^\dagger(X)}{dX}\hat{U}(X)$$

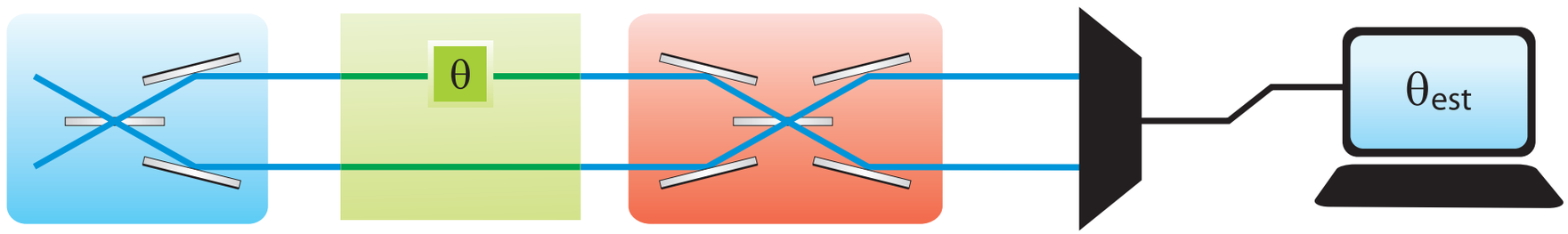
Proper framework to discuss estimation of quantities like elapsed time or harmonic oscillator phase

If $\hat{U}(X) = \exp(i\hat{O}X)$, $\hat{O}$ independent of X, then $\hat{H} = \hat{O}$

$$\delta X \geq 1/2\sqrt{\nu\left\langle\Delta\hat{H}^2\right\rangle}$$

$\Rightarrow$ Should maximize the variance to get better precision!

Optical interferometry



$\hat{n} = \hat{a}^\dagger a \rightarrow$ Generator of phase displacements

$\Rightarrow \mathcal{F}_Q(\theta) = 4\langle(\Delta\hat{n})^2\rangle_0$ where $\langle(\Delta\hat{n})^2\rangle_0$ is the photon-number variance in the upper arm.

Standard limit: coherent states $\mathcal{F}_Q(\theta) = 4\langle(\Delta\hat{n})^2\rangle_0 = 4\langle\hat{n}\rangle \Rightarrow \delta\theta \geq \dfrac{1}{2\sqrt{\langle n\rangle}}$

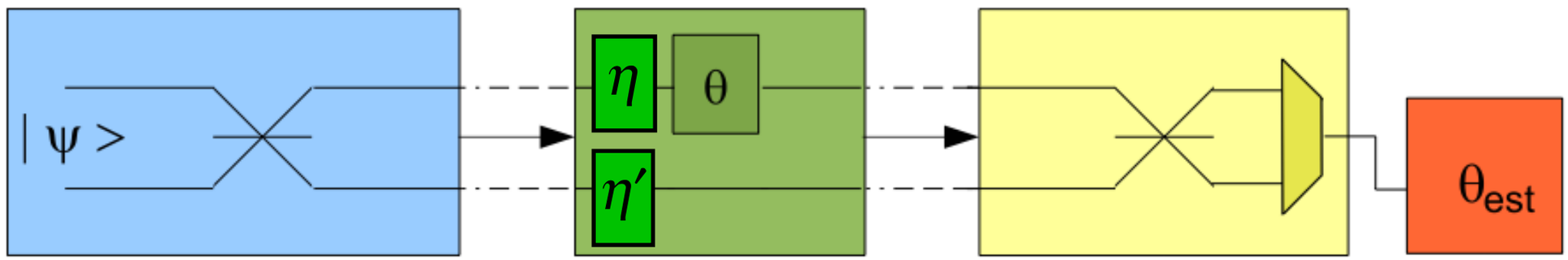
Increasing the precision: maximize variance with NOON states:

$$|\psi(N)\rangle = \left(|N,0\rangle + |0,N\rangle\right)/\sqrt{2} \rightarrow |\psi(N,\theta)\rangle = \left(|N,0\rangle + e^{iN\theta}|0,N\rangle\right)/\sqrt{2}$$

$$\left\langle(\Delta\hat{n})^2\right\rangle_0 = \frac{N^2}{4} \Rightarrow \delta\theta \geq \frac{1}{N}$$

Precision is better, for the same amount of resources.

Parameter estimation with decoherence



Loss of a single photon transforms NOON state into a separable state!

$$|\psi(N)\rangle = \frac{|N,0\rangle + |0,N\rangle}{\sqrt{2}} \rightarrow |N-1,0\rangle \text{ or } |0,N-1\rangle$$

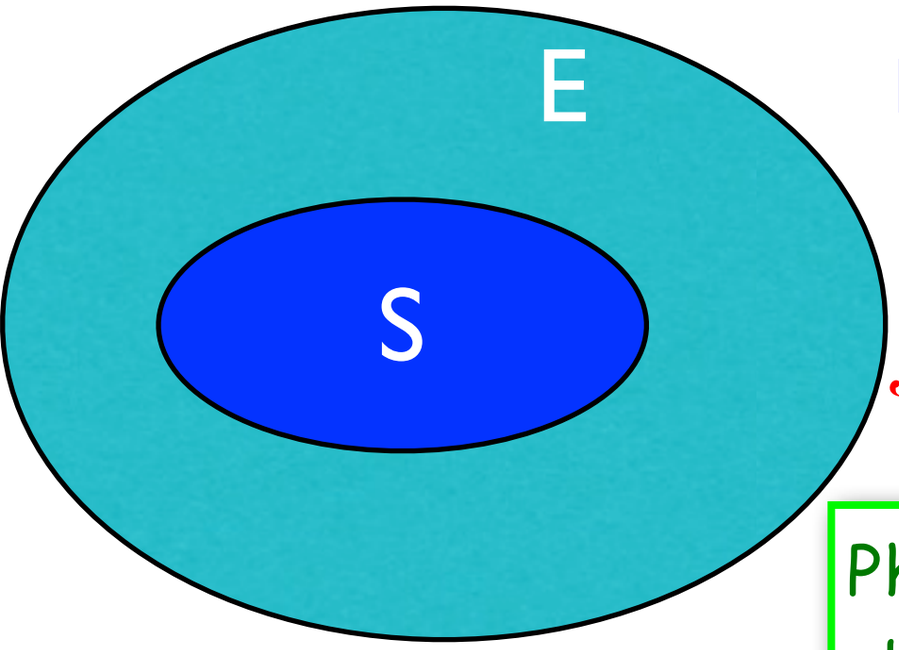
No simple analytical expression for Fisher information!

For small N, more robust states can be numerically calculated. Large N?

Parameter estimation in open systems: Extended space approach

B. M. Escher, R. L. Matos Filho, and L. D., **Nature Physics** 7, 406 (2011); **Braz. J. Phys.** 41, 229 (2011)

Given initial state and non-unitary evolution, define in S+E



$$|\Phi_{S,E}(x)\rangle = \hat{U}_{S,E}(x)|\psi\rangle_S|0\rangle_E$$ (Purification)

Then

$$\mathscr{F}_Q \equiv \max_{\hat{E}_j^{(S)}\otimes\hat{1}} F\left(\hat{E}_j^{(S)}\otimes\hat{1}\right) \leq \max_{\hat{E}_j^{(S,E)}} F\left(\hat{E}_j^{(S,E)}\right) \equiv \mathscr{C}_Q$$

Physical meaning of this bound: information obtained about parameter when S+E is monitored

Least upper bound: Minimization over all unitary evolutions in S+E - difficult problem

Bound is attainable - there is always a purification such that $\mathscr{C}_Q = \mathscr{F}_Q$

Then, monitoring S+E yields same information as monitoring S

Minimization procedure

PRL **109,** 190404 (2012) PHYSICAL REVIEW LETTERS week ending 9 NOVEMBER 2012

Quantum Metrological Limits via a Variational Approach

B. M. Escher,* L. Davidovich, N. Zagury, and R. L. de Matos Filho
Instituto de Física, Universidade Federal do Rio de Janeiro, 21.941-972, Rio de Janeiro (RJ) Brazil
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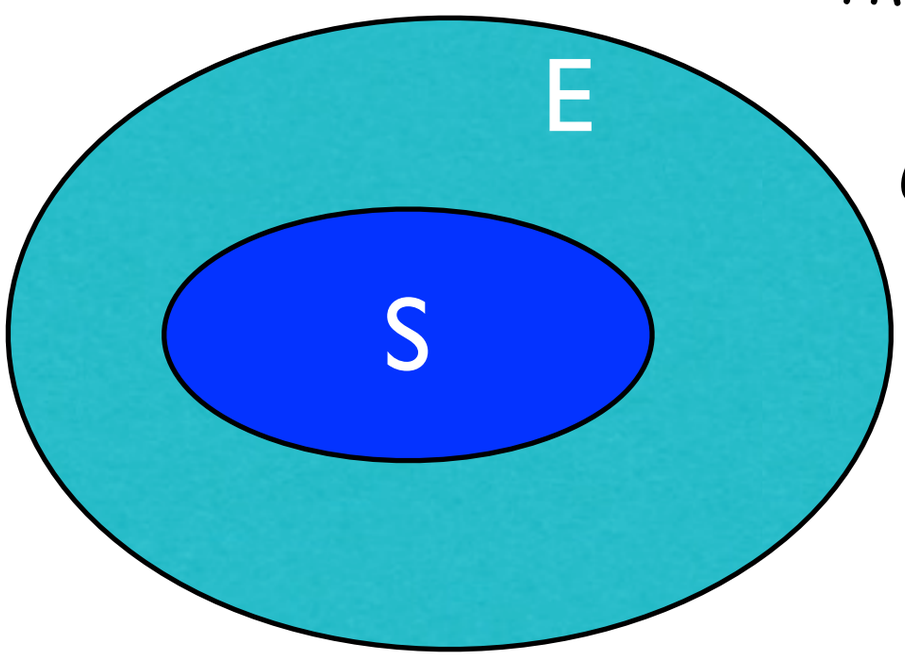


There is always an unitary operator acting only on E that connects two different purifications of ρ_S

Given $|\Phi_{S,E}(x)\rangle = \hat{U}_{S,E}(x)|\psi\rangle_S|0\rangle_E$,

$$i\frac{d|\Phi_{S,E}(x)\rangle}{dx} = \hat{H}_{S,E}(x)|\Phi_{S,E}(x)\rangle,$$

then any other purification can be written as:

$$|\Psi_{S,E}(x)\rangle = u_E(x)|\Phi_{S,E}(x)\rangle$$

"Control"

Define $\hat{h}_E(x) = i\frac{d\hat{u}_E^\dagger(x)}{dx}\hat{u}_E(x)$

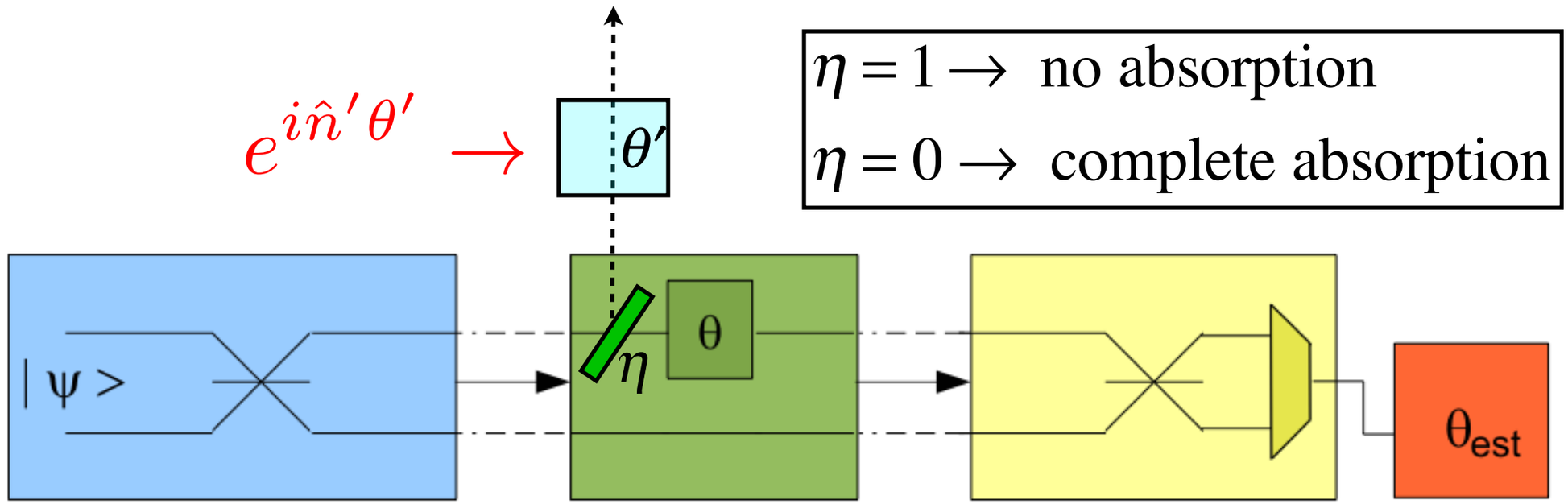
Minimize now C_Q over all Hermitian operators $h_E(x)$ that act on E

Quantum limits for lossy optical interferometry

$e^{i\hat{n}'\theta'} \rightarrow$

$\eta = 1 \rightarrow$ no absorption

$\eta = 0 \rightarrow$ complete absorption



States with well-defined total photon number: $|\psi_0\rangle = \sum_{n=0}^{N} \beta_n |n, N-n\rangle$

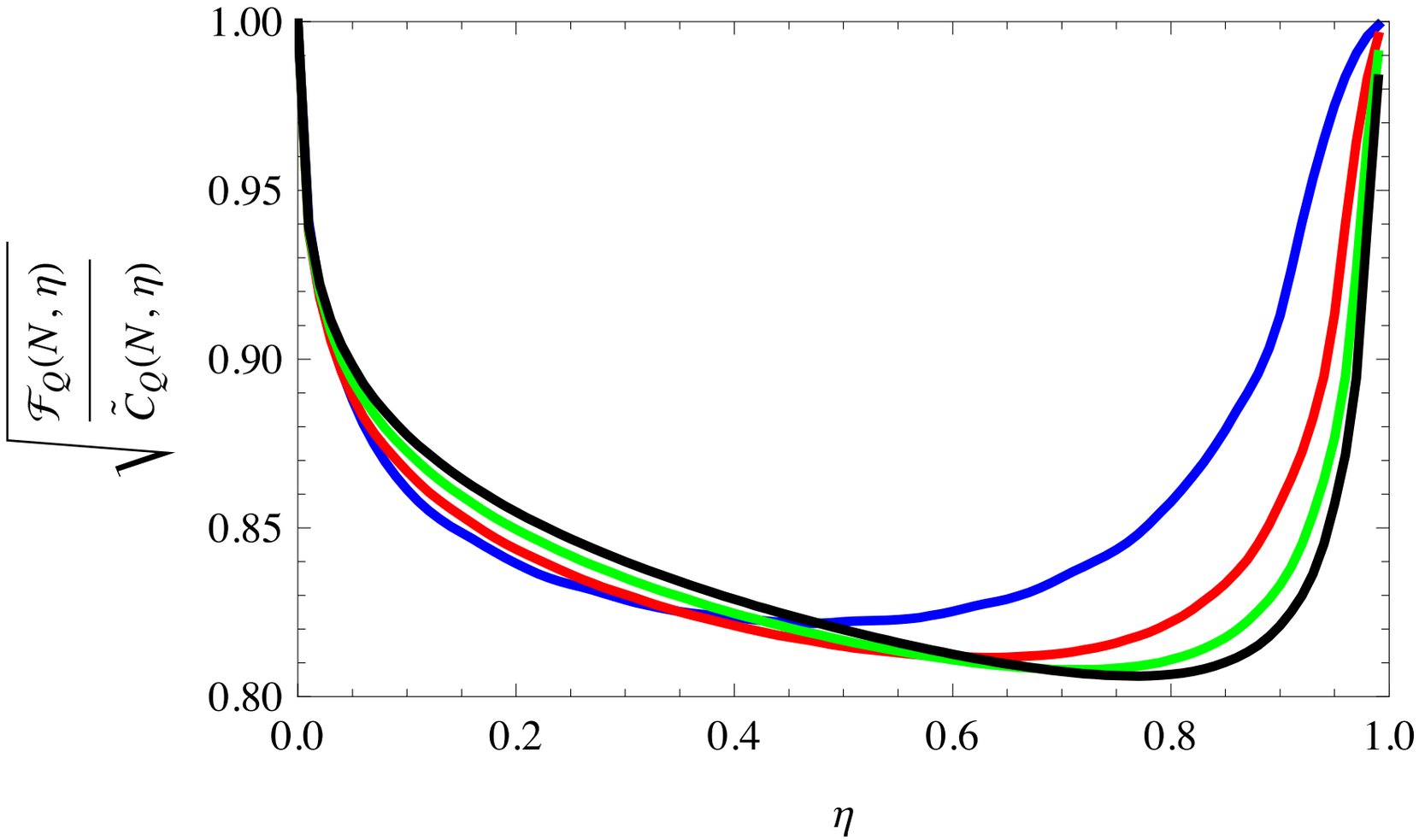
$$2\delta\theta \geq \left[1 + \sqrt{1 + \frac{1-\eta}{\eta} N}\right] / N$$

$$N \ll \frac{\eta}{1-\eta} \Rightarrow \sqrt{\nu}\,\delta\theta \geq 1/N \rightarrow$$ Heisenberg limit

$$N \gg \frac{\eta}{1-\eta} \Rightarrow \delta\theta \geq \frac{\sqrt{1-\eta}}{2\sqrt{\nu\eta N}}$$

For N sufficiently large, $1/\sqrt{N}$ behavior is always reached!

How good is this bound?



Comparison between numerical maximum value of $\mathscr{F}_Q$ and upper bound $\mathscr{C}_Q$ as a function of η, for $N = 10$ (blue), $N = 20$ (red), $N = 30$ (green), and $N = 40$ (black).

QUANTUM SPEED LIMIT

THE UNCERTAINTY RELATION BETWEEN ENERGY AND TIME IN NON-RELATIVISTIC QUANTUM MECHANICS

By L. MANDELSTAM * and Ig. TAMM

Lebedev Physical Institute, Academy of Sciences of the USSR

(Received February 22, 1945)

A uncertainty relation between energy and time having a simple physical meaning is rigorously deduced from the principles of quantum mechanics. Some examples of its application are discussed.

1. Along with the uncertainty relation between coordinate q and momentum p one considers in quantum mechanics also the uncertainty relation between energy and time.

The former relation in the form of the inequality

$$\Delta q \cdot \Delta p \geqslant \frac{h}{2}, \qquad (1)$$

An entirely different situation is met with in the case of the relation

$$\boxed{\Delta H \cdot \Delta T \sim h,} \qquad (2)$$

where ΔH is the standard of energy, ΔT — a certain time interval, and the sign $\sim$ denotes that the left-hand side is at least of the order of the right-hand one.

Leonid Mandelshtam

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Geometry of Quantum Evolution

J. Anandan [a]

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, United Kingdom

Y. Aharonov [b]

Department of Physics and Astronomy, University of South Carolina, Columbia, South Carolina 29208

Igor Tamm

Generalization to non-unitary processes? Evolved state may not become orthogonal to initial one!

Quantum speed limit for physical processes

M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, PRL **110**, 050402 (2013)

$$\arccos\sqrt{\Phi_B\left[\hat{\rho}(0),\hat{\rho}(\tau)\right]} \leq \int_0^\tau \sqrt{\mathcal{F}_Q(t)/4}\, dt$$

Bures length of geodesic
Uhlman (1992)

Bures length of actual path followed by state of the system

$\tau \rightarrow$ Minimum time for attaining fidelity $\Phi_B(0,\tau)$ between initial and final states

Attainable bound

See also A. del Campo et al., PRL **110**, 050403

Special case: Unitary evolution, time-independent Hamiltonian, orthogonal states

Mandelstam-Tamm

$$\Phi_B\left[\hat{\rho}(0),\hat{\rho}(\tau)\right] = 0, \quad \mathcal{F}_Q(t) = 4\langle(\Delta H)^2\rangle/\hbar^2 \Rightarrow \boxed{\tau\sqrt{\langle(\Delta H)^2\rangle} \geq h/4}$$

Quantum speed limit for physical processes: Purification procedure

$$\mathcal{D} := \arccos\sqrt{\Phi_B\left[\hat{\rho}(0), \hat{\rho}(\tau)\right]} \leq \int_0^\tau \sqrt{\mathcal{F}_Q(t)/4}\, dt$$

$$\Downarrow$$

$$\mathcal{D} \leq \int_0^\tau \sqrt{\mathcal{C}_Q(t)/4}\, dt = \int_0^\tau \sqrt{\langle \Delta \hat{\mathcal{H}}^2_{S,E}(t)\rangle}/\hbar\, dt.$$

$$\hat{\mathcal{H}}_{S,E}(t) := \frac{\hbar}{i}\frac{d\hat{U}^\dagger_{S,E}(t)}{dt}\hat{U}_{S,E}(t)$$

(Hamiltonian in the Heisenberg picture)

$\hat{U}_{S,E}(t)$: Evolution of the purified state corresponding to $\hat{\rho}_S$

Quantum speed limit for physical processes: Dephasing channel

Dephasing channel:

$$|0\rangle|0\rangle_E \to e^{-i\omega_0 t}\left[\sqrt{P(t)}|0\rangle|0\rangle_E + \sqrt{1-P(t)}|0\rangle|1\rangle_E\right],$$

$$|1\rangle|0\rangle_E \to e^{i\omega_0 t}\left[\sqrt{P(t)}|1\rangle|0\rangle_E - \sqrt{1-P(t)}|1\rangle|1\rangle_E\right],$$

$P(t) := (1+e^{-\gamma t})/2$ $\qquad$ $\gamma(t) \to$ Dephasing rate

Unitary evolution corresponding to the map:

$$\hat{U}_{S,E}(t) = e^{-i\omega_0 t\hat{Z}} e^{-i\theta\hat{Z}\hat{Y}^{(E)}}$$

$$\Theta(t) = \arccos\sqrt{P(t)}$$

More general unitary evolution: $\hat{\mathcal{U}}_{S,E}(t) = \hat{u}_E(t)\hat{U}_{S,E}(t)$

Minimize $\mathcal{C}_Q(t)$ over all possible evolutions $\hat{u}_E(t)$. $\mathcal{C}_Q(t)$ depends only on

$$\hat{h}_E(t) := \frac{\hbar}{i}\frac{d\hat{u}_E^\dagger(t)}{dt}\hat{u}_E(t)$$

Set $\hat{h}_E(t) = \alpha(t)\hat{X}^{(E)} + \beta(t)\hat{Y}^{(E)} + \gamma(t)\hat{Z}^{(E)}$

$\alpha(t),\ \beta(t),\ \gamma(t) \to$ Variational parameters

Quantum speed limit for physical processes: Dephasing channel

Special case: $\omega_0 = 0$

$$\mathcal{D} \leq \frac{1}{2}\sqrt{\langle \Delta \hat{Z}^2 \rangle} \arccos[\exp(-\gamma\tau/2)] \Rightarrow \gamma\tau \geq \ln \sec\left(2\mathcal{D}/\sqrt{\langle \Delta \hat{Z}^2 \rangle}\right)$$

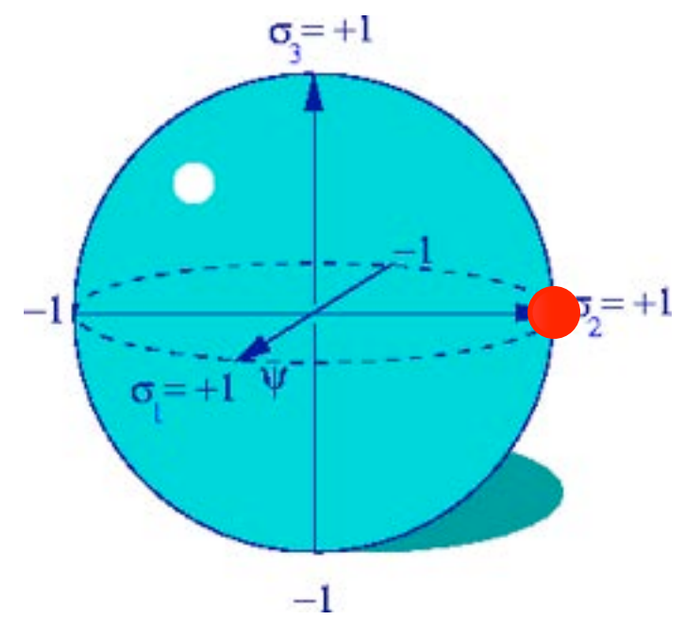
$\langle \Delta \hat{Z}^2 \rangle = 0 \Rightarrow$ Eigenstate of Z: no evolution

Maximum distance between states: $\sqrt{\langle \Delta \hat{Z}^2 \rangle}\pi/4$

Pure states with $\langle \Delta \hat{Z}^2 \rangle = 1 \Rightarrow$ Bound is saturated

Interpretation:

Evolution is along geodesic of Bloch sphere:



$$(|0\rangle + |1\rangle)/\sqrt{2} \rightarrow (|0\rangle\langle 0| + |1\rangle\langle 1|)/2$$

Quantum speed limit for physical processes: Dephasing channel

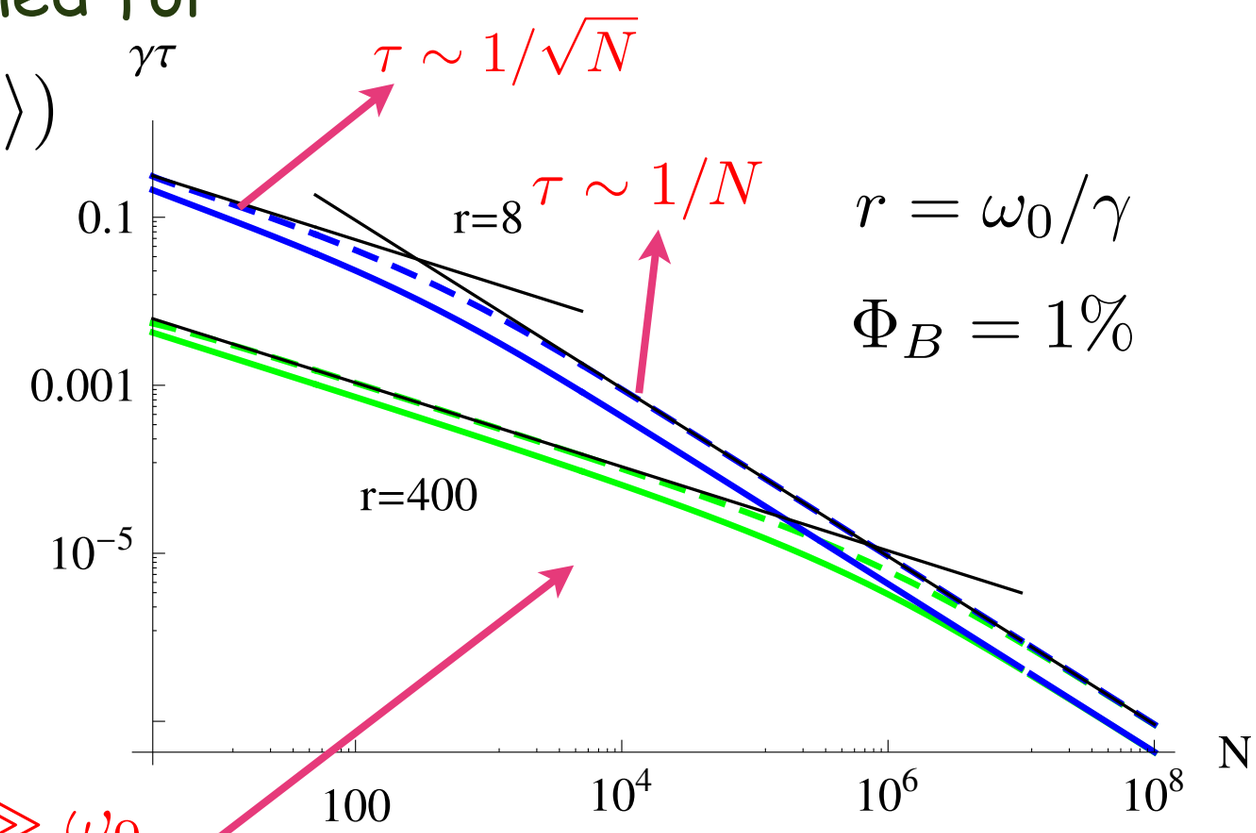
N-qubit system, each interacting with its own dephasing reservoir

Try $\hat{h}_E(t) = \sum_i [\alpha(t)\hat{X}_i^{(E)} + \beta(t)\hat{Y}_i^{(E)} + \gamma(t)\hat{Z}_i^{(E)}]$

Lower bound scales as $\tau \sim 1/N$. Attained for

GHZ states $(1/2)(|0\ldots0\rangle + e^{i\phi}|1\ldots1\rangle)$

$$\Phi_B[\hat{\rho}(0), \hat{\rho}(t)] = \frac{1 + e^{-N\gamma\tau}\cos 2N\omega_0\tau}{2}$$



Separable states:

Lower bound scales as $\tau \sim 1/\sqrt{N}$ for

$\gamma\sqrt{N} \ll \omega_0$ and as $\tau \sim 1/N$ for $\gamma\sqrt{N} \gg \omega_0$.

dashed lines

Product state, qubits initially in state

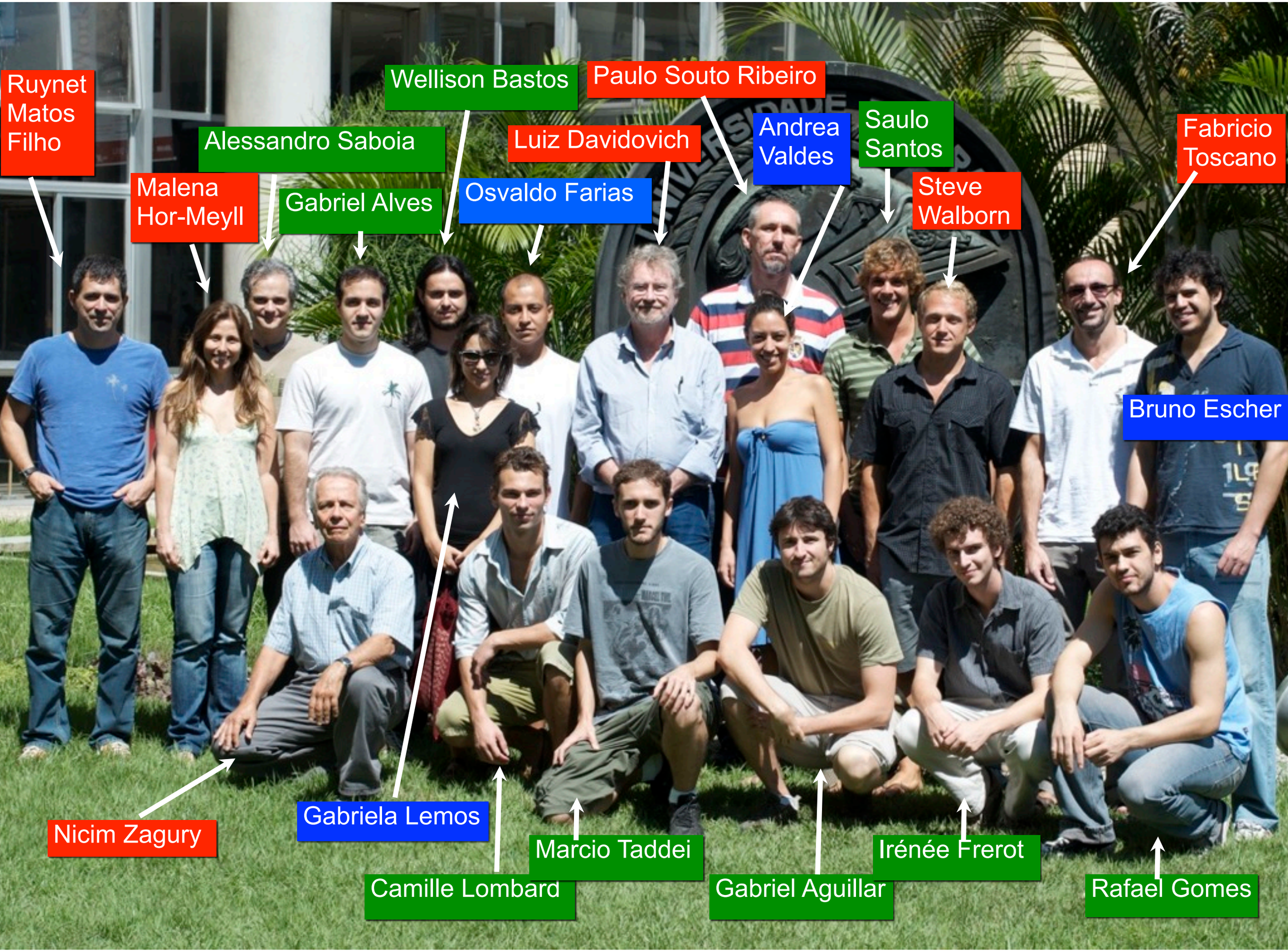
$$(|0\rangle + |1\rangle)/\sqrt{2} \Rightarrow \Phi_B = \frac{1}{2^N}\left(1 + e^{-\gamma\tau}\cos 2\omega_0\tau\right)^N$$

Summary

- General framework for estimation of parameters in noisy systems, based on expression of quantum Fisher information for purified evolution (extended space), and on "control" of environment, so as to minimize the quantum Fisher information of S+E .

- Allows analytical calculation of very good bounds on the limits of estimation.

- Bounds obtained for optical interferometry, atomic spectroscopy, minimum evolution time of open systems, and force estimation.


Ruynet Matos Filho
Wellison Bastos
Paulo Souto Ribeiro
Alessandro Saboia
Luiz Davidovich
Andrea Valdes
Saulo Santos
Fabricio Toscano
Malena Hor-Meyll
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