# Control problems in NMR spectroscopy

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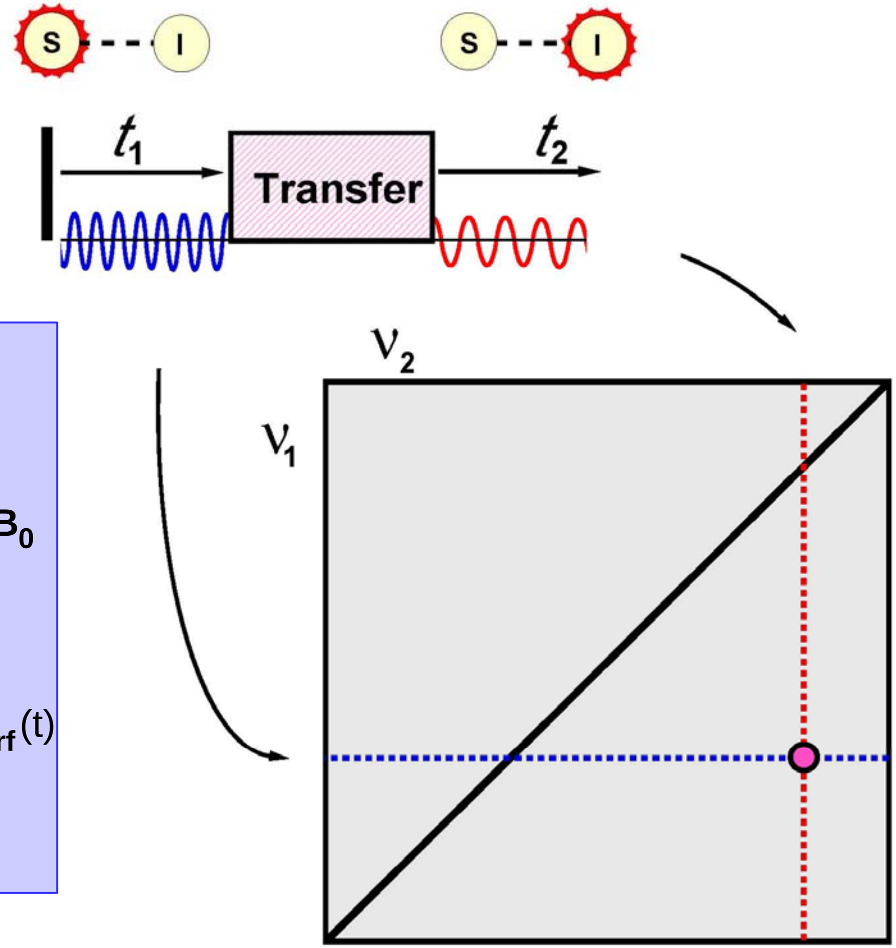
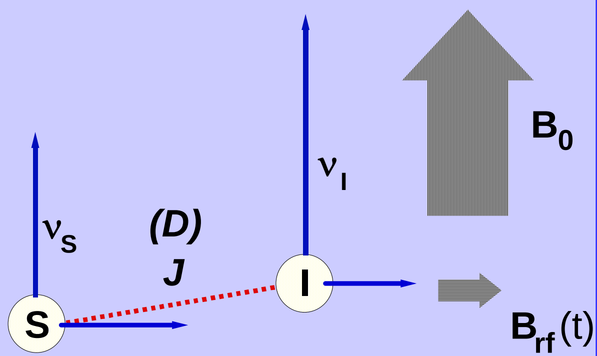
KITP Quantum Control Conf, May 2009

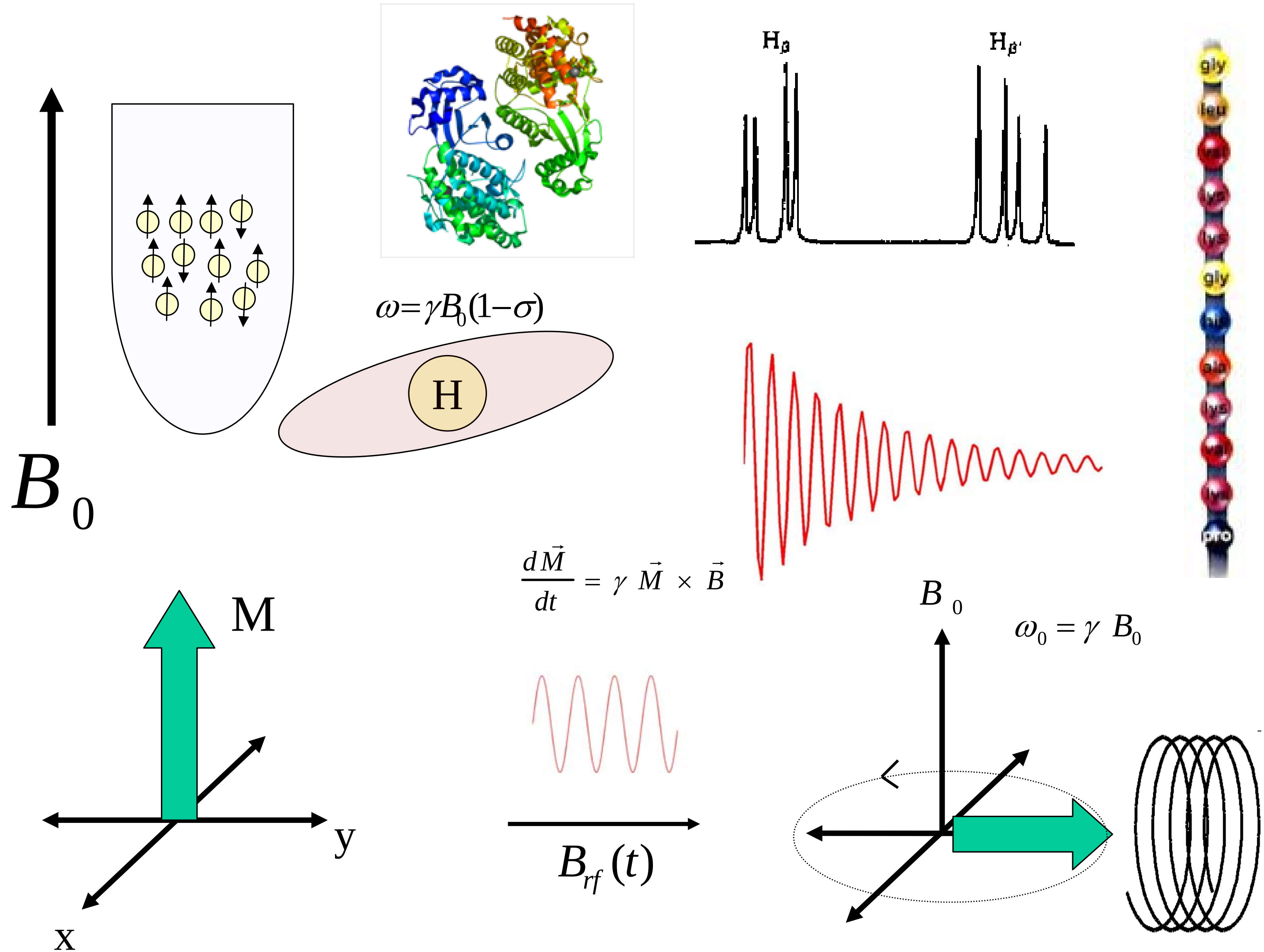
**2D NMR**




**Interactions**

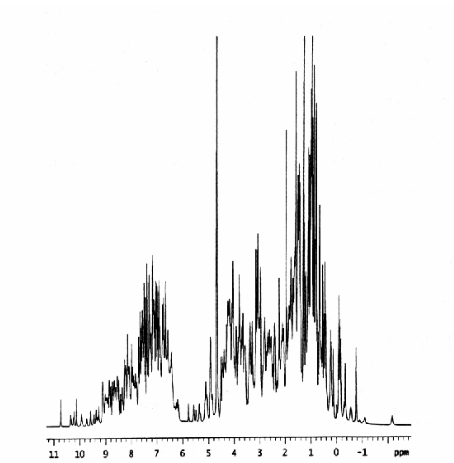



Spin Hamiltonian: $\mathbf{H_0} + \mathbf{H_{rf}}(t)$



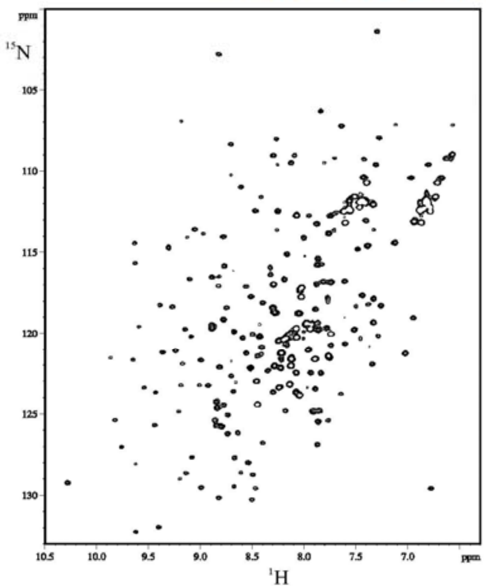
B0
ω=γB0(1−σ)
H
Hβ
Hβ'
gly
leu
gly
pro
dM/dt = γ M × B
M
y
x
Brf(t)
B0
ω0 = γ B0

# Manipulation of Coupled Spin Dynamics



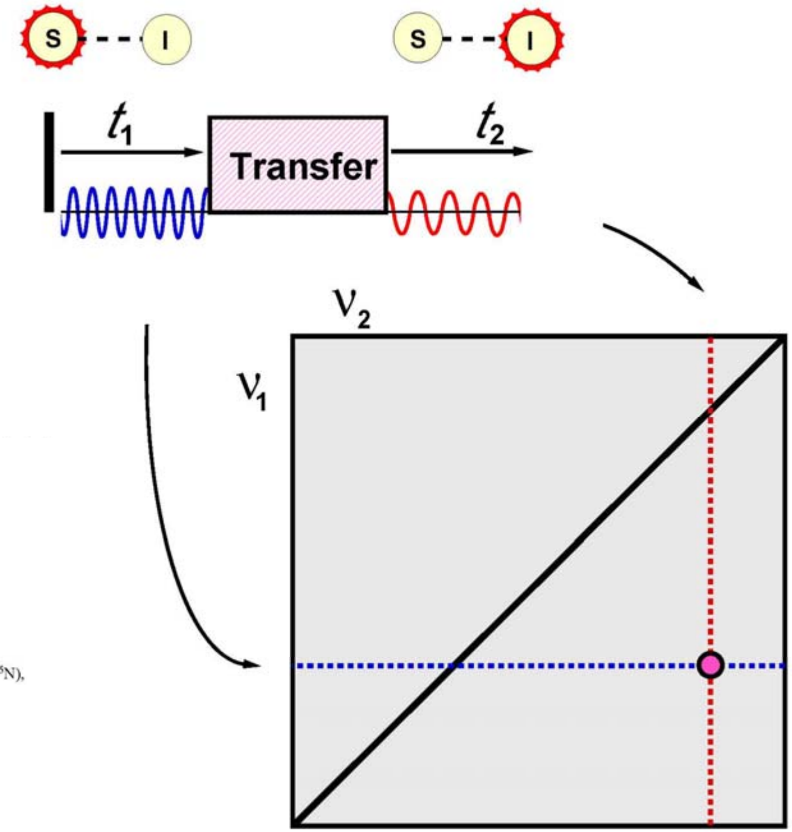
**2D NMR**

**Example: $^{15}$N-HSQC of p63**




**$^{15}$N labeling:**

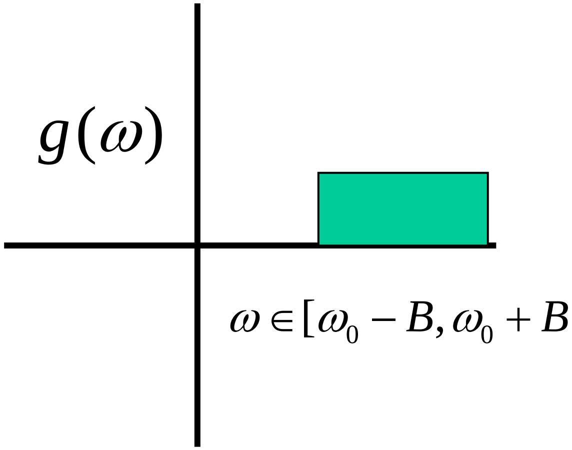
- all N atoms replaced by $^{15}$N (ca. 95 % $^{15}$N),
- characteristic fingerprint spectrum
- p63: 233 a.a. / 27 kDa
- measured at 750 MHz / 303 K

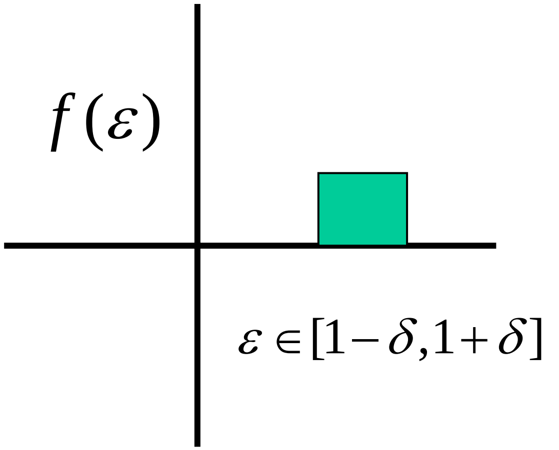



$$s(t_1, t_2) = \eta \cos(\omega_s t_1) \cos(\omega_I t_2)$$

# Problems of Broadband Excitations and Rf-Inhomogeneity

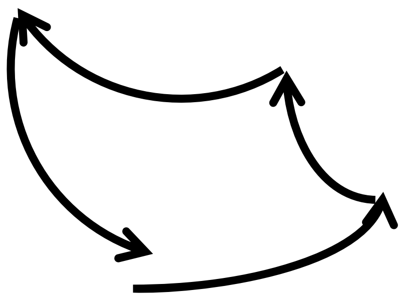
$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & -\Delta\omega & -\varepsilon u(t) \\ \Delta\omega & 0 & -\varepsilon v(t) \\ \varepsilon u(t) & \varepsilon v(t) & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



# Control Design by Area Generation

$$\frac{dX}{dt} = \varepsilon\,[u(t)\Omega_x + v(t)\Omega_y]\,X$$



$$U_\varepsilon(\Delta t) = \exp(-\varepsilon\Omega_y \Delta t)\exp(-\varepsilon\Omega_x \Delta t)\exp(\varepsilon\Omega_y \Delta t)\exp(\varepsilon\Omega_x \Delta t)$$

$$\approx I + (\Delta t)^2 \underbrace{[\varepsilon\Omega_x, \varepsilon\Omega_y]}_{\varepsilon^2\Omega_z}$$

$$U_\varepsilon(-\sqrt{\Delta t})\exp(-\varepsilon\Omega_x \Delta t)U_\varepsilon(\sqrt{\Delta t})\exp(\varepsilon\Omega_x \Delta t)$$

$$\approx I + (\Delta t)^2 \underbrace{[\varepsilon\Omega_x[\varepsilon\Omega_x, \varepsilon\Omega_y]]}_{-\varepsilon^3\Omega_y}$$

# Lie Algebras and Polynomial Approximations

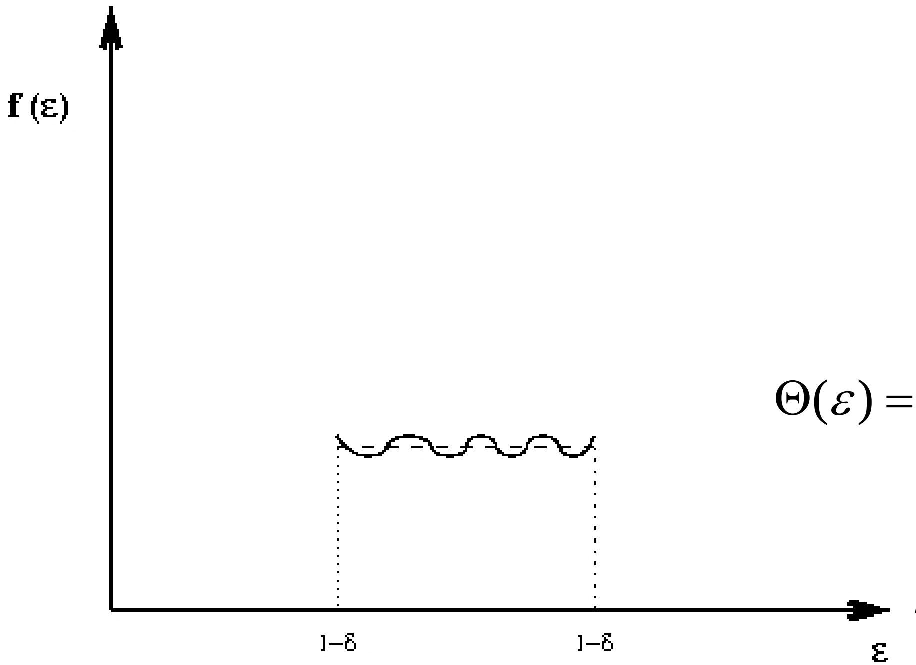
Using $\varepsilon\Omega_y, \varepsilon^3\Omega_{y,} \cdots, \varepsilon^{2k+1}\Omega_y$ as generators

$$f(\varepsilon) = \sum_k c_k \varepsilon^{2k+1}$$

Choose $f(\varepsilon)$ such that it is approx. constant for

$$\varepsilon \in [1-\delta, 1+\delta]$$

$$\exp(f(\varepsilon)\Omega_y)$$




$$\Theta(\varepsilon) = \exp(f_1(\varepsilon)\Omega_x)\exp(f_2(\varepsilon)\Omega_y)\exp(f_3(\varepsilon)\Omega_x)$$

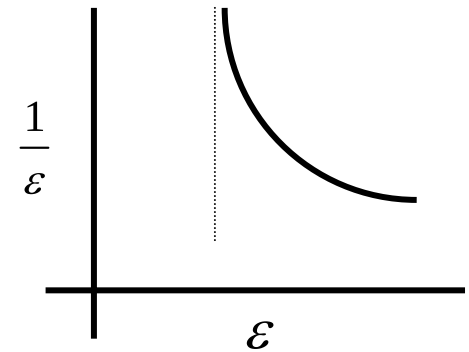
***J.S. Li and N. Khaneja, PRA (2006)***

***J.S. Li and N. Khaneja, IEEE TAC (2009)***

# Fourier Synthesis Methods for Robust Control Design

$$U_1 = \exp(k\pi\varepsilon\Omega_x)\exp(\varepsilon\frac{\beta_k}{2}\Omega_y)\exp(-k\pi\varepsilon\Omega_x)$$

$$= \exp(\varepsilon\frac{\beta_k}{2}(\cos(k\pi\varepsilon)\Omega_y + \sin(k\pi\varepsilon)\Omega_z))$$

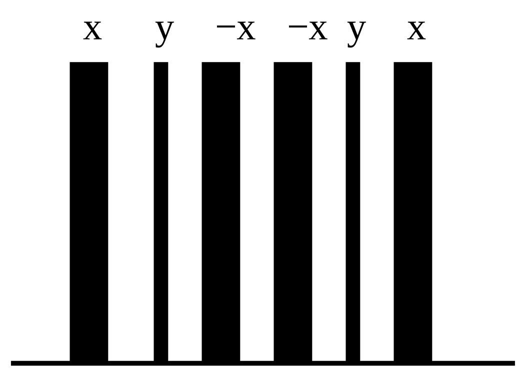



$$U_2 = \exp(-k\pi\varepsilon\Omega_x)\exp(\varepsilon\frac{\beta_k}{2}\Omega_y)\exp(k\pi\varepsilon\Omega_x)$$

$$= \exp(\varepsilon\frac{\beta_k}{2}(\cos(k\pi\varepsilon)\Omega_y - \sin(k\pi\varepsilon)\Omega_z))$$

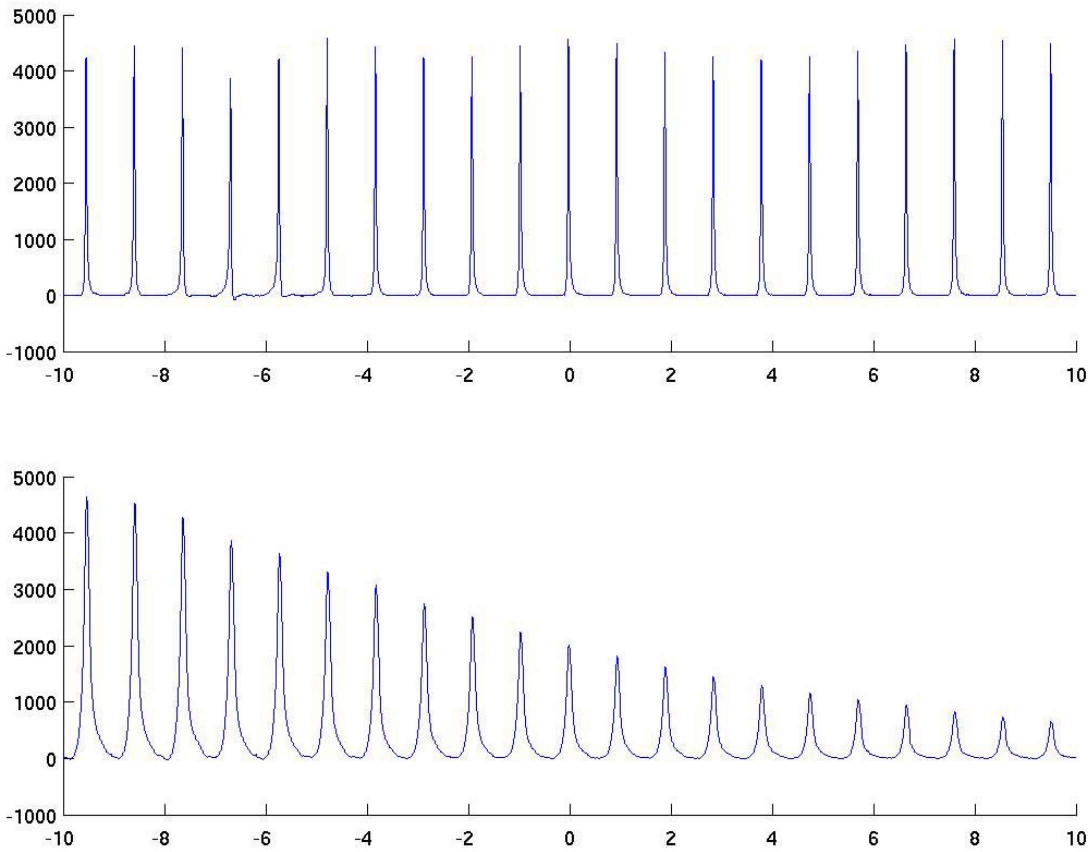
$$U_1U_2 = \exp(\varepsilon\beta_k\cos(k\pi\varepsilon)\Omega_y)$$

$$\sum_k \beta_k\cos(k\pi\varepsilon) = \frac{\theta}{\varepsilon}$$



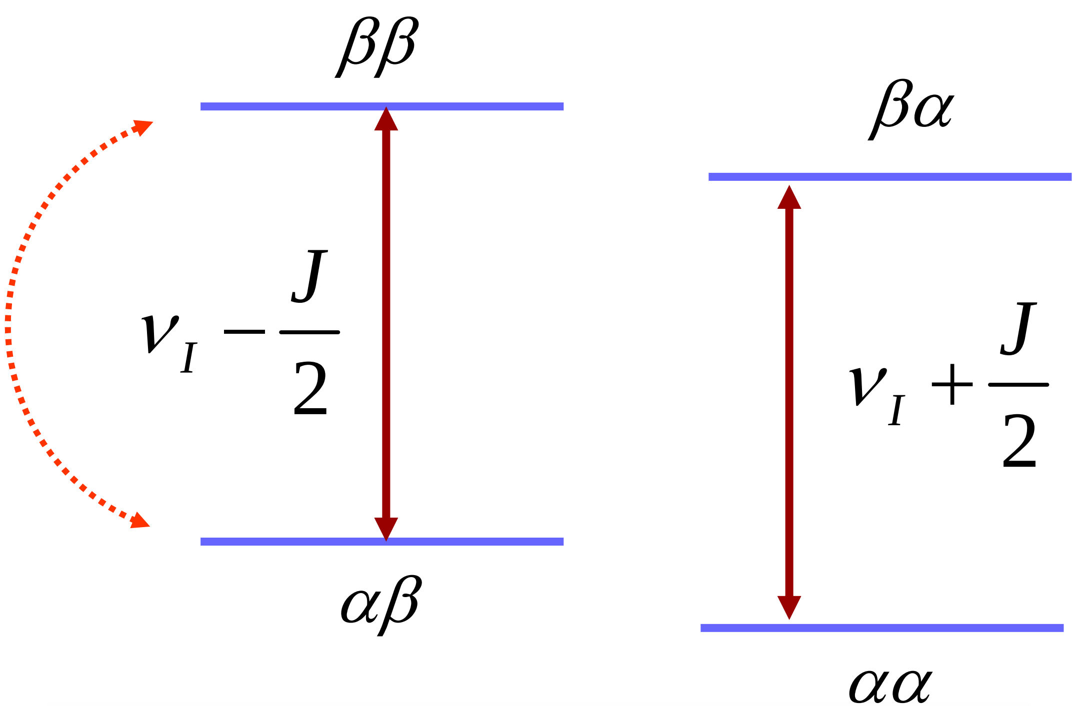

***B. Pryor, N. Khaneja Journal of Chemical Physics (2006).***

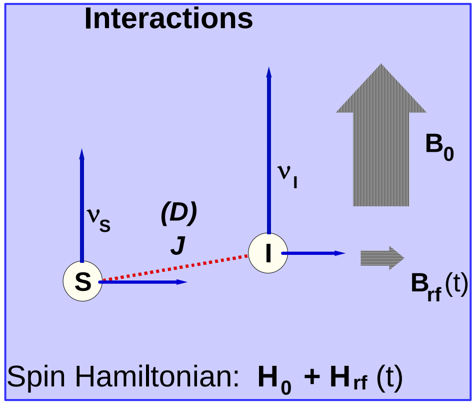
# Fourier Synthesis Methods for Robust Control

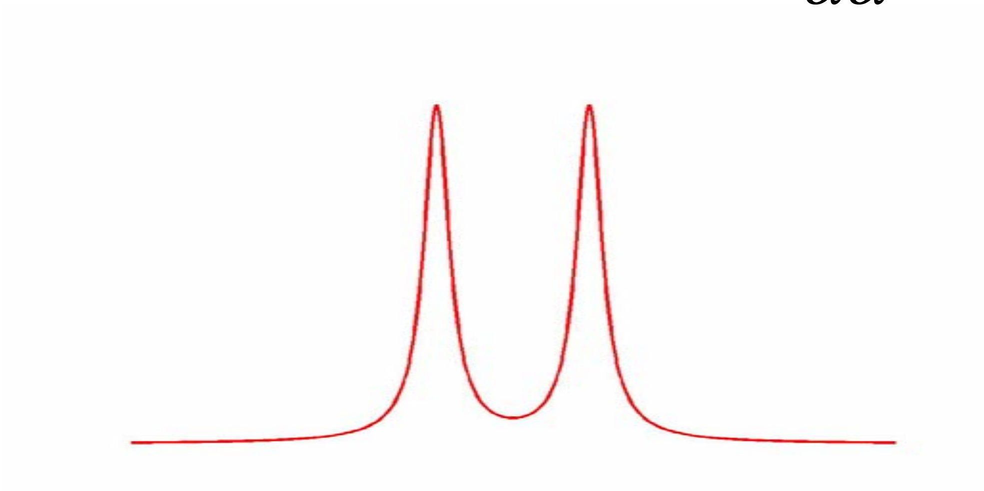


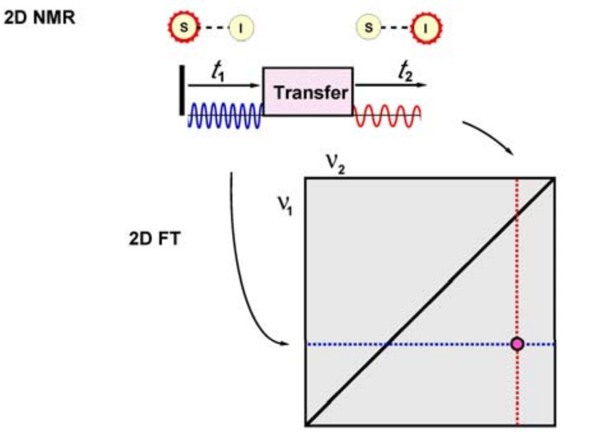

*NMR spectra at proton frequency of 500 MHz*

# Control for Minimizing Decoherence

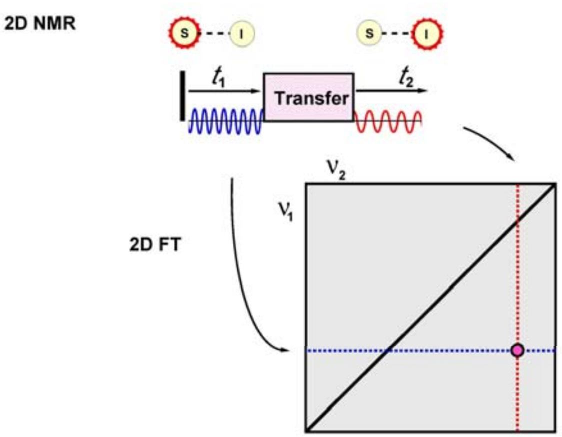


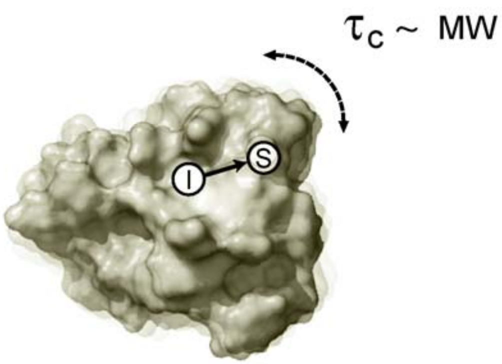



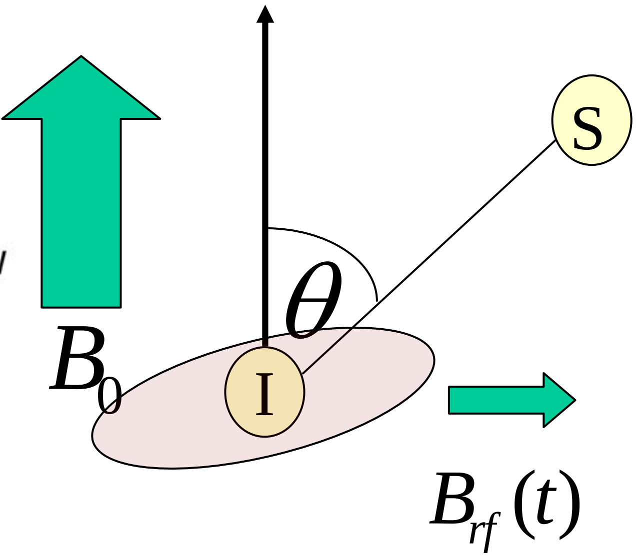





## Random collisions with solvent molecules causes stochastic tumbling of the protein molecules




$$L(\rho) = \pi k_1 [2I_zS_z, [2I_zS_z, \rho]] + \pi k_2 [I_z [I_z, \rho]] + \pi k_3 [S_z [S_z, \rho]]$$

$$k = k_1 + k_2$$

$$I_z \rightarrow 2I_zS_z$$

# Optimal Control in Presence of Relaxation

$$\frac{d}{dt}\begin{bmatrix} \langle I_z \rangle \\ \langle I_x \rangle \\ \langle 2I_yS_z \rangle \\ \langle 2I_zS_z \rangle \end{bmatrix} = \begin{bmatrix} 0 & -u(t) & & \\ u(t) & -k & -J & \\ & J & -k & -v(t) \\ & & v(t) & 0 \end{bmatrix} \begin{bmatrix} \langle I_z \rangle \\ \langle I_x \rangle \\ \langle 2I_yS_z \rangle \\ \langle 2I_zS_z \rangle \end{bmatrix}$$

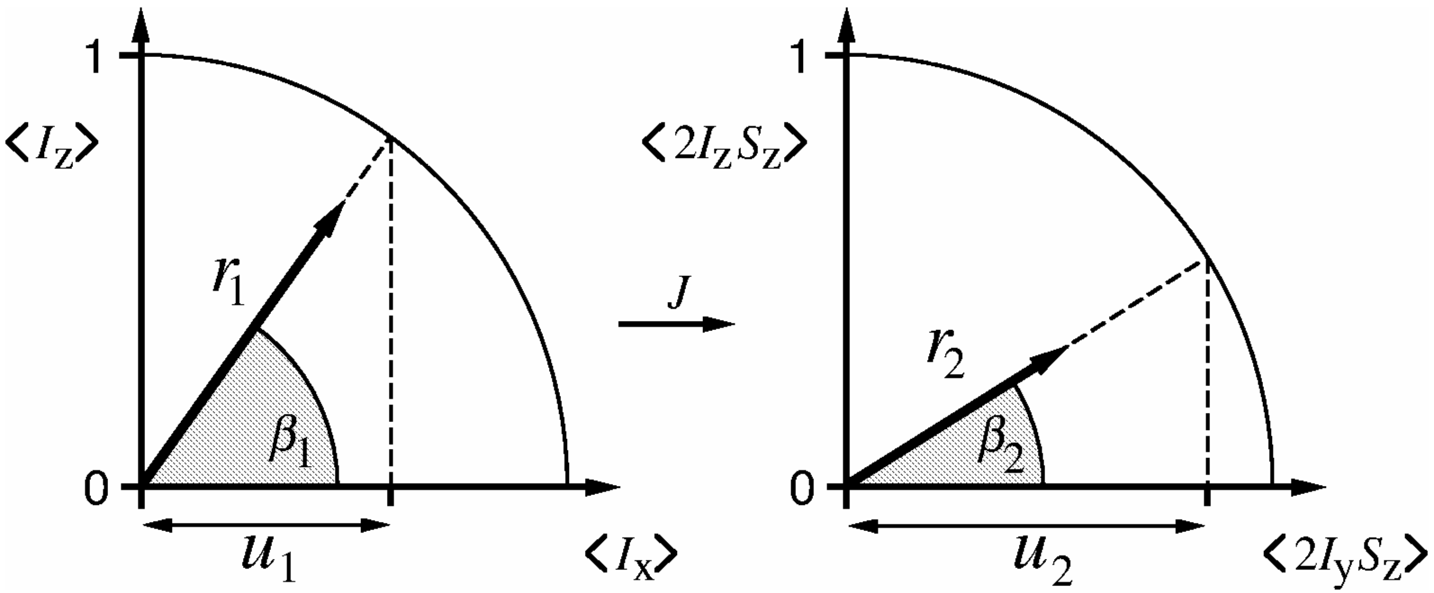
$$\xi = \frac{k}{J}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ \eta \end{bmatrix}$$

$$\eta = \sqrt{1 + \xi^2} - \xi$$

$$\frac{x_3}{x_2} = \frac{\langle 2I_yS_z \rangle}{\langle I_x \rangle} = \eta$$

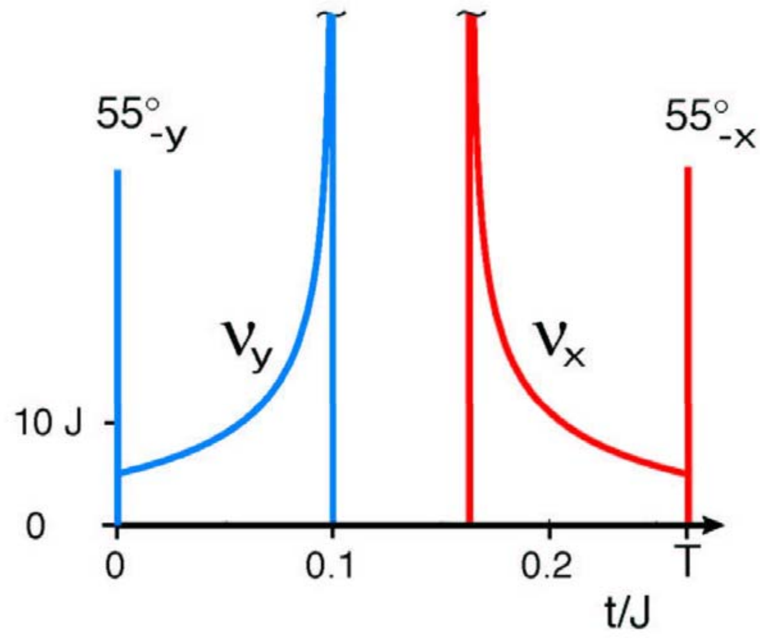
# The control problem




$$\xi = \frac{k}{J}$$

$$\frac{d}{dt}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \pi J \begin{bmatrix} -\xi u_1^2 & -u_1 u_2 \\ u_1 u_2 & -\xi u_2^2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

# ROPE Pulse Sequence




*Khaneja, Reiss, Luy, Glaser JMR(2003)*

$$\frac{d}{dt}\begin{bmatrix} \langle I_z \rangle \\ \langle I_x \rangle \\ \langle 2I_yS_z \rangle \\ \langle 2I_zS_z \rangle \end{bmatrix} = \begin{bmatrix} 0 & -u(t) & & \\ u(t) & -k & -J & \\ & J & -k & -v(t) \\ & & v(t) & 0 \end{bmatrix} \begin{bmatrix} \langle I_z \rangle \\ \langle I_x \rangle \\ \langle 2I_yS_z \rangle \\ \langle 2I_zS_z \rangle \end{bmatrix}$$

# Experimental Results

T = 250 K

J = 193 Hz

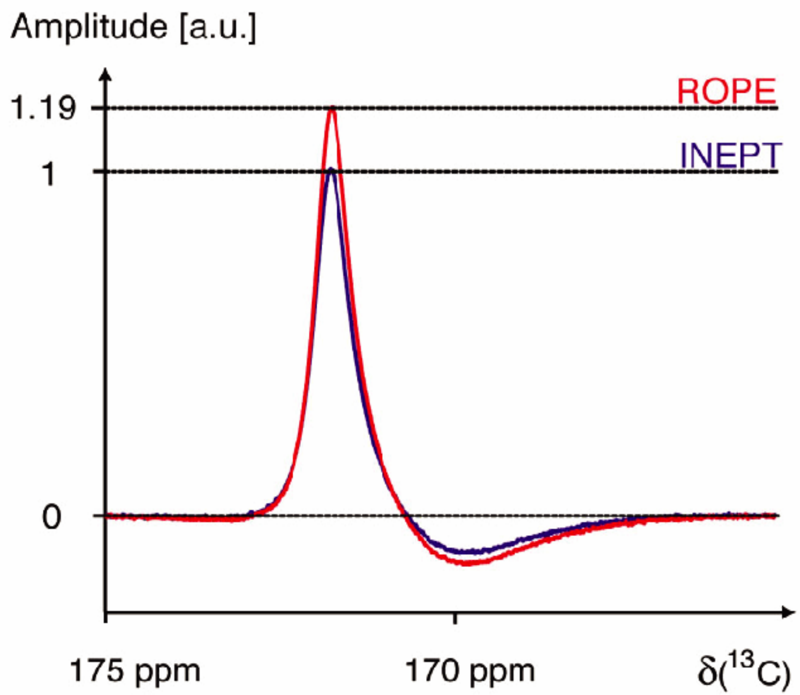
Proton frequency 500 Mhz

$T_2 = 1.4ms$

k = 227 Hz

$\xi = 1.18$

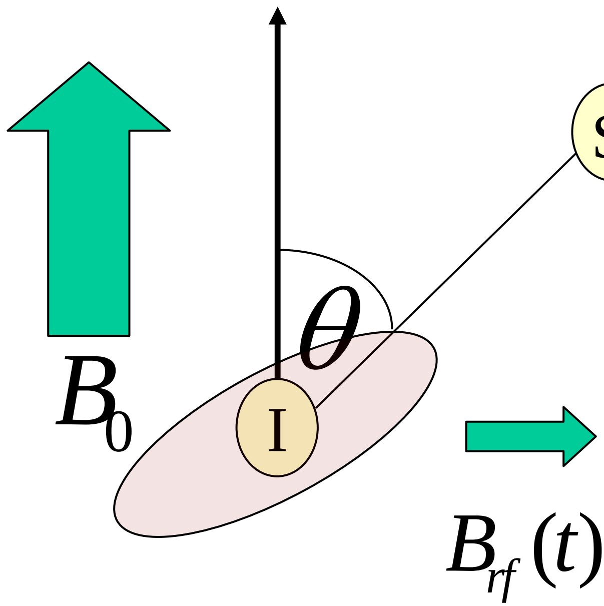
$$H_z \rightarrow H_z C_z$$

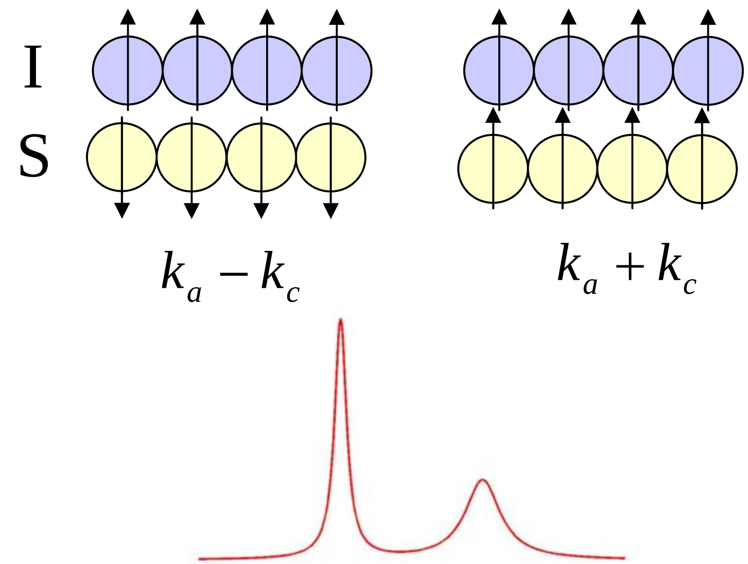



$^{13}$C-Formiate in 92% $D_6$-Glycerol and 8% $D_2O$ (T=250 K)

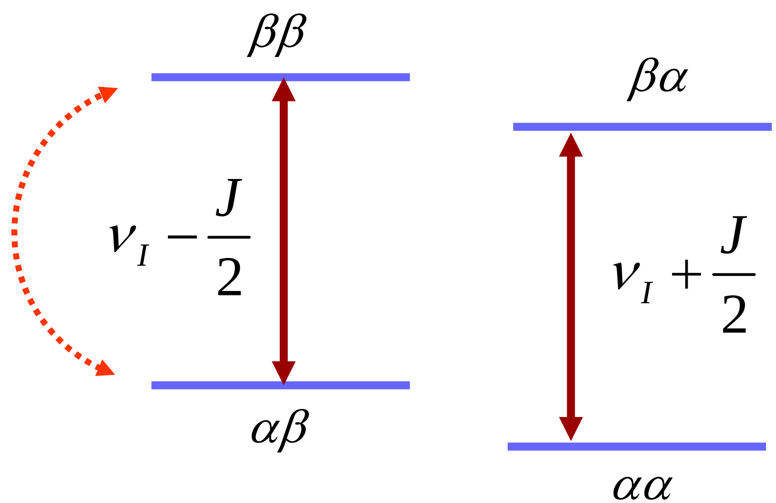
*Khaneja, Reiss, Luy, Glaser JMR(2003)*

# Cross-Correlated Relaxation

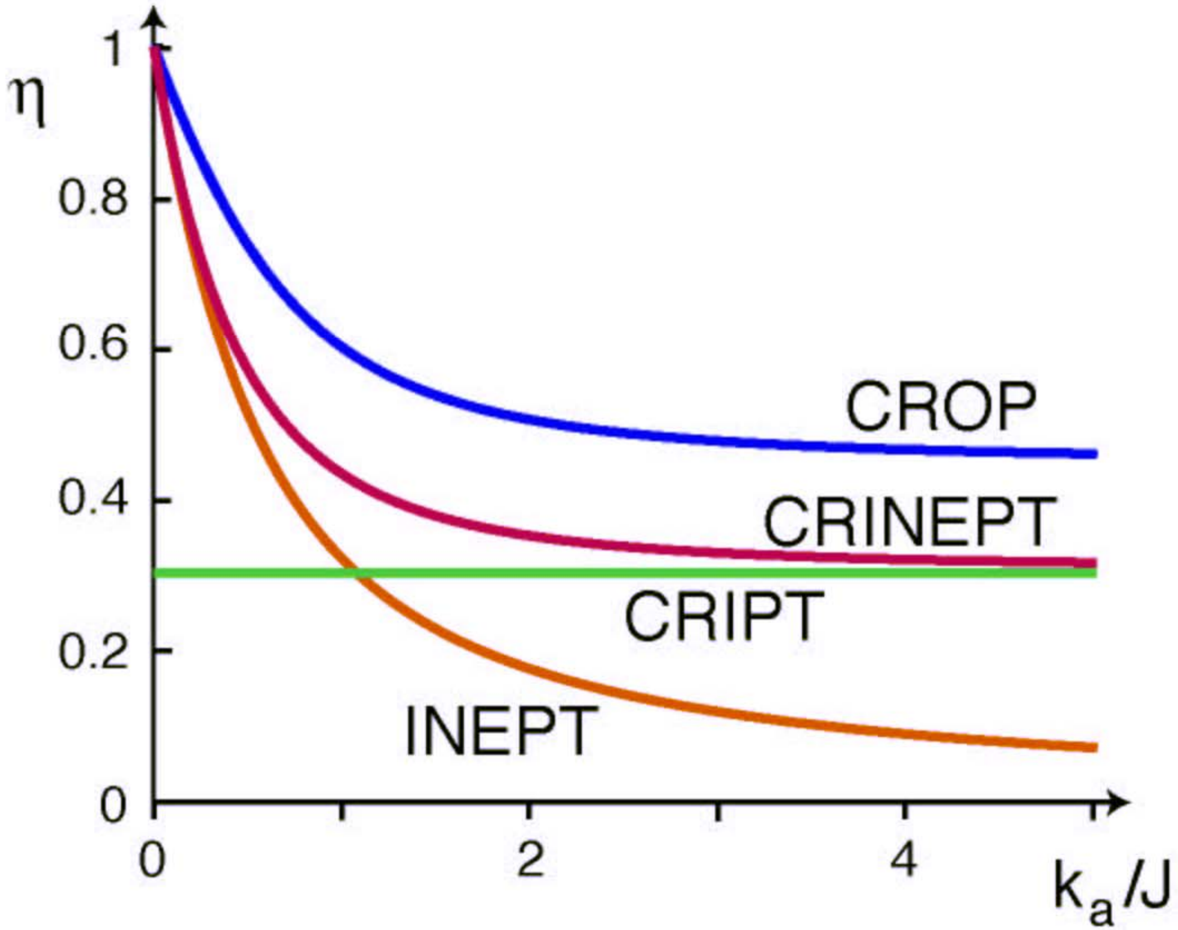



$$\frac{d}{dt}\begin{bmatrix}\langle I_z\rangle\\ \langle I_x\rangle\\ \langle I_y\rangle\\ \langle I_yS_z\rangle\\ \langle I_xS_z\rangle\\ \langle I_zS_z\rangle\end{bmatrix}=\begin{bmatrix}0 & u & v & 0 & 0 & 0\\ -u & -k_a & 0 & -J & -k_c & 0\\ -v & 0 & -k_a & -k_c & J & 0\\ 0 & J & -k_c & -k_a & 0 & -v\\ 0 & -k_c & -J & 0 & -k_a & -u\\ 0 & 0 & 0 & v & u & 0\end{bmatrix}\begin{bmatrix}\langle I_z\rangle\\ \langle I_x\rangle\\ \langle I_y\rangle\\ \langle I_yS_z\rangle\\ \langle I_xS_z\rangle\\ \langle I_zS_z\rangle\end{bmatrix}$$



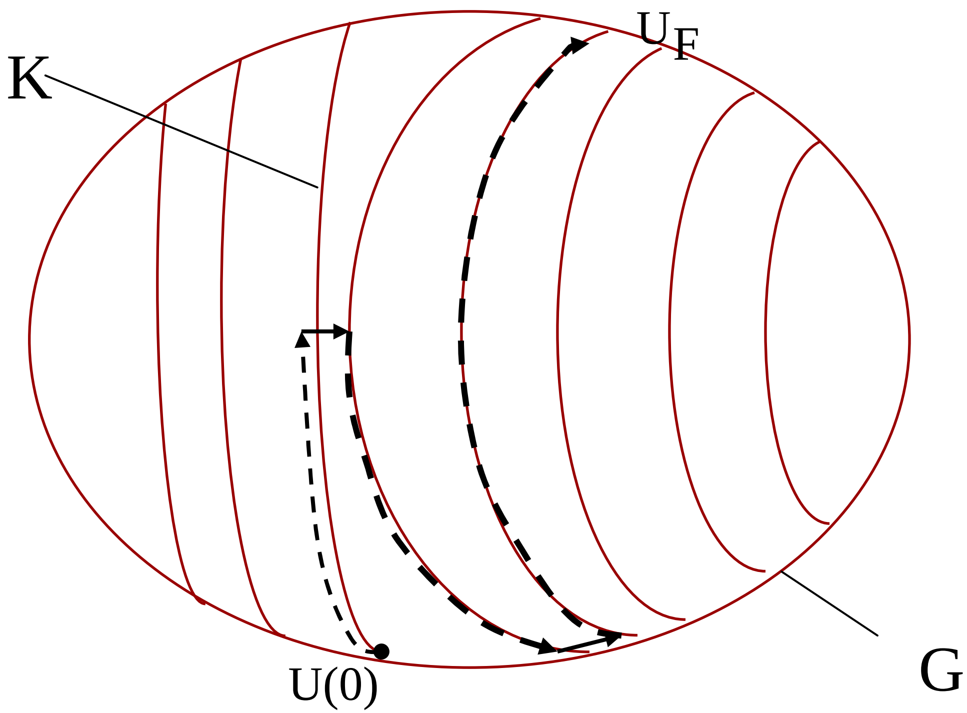
# Transfer Efficiency



# Time Optimal Control of Quantum Systems

$$k = \{-iH_j\}_{LA}$$

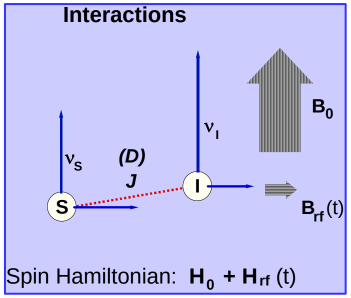
$$K = \exp(k)$$




$$g = p + k; \quad p \perp k$$

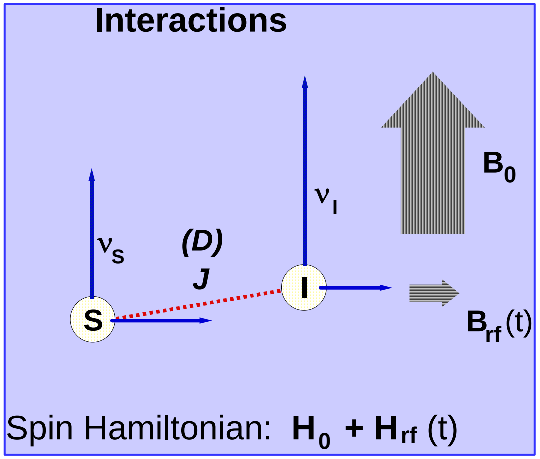
$$[p,p] \subseteq k; [k,k] \subseteq k; [p,k] \subseteq p;$$

$$\frac{dU(t)}{dt} = -i[H_d + \sum_{j=1}^{m} u_j H_j]U(t); \quad U(0) = I$$




*Physical Review A , 63, 032308, 2001*

# Cartan Decompositions , Two-Spin Systems and Canonical Decomposition of SU(4)




$$\frac{dU}{dt} = -i[\sum_{\alpha,\beta} J_{\alpha\beta} I_\alpha S_\beta + u_1 I_x + u_2 I_y + u_3 S_x + u_4 S_y]U$$

$$k = \{-i\ I_\alpha, -iS_\beta\}\ ;\quad p = \{-iI_\alpha S_\beta\}$$

$$G = SU(4); \quad K = SU(2) \otimes SU(2)$$

$$I_\alpha = \sigma_\alpha \otimes I\,;$$

$$S_\alpha = I \otimes \sigma_\alpha\,;$$

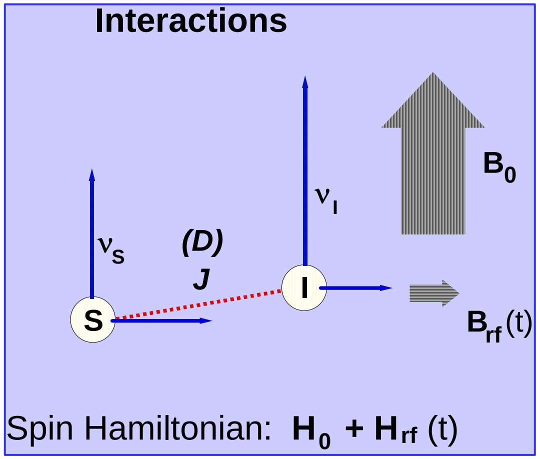
$$I_\alpha S_\beta = \sigma_\alpha \otimes \sigma_\beta\,;$$

$$a = \{-iI_x S_x, -iI_y S_y, -iI_z S_z\}$$

$$G = K\ \exp(-i(\alpha_x I_x S_x + \alpha_y I_y S_y + \alpha_z I_z S_z\ ))\ K$$

*Khaneja, Brockett, Glaser, Physical Review A , 63, 032308, 2001*

# Cartan Decompositions , Two-Spin Systems and Canonical Decomposition of SU(4)




$$U_l\left(\sum_{\alpha\beta} J_{\alpha\beta} I_\alpha S_\beta\right) U_l^\dagger \rightarrow \underbrace{\alpha_x I_x S_x + \alpha_y I_y S_y + \alpha_z I_z S_z}_{H_1}$$

$$(\alpha_x, \alpha_y, \alpha_z) \qquad \alpha_x \geq \alpha_y \geq |\alpha_z|$$

$$U_2 \exp(-iH_4 t_4) \exp(-iH_3 t_3) \exp(-iH_2 t_2) \exp(-iH_1 t_1) U_1$$

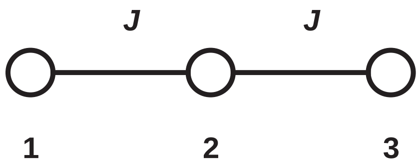
$$H_T = \beta_x I_x S_x + \beta_y I_y S_y + \beta_z I_z S_z$$

*Khaneja, Brockett, Glaser, Physical Review A , 63, 032308, 2001*

*H. Yuan and N. Khaneja , System and Control Letters 2006*

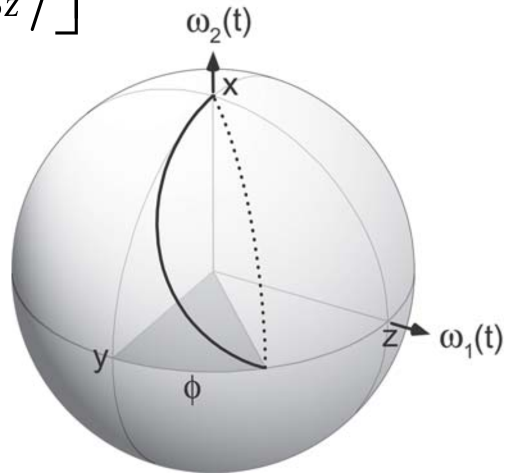
# Geometry, Control and NMR

$$\frac{d}{dt}\begin{bmatrix} \langle I_{1x}\rangle \\ \langle 2I_{1y}I_{2z}\rangle \\ \langle 2I_{1y}I_{2x}\rangle \\ \langle 4I_{1y}I_{2y}I_{3z}\rangle \end{bmatrix} = \begin{bmatrix} 0 & -1 & & \\ 1 & 0 & -u & \\ & u & 0 & -1 \\ & & 1 & 0 \end{bmatrix}\begin{bmatrix} \langle I_{1x}\rangle \\ \langle 2I_{1y}I_{2z}\rangle \\ \langle 2I_{1y}I_{2x}\rangle \\ \langle 4I_{1y}I_{2y}I_{3z}\rangle \end{bmatrix}$$




$$\tan\theta = \frac{x_3}{x_2}$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & -\cos\theta & 0 \\ \cos\theta & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

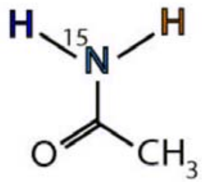



$$\theta = \omega t$$

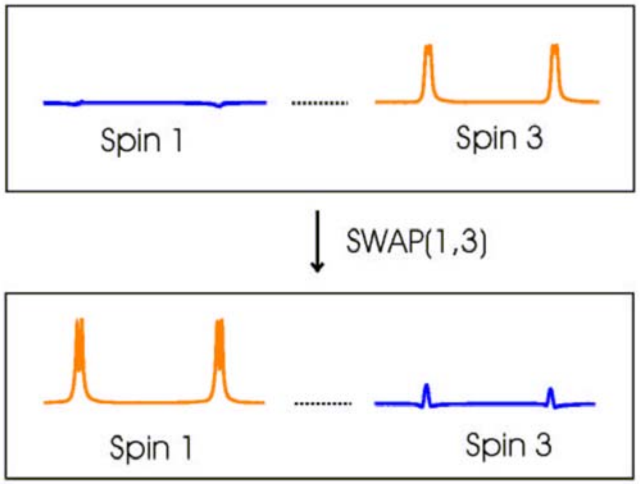
$$\exp(-i2\pi\kappa I_{1z}I_{2z}I_{3z}), \quad \frac{\sqrt{\kappa(4-\kappa)}}{2J}$$

$$\frac{(dx)^2 + (dz)^2}{y^2}$$

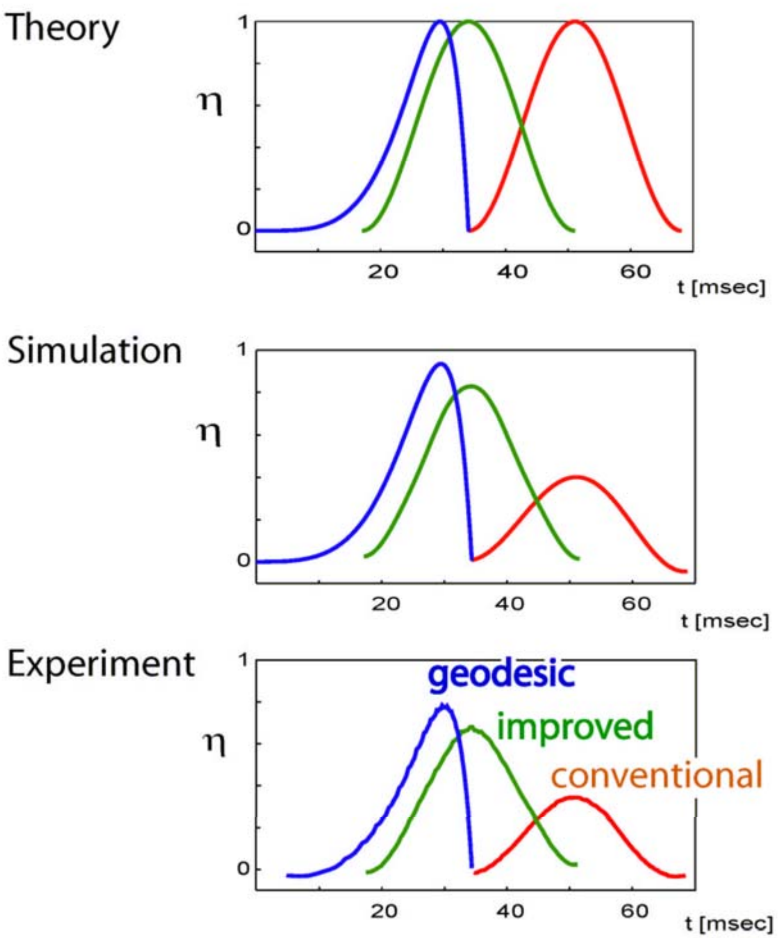
## Indirect SWAP Operation




[$^{15}$N]-Acetamide




## Efficiency η of indirect SWAP sequences

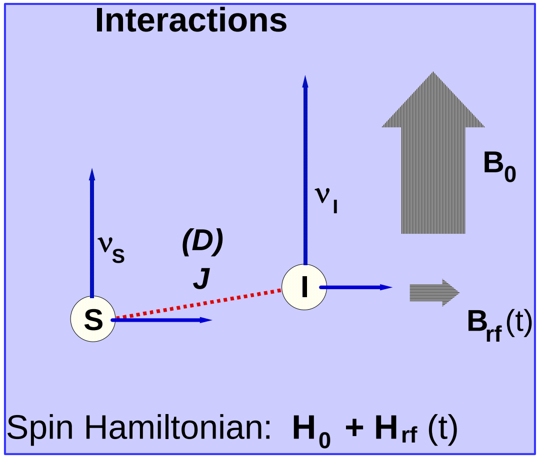



*Reiss, Khaneja, Glaser*
*J. Mag. Reson. 165 (2003)*

*Khaneja, et. al PRA(2007)*

# Another Canonical Decomposition of SU(4): Electron Nuclear Spin Dynamics

$$H_c = J I_z S_z \quad ; \quad \Omega_S \gg J \gg \Omega_I$$




$$k = -i\left\{ S_\alpha ,\ I_z S_\beta ,\ I_z \right\}$$

$$K = \exp\Big(-i \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{k}\Big) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

$$a = -i\left\{ S_z I_x ,\ I_x \right\} \qquad -i\begin{bmatrix} 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \end{bmatrix}$$

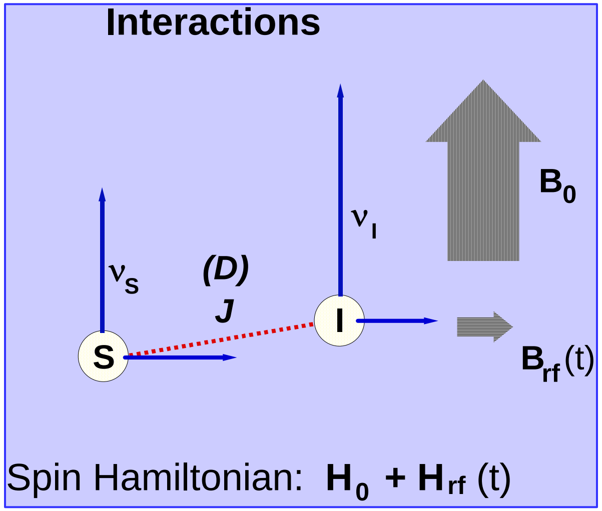
Zeier, Yuan, Khaneja, PRA (2008)

$$G = SU(4); \quad K = SU(2) \times SU(2) \times U(1)$$

$$K_1 \exp(-i\, \lambda_1 \sigma_x \otimes |0\rangle\langle 0| + \lambda_2 \sigma_x \otimes |1\rangle\langle 1|) K_2$$

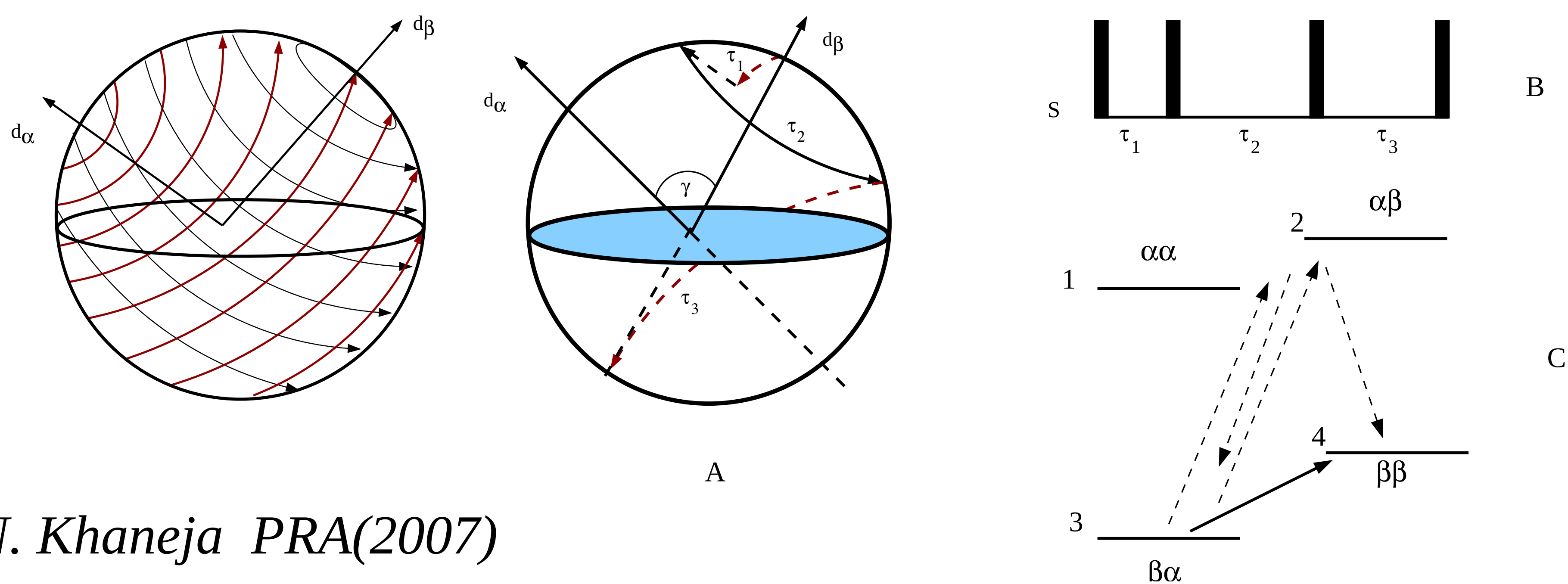
# Switched Control Problems

$$H_0 = \omega_s S_z + \omega_I I_z + S \cdot A \cdot I$$




$$d_\beta = (\omega_I + A/2)\ \hat{z} + \frac{B}{2}\hat{x}$$

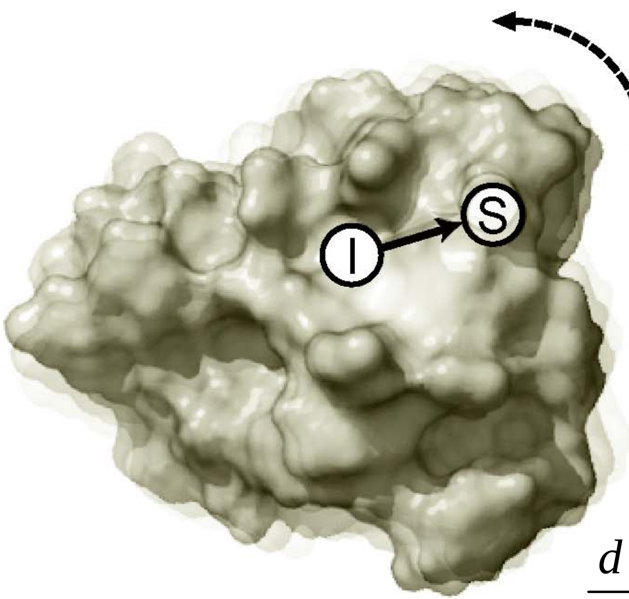
$$d_\alpha = (\omega_I - A/2)\ \hat{z} - \frac{B}{2}\hat{x}$$

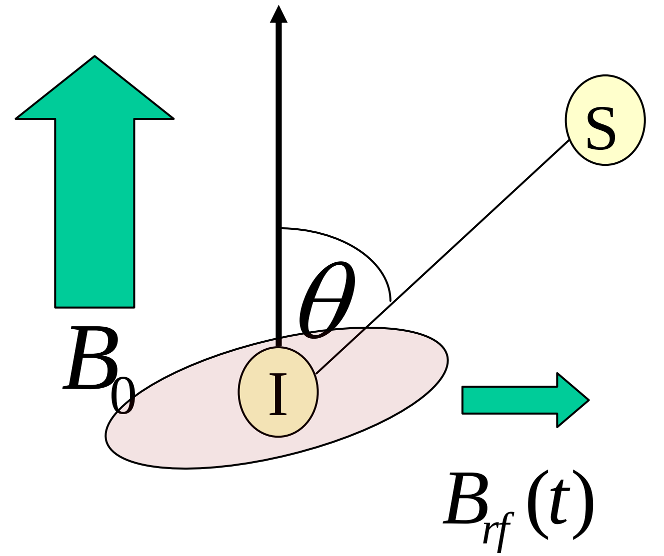



*N. Khaneja PRA(2007)*

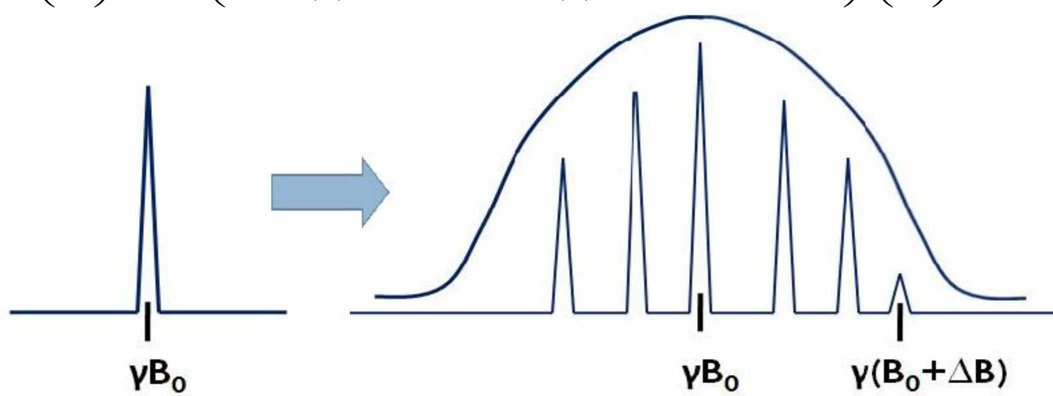
# Bilinear Control Problems with Periodic Drift: Solid State NMR

$\tau_c \sim$ MW




$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \gamma \begin{pmatrix} 0 & -(B_0 + \Delta B) & -u(t) \\ B_0 + \Delta B & 0 & -v(t) \\ u(t) & v(t) & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



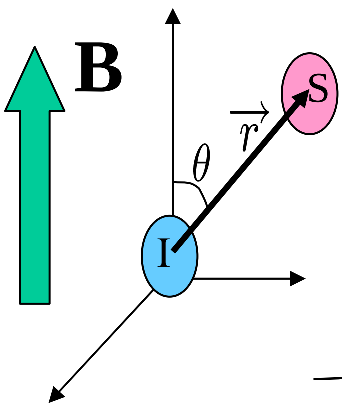
# Phase and Amplitude Modulated Recoupling

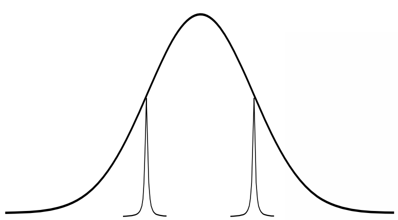
# Control Problems in solid state NMR

***J. Am. Chem. Soc., 126*** (2005)

***Chem. Phys. Letter*** (2005)

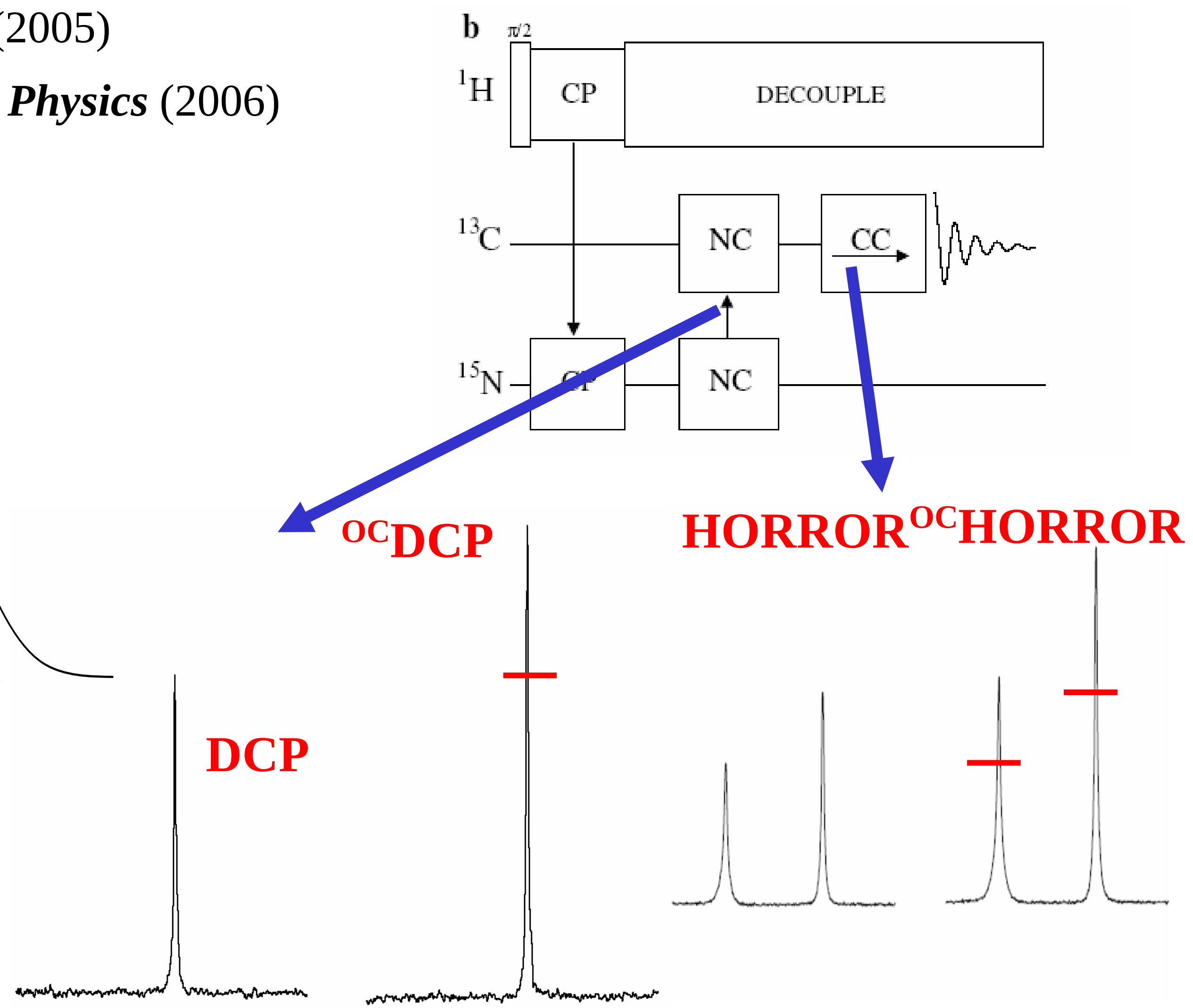
***Journal of Chemical Physics*** (2006)



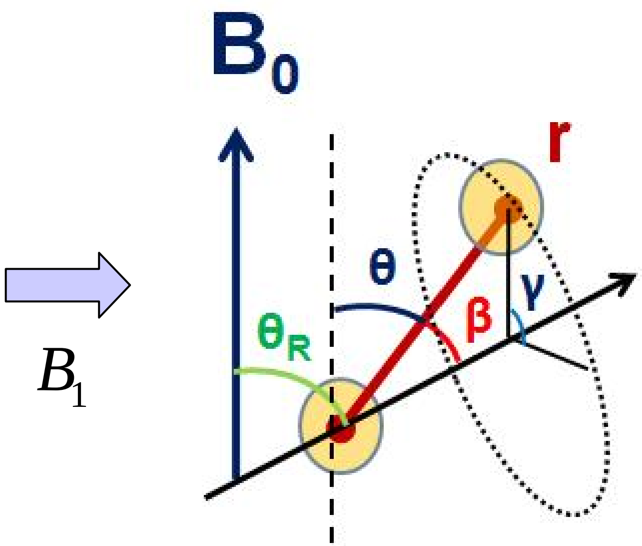



$$J = -\frac{\mu_0}{4\pi}\hbar\frac{\gamma^I\gamma^S}{r^3}\left[3(\mathrm{I}\cdot\hat{\mathrm{r}})(\mathrm{S}\cdot\hat{\mathrm{r}}) - \mathrm{I}\cdot\mathrm{S}\right]$$

$$J \sim \frac{3\cos^2(\theta)-1}{r^3}$$



Broadband Recoupling by Phase Modulations.




$$A \square \quad \Delta\omega_I, \Delta\omega_s, \omega_r$$

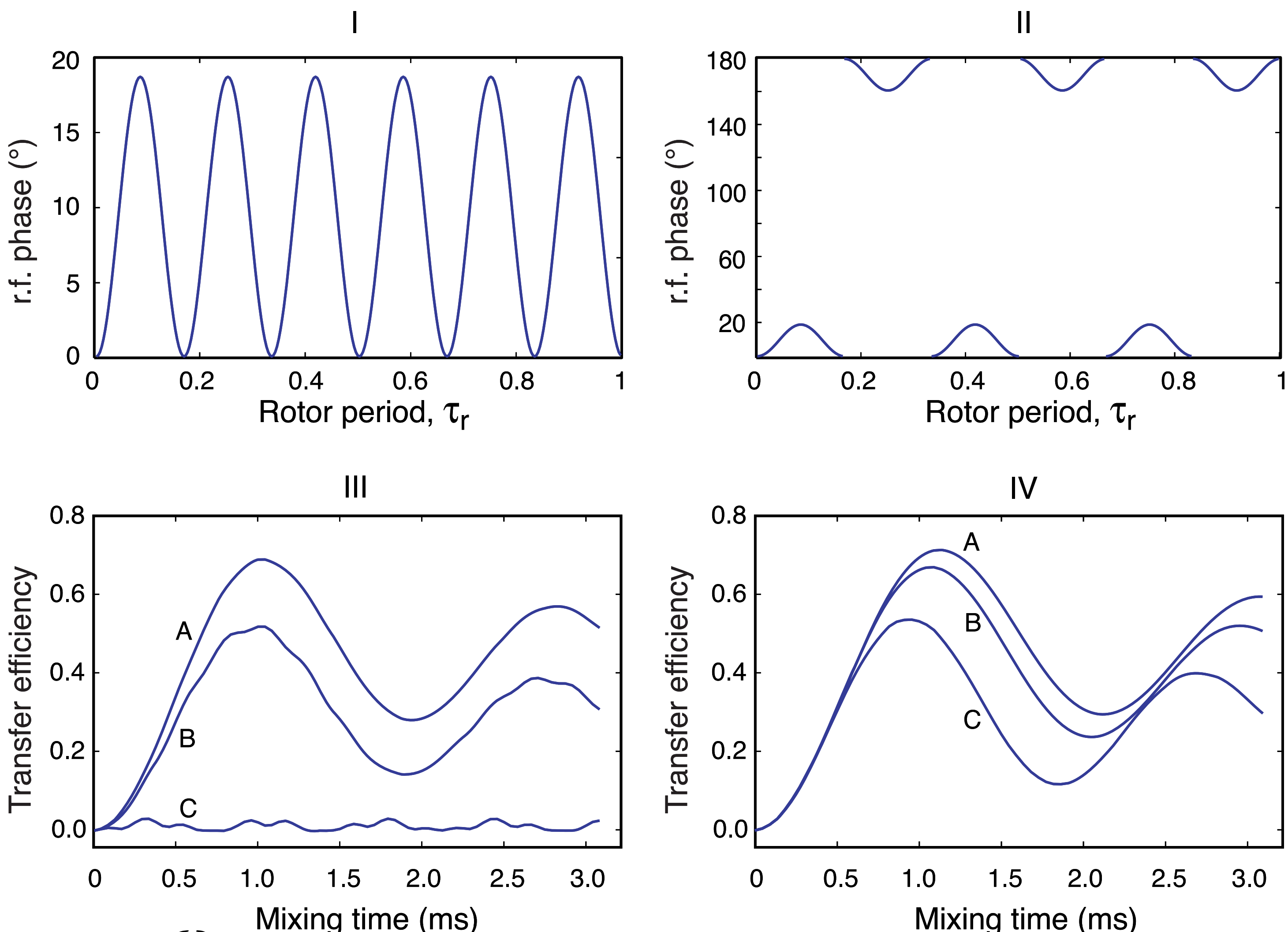
$$H = \Delta\omega_I I_z + \Delta\omega_s S_z + A(1+\varepsilon)F_x - \omega_r \sin(At)F_z$$

$$\kappa \cos(\omega_r t + \gamma)\left\{(I_z S_z + I_y S_y) + (I_z S_z - I_y S_y)\right\}$$

$$\tilde{\tilde{H}} = \frac{\omega_r}{2} F_y + \varepsilon A F_x + \kappa \cos(\omega_r t + \gamma)\left\{(I_z S_z + I_y S_y)\right\}$$

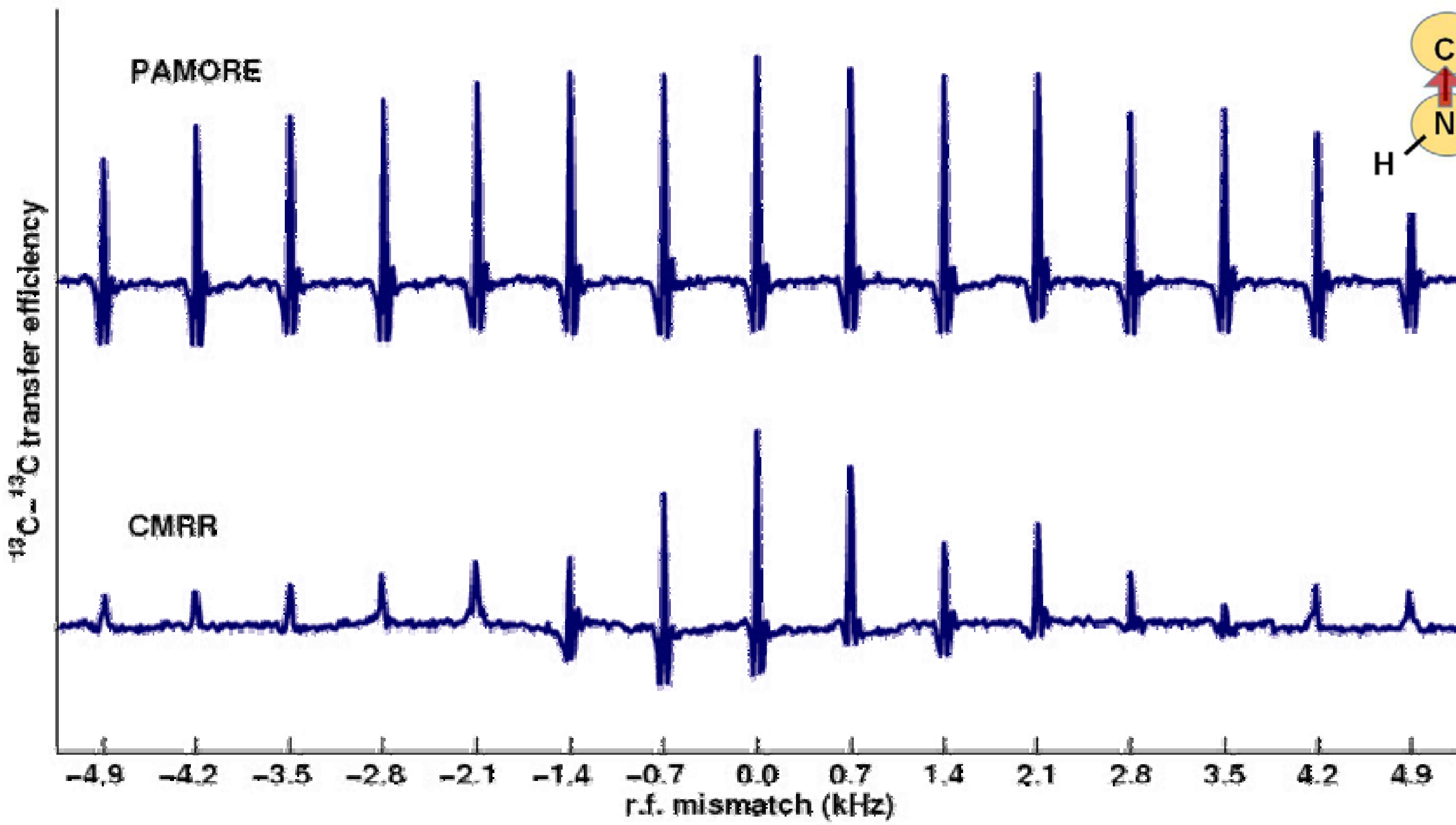
**G. De Paëpe, et al. *J.Amer.Chem.Soc.* 128, 1776-1777(2006).**

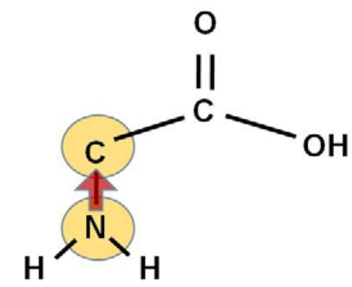
# CMRR Phase Modulations




$$\phi(t) = \frac{\omega_r}{A}(1 - \cos(At)) \qquad \phi(t + \tau_c) = 180 - \phi(t)$$

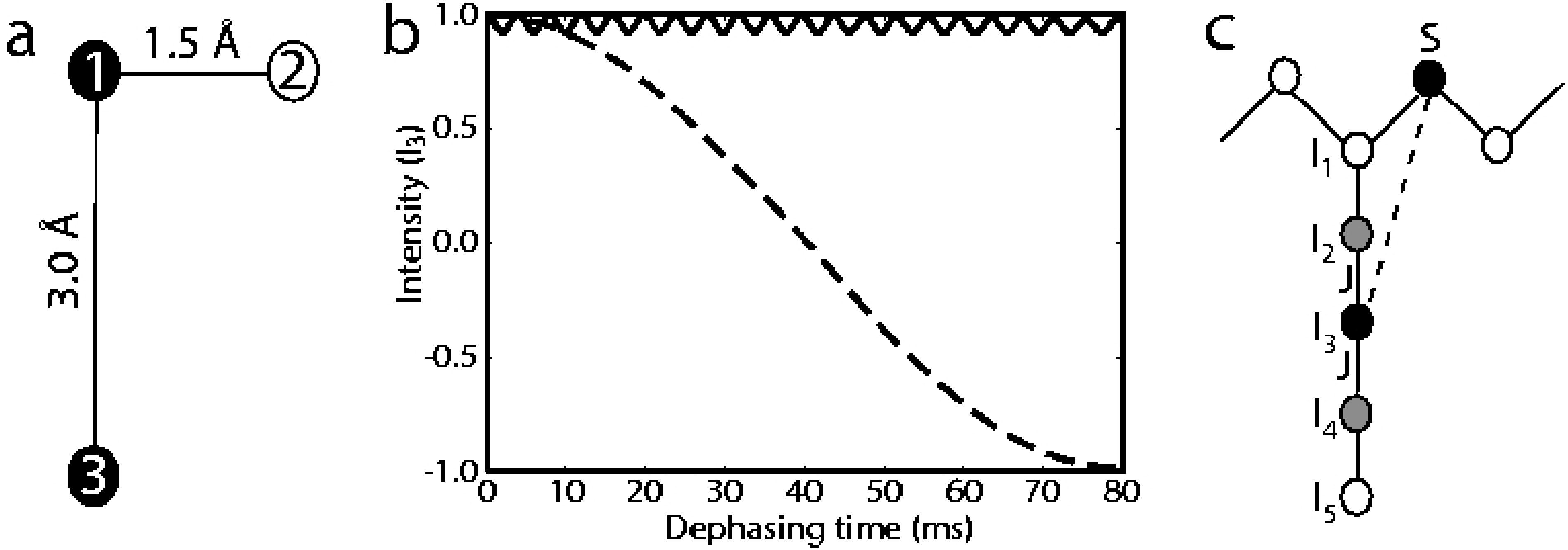
# Experimental Results (Homonuclear Transfer) $C_\alpha \rightarrow CO$




J. Lin, M. Bayro, R.G. Griffin, and N. Khaneja
*J.Magn.Reson.* 145, 97 (2009)

*360 Mhz*
*8 Khz spinning*

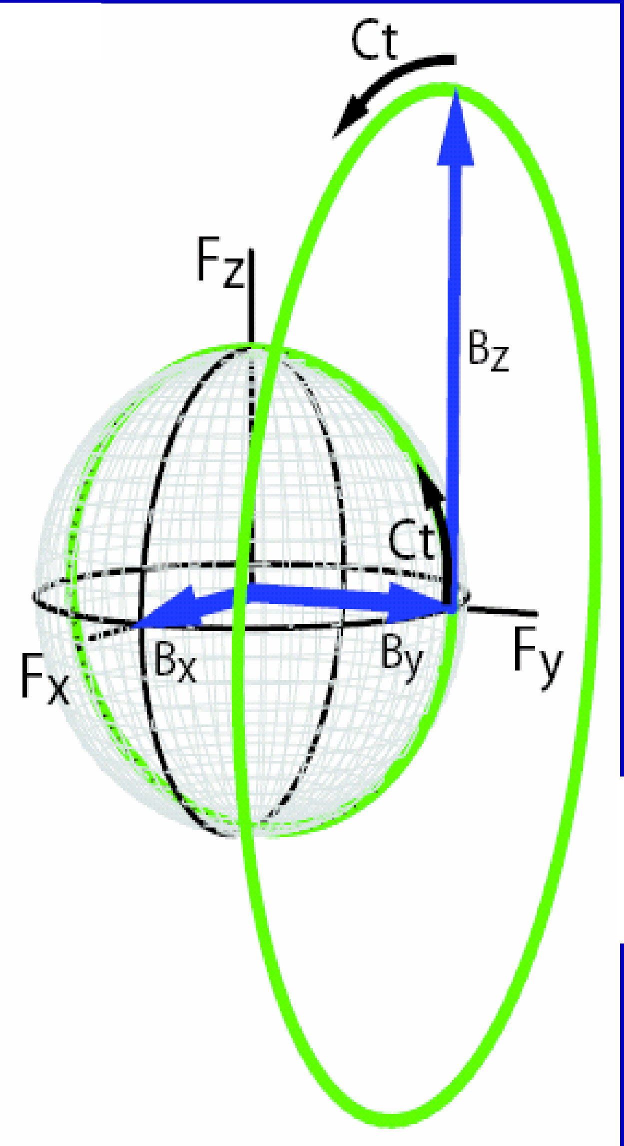



$$\kappa_{12}\overbrace{\{(I_{1z}I_{2z}-I_{1y}I_{2y})\cos\gamma_{12}+(I_{1z}I_{2y}+I_{1y}S_{2z})\sin\gamma_{12}\}}^{H_1}+$$
$$\kappa_{13}\underbrace{\{(I_{2z}I_{3z}-I_{2y}I_{3y})\cos\gamma_{13}+(I_{2z}I_{3y}+I_{2y}S_{3z})\sin\gamma_{13}\}}_{H_2}$$

$$[H_{1,}\quad H_2]\neq 0$$

***Dipolar Truncation Problem***

# TOFU: Multiple Frames and Effective Hamiltonians




$$H_0(t) = \omega_I(t) I_z + \omega_S(t) S_z + \omega_{IS}(t)(3 I_z S_z - \overline{I} \cdot \overline{S})$$

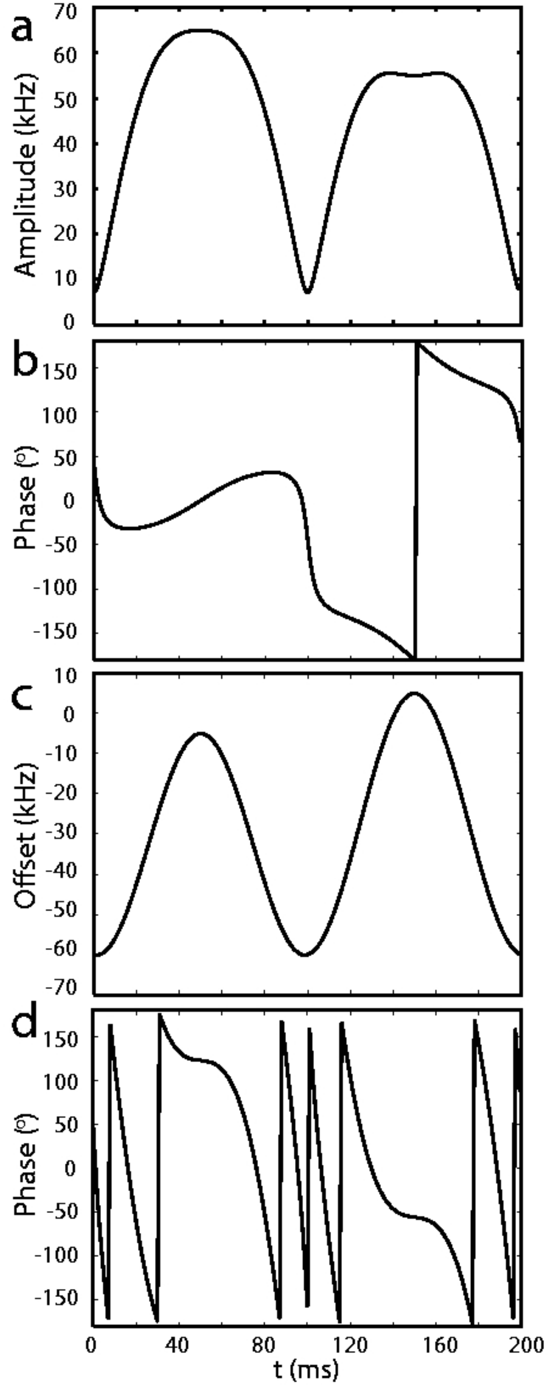
$$H_{rf}(t) = A(t)\cos\theta(t) F_x + A(t)\sin\theta(t) F_y + \delta\omega(t) F_z$$

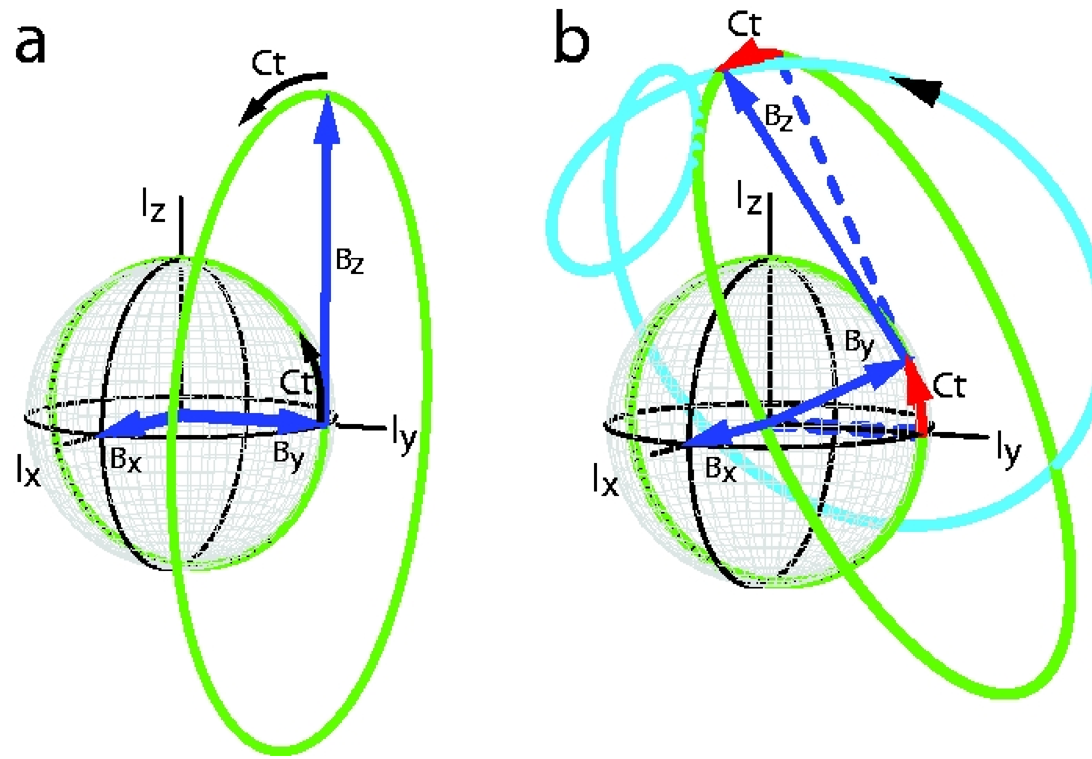
$$H_{rf}(t) = C F_x + C \exp(-iCtF_x)\, F_y\, \exp(iCtF_x) + B \exp(-iCtF_x)\exp(-iCtF_y)\, F_z\, \exp(iCtF_y)\exp(iCtF_x)$$

$$C F_y + B \exp(-iCtF_y) F_x \exp(iCtF_y)$$

$$B F_z$$

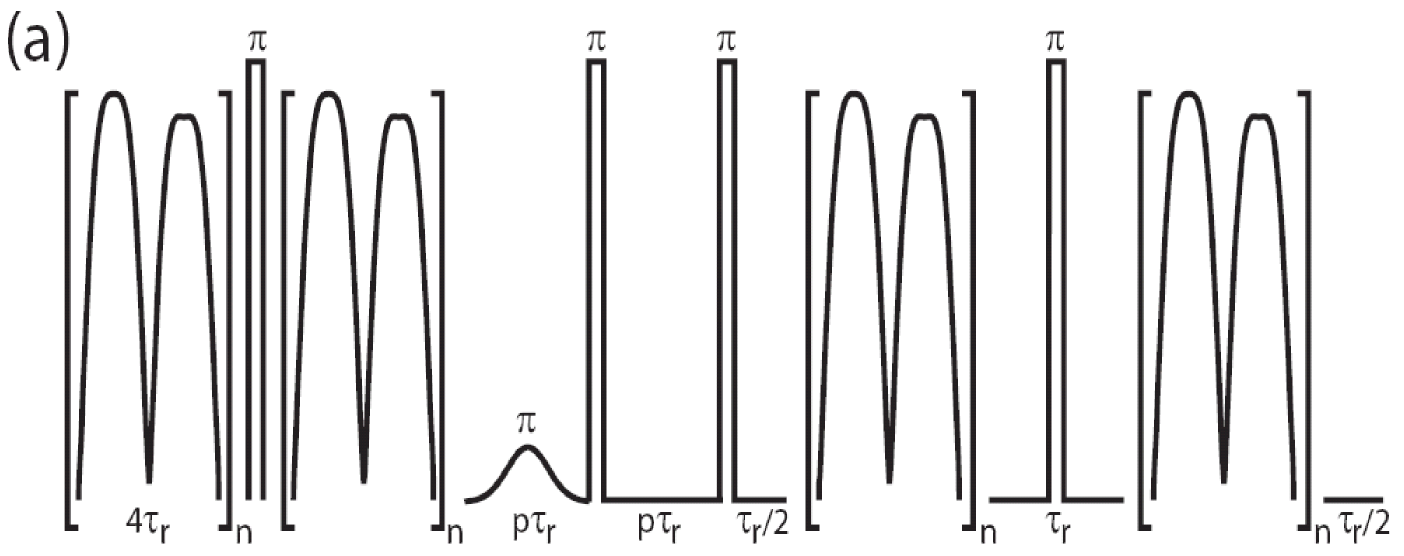
Fig. 2

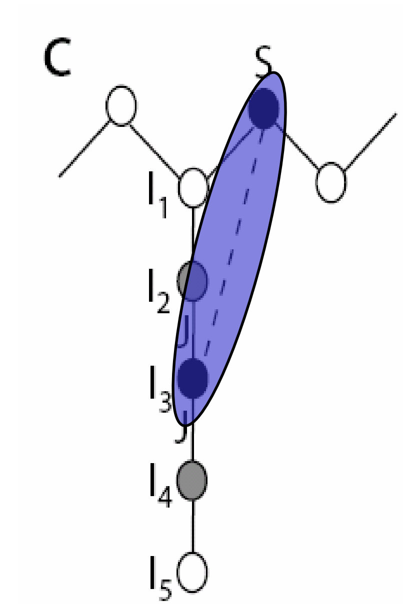



$$C = \omega_r / 4 \quad ; B = 3\omega_r$$

# Triple Oscillating Field techniqUe + Rotor assisted dipolar recoupling (RADAR)




$I_zS_z$ recoupling

RADAR

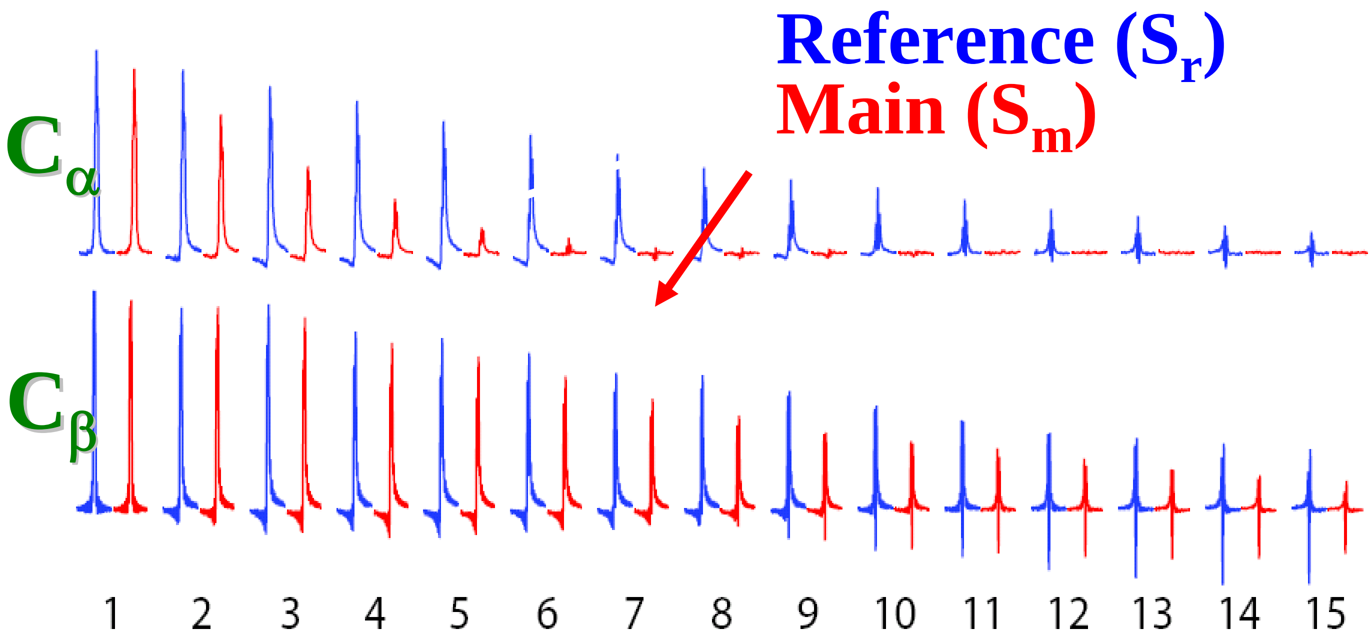
$I_zS_z$ recoupling

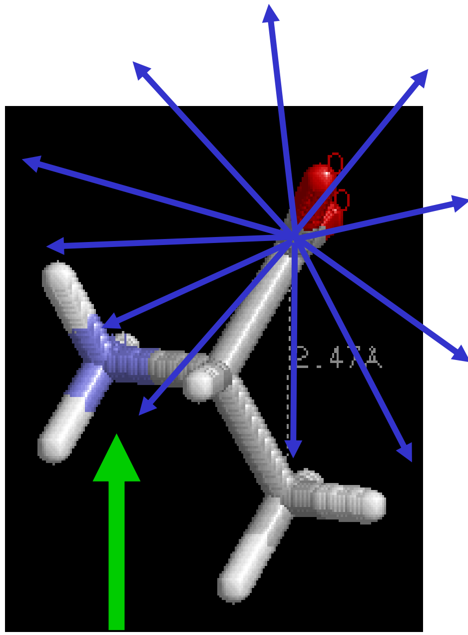
Refocusing of all dipolar couplings

Selective inversion

Dipolar couplings to C' active + surrounding J's active + relaxation

# TOFU + RADAR: U-$^{13}$C,$^{15}$N-*L*-alanine




$C_\beta$

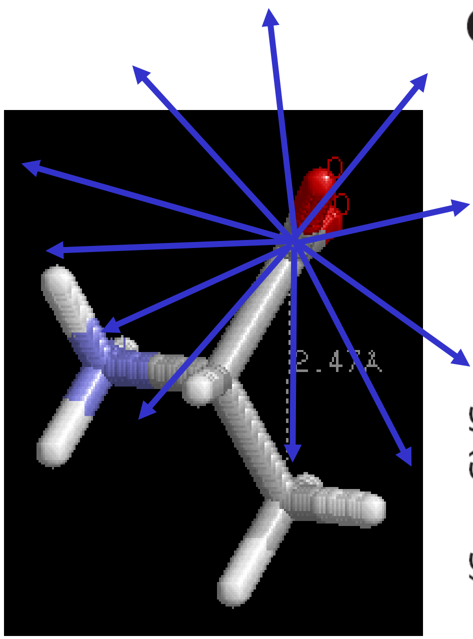
**FRESNEL**

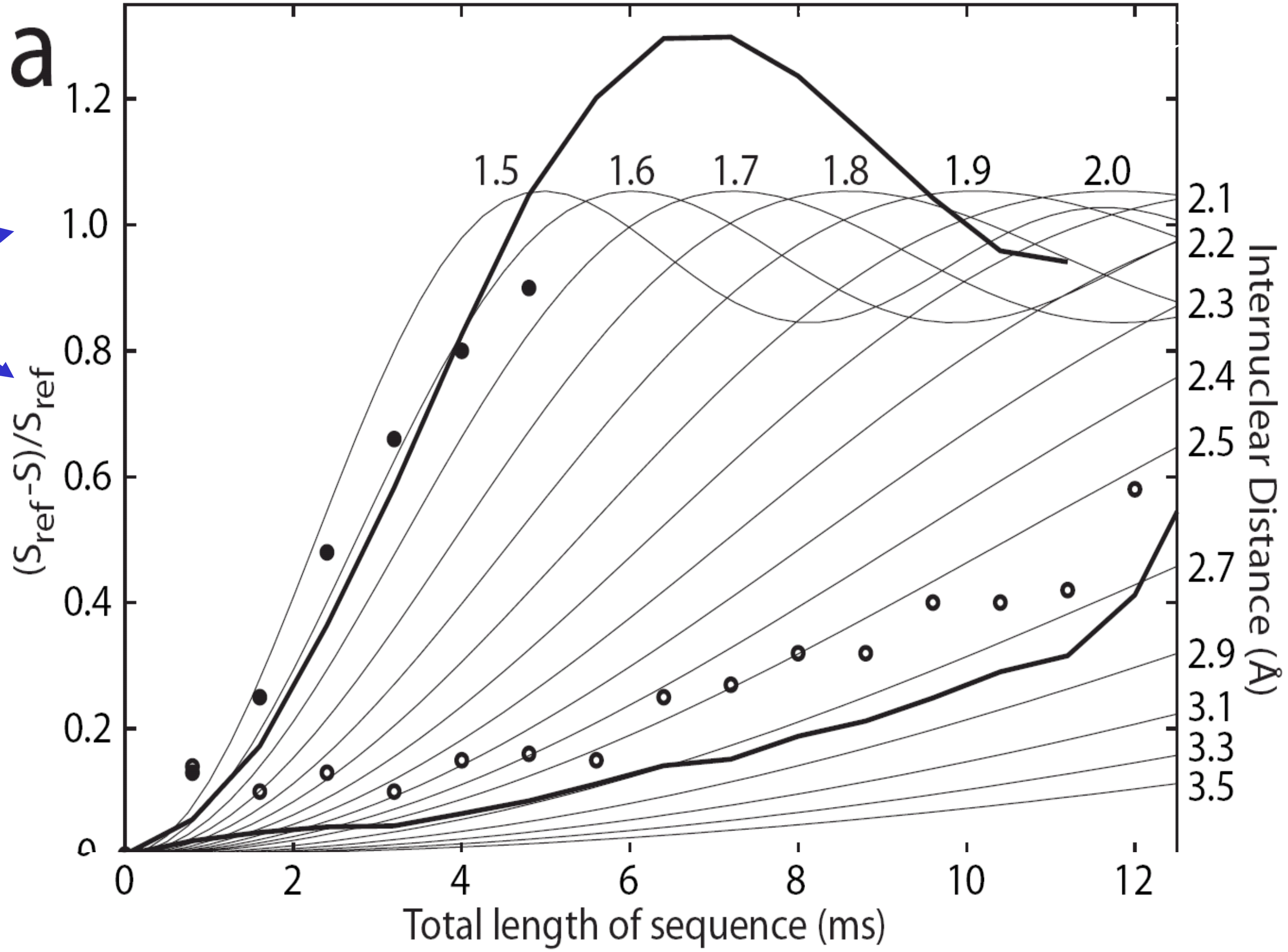
$$\eta(T) = \frac{S_r(t) - S_m(t)}{S_r(t)} = 1 - \int \cos(\omega_k^{DD} T)$$

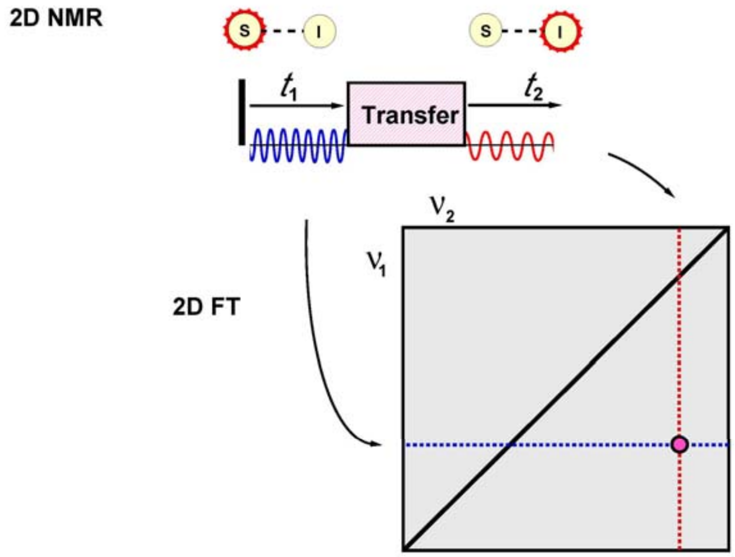
**Easy read-out**

# Accurate distances by ssNMR:

## U-$^{13}$C,$^{15}$N-*L*-alanine






*Heteronuclear Decoupling*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & -J & & \\ J & & -u(t) & \\ & u(t) & & -\omega \\ & & \omega & \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\max \int x_1(\tau)d\tau \qquad \omega \in [-B, B]$$

# *Heteronuclear Decoupling by Multiply Modulated Fields*

$$H = \omega_{\mathrm{I}}\ \mathrm{I_z} + 2\,\pi\ \mathrm{J\ I_z\ S_z} + \omega_S\ \mathrm{S_z}$$

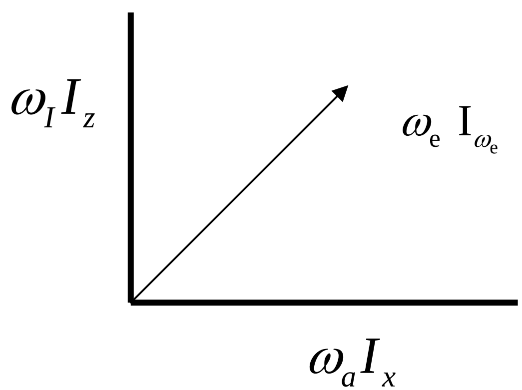
$$H = \omega_{\mathrm{I}}\ \mathrm{I_z} + 2\,\pi\ \mathrm{J\ I_z\ S_z}\ ; \quad \omega_{\mathrm{I}} \in [-B, B]$$

$$H_{rf} = \omega_{\mathrm{a}}\ \mathrm{I_x} + (2\omega_{\mathrm{b}} \sin \nu_b t + 4\omega_c \cos \nu_b \sin \nu_c t + \cdots) I_y$$

$$\longrightarrow \exp(-\nu_b I_{\omega_e}) \longrightarrow$$

$$H = \omega_{\mathrm{e}}\ \mathrm{I}_{\omega_{\mathrm{e}}} + 2\,\pi\ \mathrm{J\ I}_{\omega_{\mathrm{e}}}\ \mathrm{S_z} + \omega_{\mathrm{b}} I_f + 2\omega_c \sin \nu_c t\ I_y$$

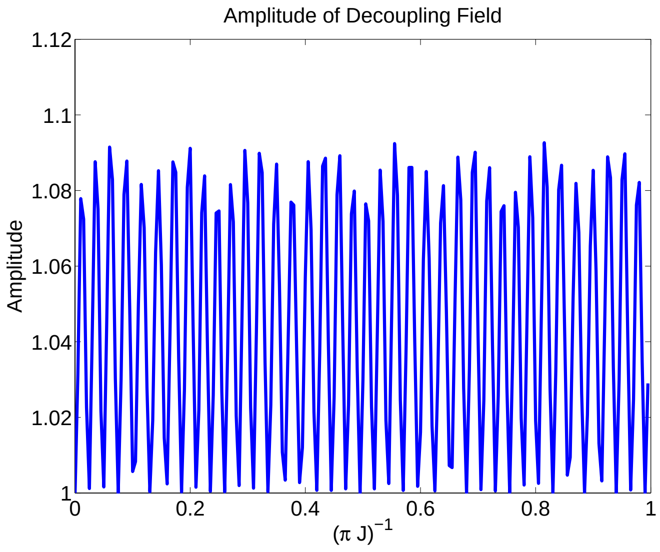
$$I_f = I_{\omega_e} \times I_y$$

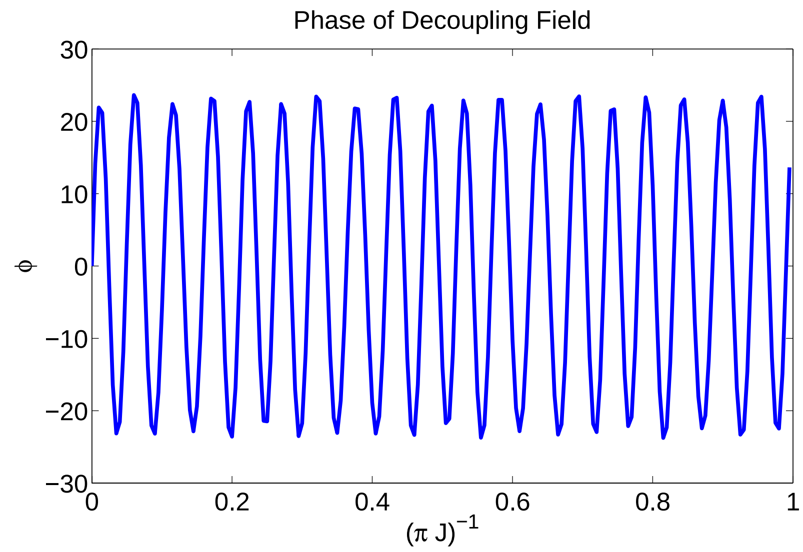


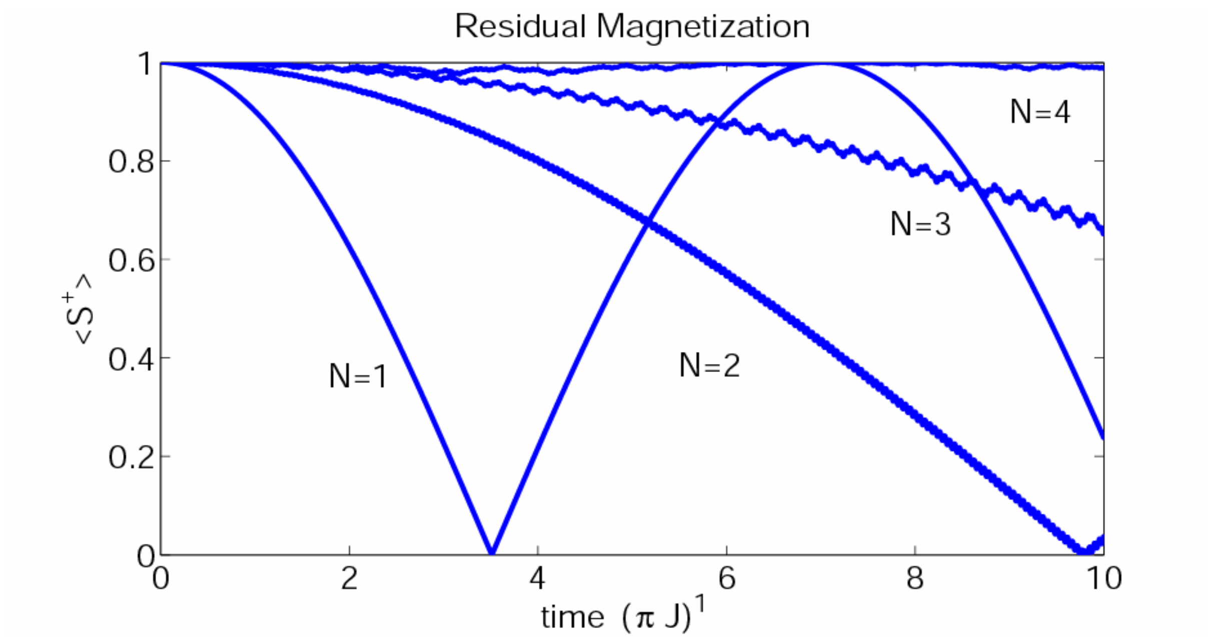

$$\omega_e \in [-C, C]$$

$$\nu_b = \frac{\sqrt{\omega_a^2 + \omega_I^2} + \omega_a}{2}$$

$$C = \frac{\sqrt{\omega_a^2 + \omega_I^2} - \omega_a}{2}$$




$$\frac{A_{eff}}{B} \approx 1$$

$$\frac{(\pi J)}{B} \approx .01$$

# Collaborators

- Steffen Glaser
- Niels Nielsen
- Gerhard Wagner
- Robert Griffin

- James Lin
- Haidong Yuan
- Robert Zeier
- Jr Shin Li
- Jamin Sheriff
- Philip Owrutsky
- Mai Van Do