Finite Control of Quantum Systems

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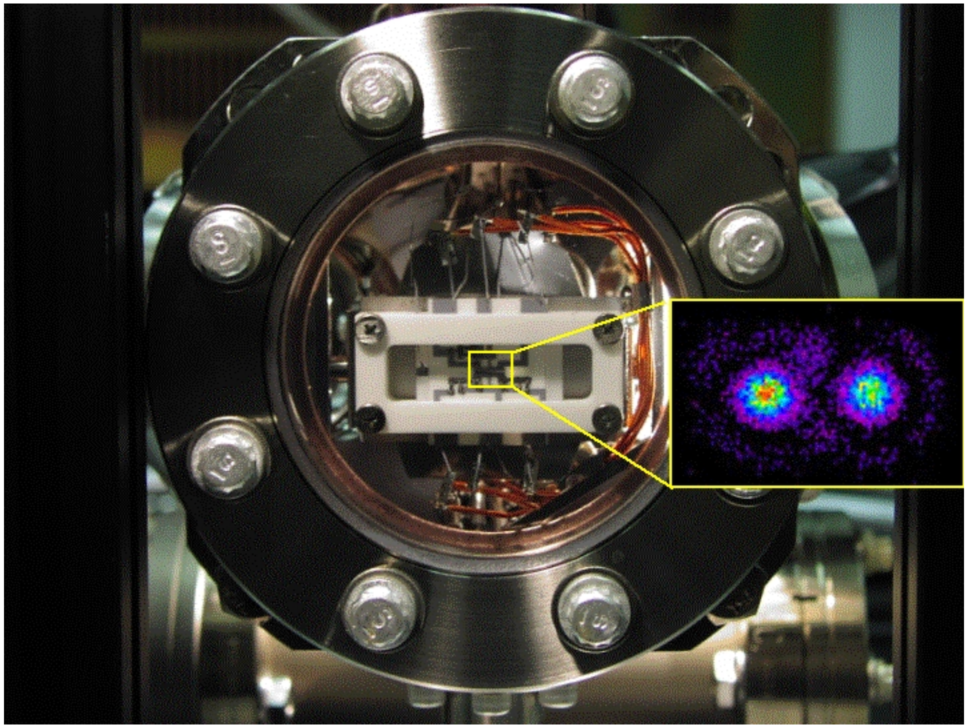
Work with Brockett and Rangan, Bucksbaum, Monroe

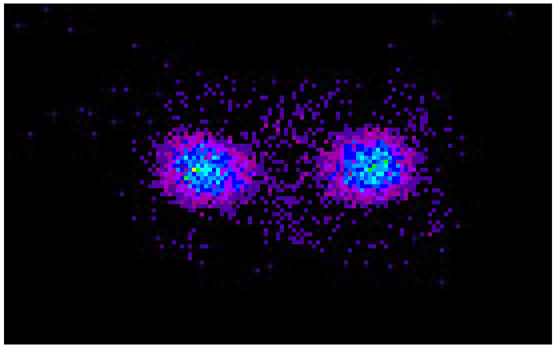
- **Ion traps**
- **Control of Spin/Oscillator Systems in Infinite Dimensions**
- **The Eberly Law method**

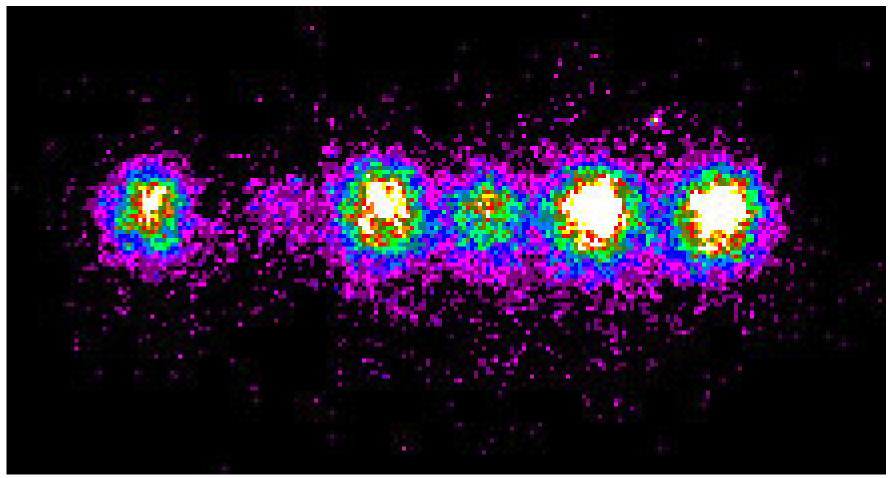
- Trapped Ions

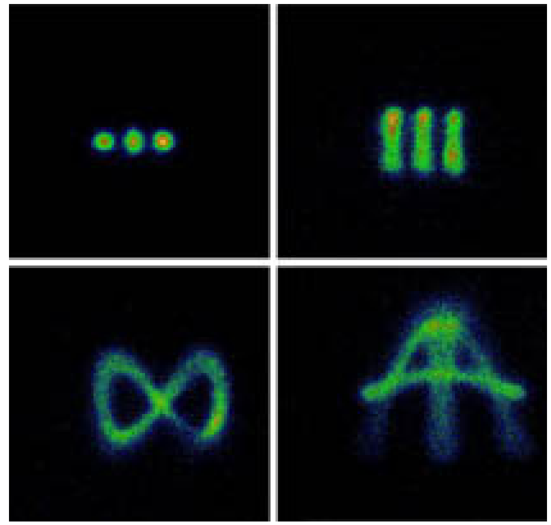
Consider quantum control of a scalable quantum-computing paradigm — a crystal of trapped ions. The two-level atom (qubit) coupled to a harmonic oscillator is an example of a quantum system with an infinitely large number of accessible eigenstates.

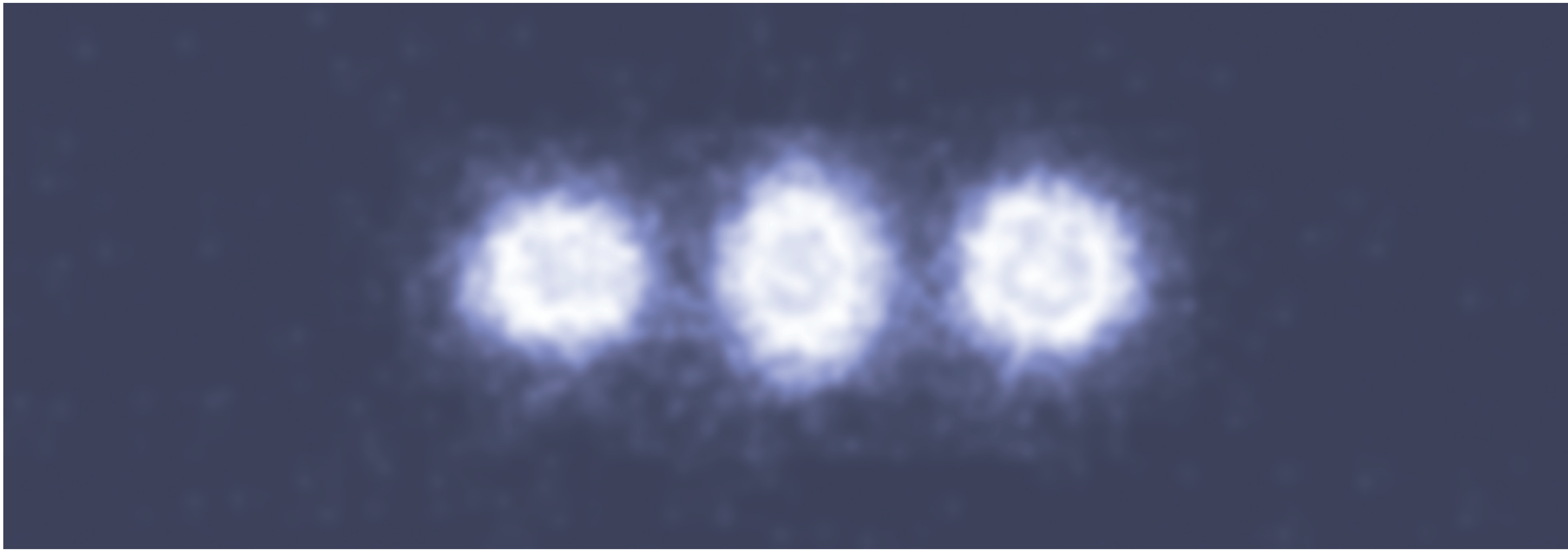
This work was motivated by the need to develop fast control schemes to produce entangled states of qubits. Such entangled states could then lead to interesting quantum states of the coupled spin-motion system.











Related papers:
Rangan, Bloch, Monroe and Bucksbaum, PRL 113004 (2004).

Brockett, Rangan and Bloch, Proc. 42nd CDC (2003).

Rangan and Bloch, JMP 2006.

Bloch, Brockett and Rangan, IEEE Transaction on Automatic Control, to appear.

A trapped-ion qubit is most readily formed of two hyperfine states of a laser-coolable ion, separated by a frequency $\omega_0/2\pi$ in the several GHz range. Qubits are coupled via the vibrational modes of the ions' motion, which can be treated as quantum harmonic oscillators. The quantized vibrational energy levels separated by a frequency $\omega_m/2\pi$ in the MHz range create sidebands in the spectrum of the ion. The hyperfine 'qubit' states are addressed by a pair of optical beams.

Some infinite-dimensional systems can be made to be effectively finite-dimensional by either bandwidth limits imposed by the control fields (Rangan), or by turning off specific transitions in order to truncate the Hilbert space (Rangan, Bloch, Bucksbaum, Monroe), and the controllability of such systems can be analyzed using finite-dimensional methods.

We are interested in the quantum systems that are modelled as finite-dimensional for quantum computing purposes, when in fact they are infinite-dimensional. Much recent work...

•Infinite-Dimensional Controllability

Controllability results for infinite-dimensional systems are seldom just straightforward extensions of the finite-dimensional ones, and in particular this is true for bilinear systems. Recently, there has been significant interest in the class of bilinear systems because of their relevance to quantum control. In the following we illustrate the limitations of applying the tools of finite-dimensional systems analysis to certain classes of infinite-dimensional systems.

•Limitations of Lie Algebraic analysis

Lie algebraic structure often gives us insights into controllability of a quantum system, but for infinite-dimensional systems, insight is limited.

In the well-known example of a resonantly-driven quantum harmonic oscillator (the evolution is given by

$$\frac{\partial\psi}{\partial t} = \left(\omega\frac{i}{2}\left(\frac{\partial^2}{\partial x^2} - x^2\right) - iu(t)x\right)\psi. \tag{0.1}$$

Here, the bilinear control term $u(t)x$ arises because of the dipole interaction between the field and harmonic oscillator.

The two operators of interest, $A = \frac{i}{2}\left(\frac{\partial^2\psi}{\partial x^2} - x^2\right)$ and $B = -ix$ generate a Lie algebra of skew-hermitian operators that is just four-dimensional.

Thus, we expect that the control of this system will be limited. For example, it is well-known that it is not possible to transfer the number state $x(0) = |0\rangle$ to $x(T) = |n\rangle$ for $n > 0$,

Even when we encounter infinite-dimensional systems for which the Lie algebra also is infinite-dimensional the statements one can make about the controllability are also limited. More work required to say with precision exactly what the reachable states are.

•Finite Controllability

Describe an elementary but useful theorem about controllability on finite-dimensional subspaces of a complex Hilbert space.

Definition 0.1 ***We will say that a finite-dimensional system evolving in the space of complex*** n***-vectors*** x***,***

$$\dot{x} = \sum_i u_i G_i x$$

with skew-Hermitian operators G_i***, is* unit vector controllable *if any unit length vector*** x_0 ***can be steered to any second unit length vector*** x_f ***in finite time.***

Theorem 0.2 (Finitely Controllable Infinite Dimensional Systems) ***Consider a complex Hilbert space $\mathcal{X}$ together with a nested set of finite-dimensional subpaces $\mathcal{H} = \{\mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{H}_3 ...\}$. Consider***

$$\dot{x} = (\sum_{i=1}^{m} u_i G_i)x.$$

Assume that $\mathcal{H}_1$ is an invariant subspace for a subset $\mathcal{G}_1$ of the set $\{G_i\}$ and that the system is unit vector controllable on $\mathcal{H}_1$ using only this subset of the G_i. If for each $\mathcal{H}_\alpha$ $\alpha \neq 1$ there is a subset $\mathcal{G}_\alpha$ of $\{G_i\}$ that leaves $\mathcal{H}_\alpha$ invariant and if for any unit vector in $\mathcal{H}_\alpha$ the orbit generated by exp $(\mathcal{G}_\alpha)$ contains a point in one of the lower dimensional subspaces $\mathcal{H}_\beta$ then any unit vector in any of the $\mathcal{H}_i$ can be steered to any other unit vector in any other $\mathcal{H}_j$ using a finite number of piecewise constant controls.

- Remark:

Given a system and a nested set of finite dimensional subspaces it will be said to be finitely controllable *if it can be transfered from any point in one of the subpaces to any other point in that subspace with a trajectory lying entirely within the subspace.*

•Discussion:

Let l_2 denote the Hilbert space of infinite vectors whose entries are square summable which corresponds to the entire state space of our quantum system. Let l_0 denote the subspace consisting of those elements with only a finite number of nonzero entries, which corresponds in our setting to a finite superposition of states.

Suppose A and B are given by

$$A = \begin{bmatrix} B_0 & 0_{22} & 0_{22} & 0_{22} & ... \\ 0_{22} & B_0 & 0_{22} & 0_{22} & ... \\ 0_{22} & 0_{22} & B_0 & 0_{22} & ... \\ 0_{22} & 0_{22} & 0_{22} & B_0 & ... \\ ... & ... & ... & ... & ... \end{bmatrix} ; \tag{0.2}$$

$$B = \begin{bmatrix} 0_{11} & 0_{12} & 0_{12} & 0_{12} & ... \\ 0_{21} & B_0 & 0_{22} & 0_{22} & ... \\ 0_{21} & 0_{22} & B_0 & 0_{22} & ... \\ 0_{21} & 0_{22} & 0_{22} & B_0 & ... \\ ... & ... & ... & ... & ... \end{bmatrix}, \tag{0.3}$$

with 0_{ij} denoting an i by j matrix of zeros and

$$B_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{0.4}$$

The operators A and B leave l_0 invariant but the exponential of their sum does not.

One can provide a direct argument not involving Lie theoretic techniques that shows that any unit vector can be transferred to e_1. The idea, motivated by the analysis of Law and Eberly, is to alternate the use of A and B and reason that if we start with a vector with $x_n \neq 0$ and $x_{m>n} = 0$ we can use a control with $v = 0$ (or $u = 0$ depending on whether n is even or odd) to reduce the vector to one for which x_n is zero, then use a control with $u = 0$ to reduce the vector to one with $x_{n-1} = 0$ without changing x_n, etc.

- **Physical Systems**

Apply the Finite Controllability Theorem to determine the reachable set of states of some infinite-dimensional quantum systems.

- **System 1: Quantum harmonic oscillator Discuss this well-known system first. The controllability algebra is finite-dimensional and, in particular, the system does not satisfy the conditions needed for the application of the Finite Rank Controllability Theorem. Useful for setting up the formalism used in subsequent examples that are finitely controllable.**

If the control is a sinusoidal resonant driving field (of frequency equal to the harmonic oscillator frequency ω_m) as shown in the transfer graph Fig. 0.1, then the evolution is via

$$\frac{\partial \psi}{\partial t} = \left(\omega_m \frac{i}{2} \left(\frac{\partial^2}{\partial x^2} - x^2 \right) - iu(t)x \right) \psi. \tag{0.5}$$

Here, the control term $u(t)x$ arises because of the dipole interaction between the field and harmonic oscillator. The operators of interest are $A = \frac{i}{2}\left(\frac{\partial^2}{\partial x^2} - x^2\right)$ and $B = -ix$. A and B generate a Lie algebra of skew-hermitian operators that is just four-dimensional ($C = [A, B] = \frac{\partial}{\partial x}$, $D = [B, C] = iI$, where I is the identity operator). This in itself tells us that the resonantly driven harmonic oscillator is not controllable.

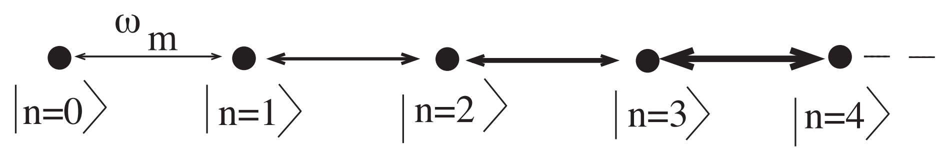


Figure 0.1: Graphical representation of the quantum harmonic oscillator driven by a sinusoidal resonant field. Note that while the strengths of the transition couplings increase as the square root of the quantum number n as shown by the boldness of the connections between energy levels, the transition frequency between each level is the same.

As is well-known, the spectrum of A is discrete.

If we describe the evolution in terms of an eigenfunction expansion, with the basis being $|n\rangle$'s, the eigenfunctions of $\partial^2/\partial x^2 - x^2$, then the evolution is via

$$\begin{aligned} \dot{x}_n &= -i\omega_m(n+\frac{1}{2})x_n \\ &\quad -u(t)\frac{i}{\sqrt{2}}(\sqrt{n-1}x_{n-1} - \sqrt{n}x_{n+1}). \end{aligned} \tag{0.6}$$

Although the eigenstates of the harmonic oscillator can be written as an infinite set of nested finite subspaces, it is seen that the operator B connects space $\mathcal{H}_i$ to *both* $\mathcal{H}_{i-1}$ and $\mathcal{H}_{i+1}$. Thus finite superpositions of eigenstates may not be reached by resonantly driving the harmonic oscillator, consistent with the fact that the requirements of the Finite Controllability Theorem are not met. Physically, this is due to the degeneracy of spacings between the eigenstates and the fact that the control vector field simultaneously illuminates all states.

- **System 2: Spin-half particle in a quadratic potential**

In contrast to the harmonic oscillator, the model of a spin-half particle coupled to a harmonic oscillator with suitable controls turns out to be finitely controllable. This model is a good representation of an ion with two essential internal states trapped in a quadratic potential.

Show below that this system satisfies the conditions of the Finite Controllability Theorem.

Moreover, one can also provide an algorithm for explicit control.

The spin-$\frac{1}{2}$ model represents a two-level atomic ion with an energy splitting $\hbar\omega_0$, where the frequency $\omega_0/2\pi$ is in the several GHz range. The atomic levels are coupled to the motion of the ion in a harmonic trap. These quantized vibrational energy levels are separated by a frequency $\omega_m/2\pi$ in the MHz range.

Law and Eberly showed by coupling the harmonic oscillator with a two-level system it is possible to arrive at a system which is much more controllable than the harmonic oscillator. At an intuitive level, this can be seen simply as a consequence of the fact that the addition of a spin degree of freedom breaks the infinite degeneracy associated with the harmonic oscillator and allows the system to resonate with more than one frequency. This allows the transfer of population from any eigenstate to any other eigenstate by sequentially applying the two frequencies.

An eigenstate of the spin-half system coupled to a quantum harmonic oscillator is denoted by $|S, n\rangle$, where the first index refers to the "spin" state of the system, and the second index is the number state of the harmonic oscillator. An applied field causes transitions between the eigenstates of the coupled spin-oscillator system. A monochromatic field of angular frequency $\omega = \omega_0$ causes resonant transitions between states $|\downarrow, n\rangle$ and $|\uparrow, n\rangle$ (carrier or spin-flip transitions). A monochromatic field of angular frequency $\omega = \omega_0 - \omega_m$ causes resonant transitions between states $|\downarrow, n\rangle$ and $|\uparrow, n-1\rangle$, i.e., produces so called red sideband (that is with angular frequency $\omega = \omega_0 - \omega_m$) transitions.

These transitions are graphically depicted in Fig. 0.2 with the thickness of the edges qualitatively representing the strength of the coupling between the states. When both fields (carrier and red sideband) are applied *simultaneously*, the eigenstates of the system are sequentially connected. Therefore, we look at the trapped-ion model controlled only by these two fields.

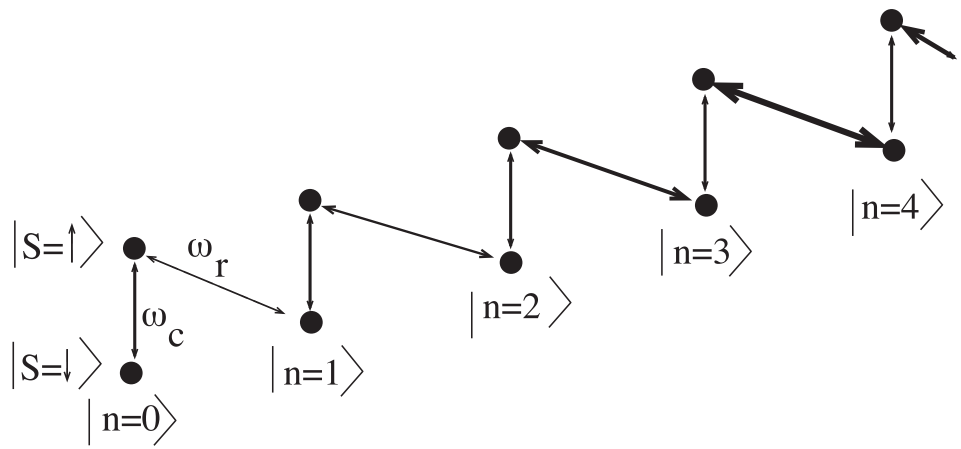


Figure 0.2: Graphical representation of the coupled spin-half quantum harmonic oscillator system driven by sinusoidal resonant fields of angular frequency ω_c and ω_r as shown. When $\eta \ll 1$, the strengths of the ω_c transition couplings are independent of the harmonic oscillator quantum number n, whereas the strengths of the ω_r transition couplings increase as the square root of n as shown by the boldness of the coupling lines. Note that there is no direct coupling between two consecutive oscillator states with fixed spin.

Now we write the evolution equation of the spin-half coupled to harmonic oscillator driven by two fields that drive the carrier and red sideband transitions. The amplitudes corresponding to the fields that cause the carrier and red transitions are dubbed E_c and E_r respectively. In the interaction picture and in the energy eigenbasis, the evolution equation is written as

$$\dot{Y} = (u(t)B_c + v(t)B_r)Y. \tag{0.7}$$

The controls $u(t)$ and $v(t)$ are related to the applied fields via the equations

$$u(t) = c_1 E_c(t) = 0.25\mu \exp(-\eta^2/2) E_c(t), \tag{0.8}$$

$$v(t) = c_2 E_r(t) = 0.25\eta\mu \exp(-\eta^2/2) E_r(t). \tag{0.9}$$

Here η, the so-called Lamb-Dicke parameter, is the product of k, wave vector of the light, and x_0, the amplitude of the zero-point motion of the particle in the harmonic potential (or the spatial extent of the ground state harmonic oscillator wave function). By ordering the eigenstates as $|\uparrow, 0\rangle,\ |\uparrow, 1\rangle,\ \ldots, |\downarrow, 0\rangle,\ |\downarrow, 1\rangle,\ \ldots$, the control matrices are written as

$$B_c = \left(\begin{array}{c|c} 0 & iL_0 \\ \hline iL_0^T & 0 \end{array}\right). \tag{0.10}$$

$$B_r = \left(\begin{array}{c|c} 0 & L_1 \\ \hline -L_1^T & 0 \end{array}\right). \tag{0.11}$$

The upper-triangular matrices L_0 and L_1 are defined as

$$L_0 = \begin{pmatrix} L_0(\eta^2) & 0 & 0 & \dots \\ 0 & L_1(\eta^2) & 0 & \dots \\ 0 & 0 & L_2(\eta^2) & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{0.12}$$

$$L_1 = \begin{pmatrix} 0 & L_0^{(1)}(\eta^2) & 0 & \dots \\ 0 & 0 & L_1^{(1)}(\eta^2) & \dots \\ 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{0.13}$$

•Controllability: Lie Algebra

The control of the trapped-ion system is often studied in two different limiting cases - one in which the extent of zero-point motion of the spin-half particle in the harmonic potential x_0 is much smaller than the wavelength of the applied light $2\pi/k$, i.e., $\eta \ll 1$ (the Lamb-Dicke limit), and the other in which $\eta \simeq 1$ (beyond the Lamb-Dicke limit). The case in which $\eta \simeq 1$ is more general than the case of the Lamb-Dicke limit, but requires a more sophisticated analysis. We study initially the Lamb-Dicke limit in which the Lamb-Dicke parameter $\eta \ll 1$.

The terms in equations (0.12) and (0.13) are expanded to first order in η. The control Hamiltonians can then be expressed in operator form as

$$B_c = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \text{ and} \tag{0.14}$$

$$B_r = \eta \begin{bmatrix} 0 & a \\ -a^\dagger & 0 \end{bmatrix}, \tag{0.15}$$

In order to compute the Lie algebra, let us consider T, an operator acting on a complex Hilbert space. We associate with T a skew-hermitian operator acting on $\mathcal{H} \oplus \mathcal{H}$ defined by

$$J(T) = \begin{bmatrix} 0 & T \\ -T^\dagger & 0 \end{bmatrix}. \tag{0.16}$$

For convenience, let $K(T)$ be another operator defined in a similar way as

$$K(T) = \begin{bmatrix} T & 0 \\ 0 & -T \end{bmatrix}. \tag{0.17}$$

The control operators we are interested in for the purposes of determining the structure of the Lie algebra are given by $B_c = J(iI)$ and $B_r = \eta J(a)$. We have

Lemma 0.3 ***The Lie algebra generated by $J(iI)$ and $J(T)$ includes the operators***

$$J(W^{2p});\ p = 1, 2, 3, \cdots\ ;\ K(W^{2p+1});\ p = 0, 1, 2, \cdots\ , \tag{0.18}$$

where, $W = i(T + T^{\dagger})$***.***

Proof:

A calculation shows that $[J(T), J(iI)] = K(W)$ and further, $[J(iI), K(W)] = -2iJ(W)$. We can then check that

$$ad^p_{J(W)}(K(W)) = (-2)^p \left\{ \begin{array}{l} J(W^{p+1}),\ \text{if p is odd} \\ K(W^{p+1}),\ \text{if p is even} \end{array} \right\}. \tag{0.19}$$

These calculations make it clear that if the powers of W are independent then $J(iI)$ and $J(T)$ do not generate a finite-dimensional algebra. Thus if T is nonzero only on the diagonal immediately above the main diagonal (which is true for the operator a), and if every term on this upper-diagonal is nonzero, then the successive powers of W are independent and the algebra is infinite-dimensional.

This is the case for the coupled spin-half harmonic oscillator system.

Of course, this calculation only shows that this system, unlike the harmonic oscillator, does not generate a finite-dimensional controllability Lie algebra. More work is required to say with precision exactly what the reachable states are.

Note: In the case where the Lamb-Dicke limit does not apply, the Lie algebra will still be infinite-dimensional but the terms are more complicated.

•Finite controllability

Now discuss how finite controllability works in this infinite-dimensional setting.

From Fig. 0.2, it is seen that the sequentially connected eigenstates can be looked at as an infinite set of finite-dimensional subspaces with the ground state $|\downarrow, 0\rangle$ being equal to $\mathcal{H}_1$. Further, when operators B_c and B_r are applied *sequentially*, each subspace H_i can be transferred to $\mathcal{H}_{i-1}$. Thus the criteria for finite controllability are met. By sequential application of the two operators, any finite superposition of eigenstates can be transferred to the ground state in finite time.

The application of these statements to the spin-half in quadratic potential example is best understood by writing the control matrices B_c **and** B_r **in a re-ordered basis as follows: The eigenstates can be ordered as** $|\uparrow, 0\rangle,\ |\uparrow, 1\rangle,\ \ldots, |\downarrow, 0\rangle,\ |\downarrow, 1\rangle,\ \ldots$**.**

In the interaction picture, the Schrödinger equation is written as

$$\dot{Y} \;=\; (u(t)B_c + v(t)B_r)Y, \tag{0.20}$$

where $u(t)$ **and** $v(t)$ **are defined as before. Then,**

$$B_c \;=\; i\left(\begin{array}{cc|cc|cc|c} 0 & L_0 & 0 & 0 & 0 & 0 & \ldots \\ L_0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \hline 0 & 0 & 0 & L_1 & 0 & 0 & \ldots \\ 0 & 0 & L_1 & 0 & 0 & 0 & \ldots \\ \hline 0 & 0 & 0 & 0 & 0 & L_2 & \ldots \\ 0 & 0 & 0 & 0 & L_2 & 0 & \ldots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

$$B_r = \left(\begin{array}{c|cc|cc|cc}
0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\hline
0 & 0 & L_0^{(1)} & 0 & 0 & 0 & \dots \\
0 & -L_0^{(1)} & 0 & 0 & 0 & 0 & \dots \\
\hline
0 & 0 & 0 & 0 & L_1^{(1)} & 0 & \dots \\
0 & 0 & 0 & -L_1^{(1)} & 0 & 0 & \dots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right) .$$

L_i's and $L_i^{(1)}$'s are Laguerre polynomials of the zeroth and first order, all with argument η^2.

•Explicit finite controllability scheme

The property that both control vector fields are never used simultaneously is exploited by Law and Eberly and Kneer and Law in order to devise a explicit scheme for the production of a finite superposition of eigenstates from another finite superposition in the control of a spin-half particle coupled to a harmonic oscillator (in the Lamb-Dicke limit). It shows that if x can be transferred to y by a series of such "single nonzero u_i" moves then the transfer from y to x is also possible.

Specifically the Law-Eberly scheme to transfer any eigenstate $|i\rangle$ to any other eigenstate $|j\rangle$ involves the alternate use of transitions generated by spin reversal (π-pulses of E_c) and transitions generated by π-pulses of E_r which convert from a state in which the oscillator has energy E_i and spin down to a state in which the energy of the oscillator in altered by one unit and the spin is flipped as well .

For example suppose we wish to drive a state from the $|\downarrow, n\rangle$ to $|\uparrow, n-2\rangle$. This can be done using B_r to drive the system from $|\downarrow, n\rangle$ to $|\uparrow, n-1\rangle$, B_c to drive the system from $|\uparrow, n-1\rangle$ to $|\downarrow, n-1\rangle$ and finally B_r to go from $|\downarrow, n-1\rangle$ to $|\uparrow, n-2\rangle$.

We note that this scheme works both in the Lamb-Dicke limit and beyond the Lamb-Dicke limit. In the Law-Eberly scheme, the π-pulses of E_c are all of the same time duration because in the Lamb-Dicke limit, all the carrier transitions are equally strong. However, the coupling strengths of the red-sideband transitions are proportional to $\sqrt{n}$, and therefore the π-pulses of E_r are shorter in duration as eigenstates of higher n are addressed. In order to generate an arbitrary superposition of a finite number of eigenstates, starting from another arbitrary superposition, an additional trick is to go through the ground state of the system which acts as a "pass state".

It is possible to provide an explicit algorithm which will drive the system from any finite superposition to any other finite superposition.

To prepare an arbitrary finite superposition, the simplest path is to take the system through the ground state. One assumes that the desired state is the initial state and then designs a sequence of alternating pulses of the E_c and E_r fields that would take this state to the ground state $|\downarrow, 0\rangle$. The actual sequence that produces the superposition is the time-reversed sequence that was designed.

For example, if the desired superposition is $(|\uparrow, 3\rangle + |\downarrow, 2\rangle)/\sqrt{2}$**, the sequence of pulses that will transfer this state to the ground state is** $E_c^{(1)}(\pi)\ E_r^{(2)}(\phi_2)\ E_c^{(3)}(\phi_3)\ E_r^{(4)}(\phi_4)\ E_c^{(5)}(\phi_5)\ E_r^{(6)}(\phi_6)\ E_c^{(7)}(\phi_7)$**. The action of each pulse is the following:** $E_c^{(1)}$ **is a** π **pulse of the carrier field that moves the state** $|\uparrow, 3\rangle$ **to** $|\downarrow, 3\rangle$**. (Simultaneously, the population in** $|\downarrow, 2\rangle$ **is transferred to** $|\uparrow, 2\rangle$**).** $E_r^{(2)}$ **is a pulse of the red-sideband field that moves between the states** $|\downarrow, 3\rangle$ **and** $|\uparrow, 2\rangle$**. Since there is already a superposition of the two states, the duration of the red-sideband field is shorter than that of a** π**-pulse. Simultaneously, a superposition of** $|\downarrow, 2\rangle$ **and** $|\uparrow, 1\rangle$ **is created. The next transition** $E_c^{(3)}(\phi_3)$ **transfers population between** $|\uparrow, 2\rangle$ **and** $|\downarrow, 2\rangle$**, and again is shorter than a** π **pulse. This sequence progresses till all the population is in** $|\downarrow, 0\rangle$**. The actual sequence is the time-reversed sequence of the one that is described above — this creates the desired superposition from the initial ground state.**

If one were to transfer an arbitrary initial superposition to an arbitrary final superposition of eigenstates, one employs the above algorithm twice. The sequences A and B that take the system from the initial and final superpositions respectively to the ground state are first calculated. Then the sequence A is first applied taking all the population to the ground state. The time time-reversed sequence of B is then applied which takes the population to the desired final superposition. Clearly, this scheme works in finite time only if the initial and final states are both superpositions of a *finite* number of states.

Note that finite superpositions are dense in the Hilbert space of all possible states. Hence from our Lie algebra analysis and the use of the Law-Eberly algorithm we have

Proposition 0.4 ***The span of the Lie algebra generated by the operators B_c and B_r for the quantum control system in Eq. (0.7) is infinite-dimensional and the reachable set, which is dense in the Hilbert space of all states, includes all finite superpositions.***

Note also that the proof of controllability that Law and Eberly give of what they term "arbitrary control" might be more accurately described as demonstrating that any state in l_0 can be mapped to any other state in l_0, staying within l_0