# Black Hole Entropy, *q*-Deformed *2d* YM, and Non-Perturbative Topological Strings

Hirosi Ooguri (Caltech)

based on:

Strominger, Vafa + H.O. hep-th/0405146

Aganagic, Saulina, Vafa + H.O. to appear

# Topological String Theory

(1) Start with a Calabi-Yau 3-fold.

$\mathcal{N}=2$ non-linear sigma-model

$$\Sigma_g \to X$$

Riemann surface $\qquad$ $CY_3$

(2) Topologically twist the sigma-model.

(3) Couple it to the topological gravity on $\Sigma_g$ .

$$F_g = \int_{\mathcal{M}_g} \left\langle \left| \prod_{i=1}^{3g-3} (\eta_i, G^-) \right|^2 \right\rangle$$

Beltrami differential $\qquad$ supercurrent

$$Z^{top}_{pert.} = \exp\left[ \sum_{g=0}^{\infty} \lambda^{2g-2} F_g \right]$$

The series is typically not summable.

# Applications of Topological String Theory

Topological string computes superpotential terms in the low energy effective theory of Type II string compactified on a Calabi-Yau 3-fold.

BCOV/1994

Matrix model computation of
4d gauge theory superpotentials

Dijkgraaf + Vafa/2002

Microscopic understanding of large N duality between gauge theory and string theory.

Gopakumar + Vafa/1998, Vafa + H.O./1999, 2002

Relations to integrable models in various dimensions:

N=4 super-YM in 4d,
Chern-Simons gauge theory in 3d,
pure YM in 2d,
non-critial string theory in 1d, ...

Applications to mathematics:

Gromov-Witten invariants,
homological mirror symmetry,
knot invariants, ...

# What do we wish for a non-perturbative definition of topological string theory?

We want "?" that reproduces the perturbative series

$$Z^{top}_{pert.} = \exp\left[ \sum_{g=0}^{\infty} \lambda^{2g-2} F_g \right]$$

when expanded in the string coupling constant.

This is a minimum requirement.
Such an object may not be unique.

"?" has to be physically meaningful.

Is there a physical reason
for its non-perturbative existence?

"?" has to be useful.

Does it Come with a method to compute it?

# Black Hole Entropy

Consider type IIA string theory on a Calabi-Yau 3-fold.

Wrapping D branes on cycles ==> BPS black holes in 4d

$p^0$ = magnetic D6 charges on CY
$p^i$ = magnetic D4 charges on 4-cycles $\alpha^i$ in CY
$q_i$ = electric D2 charges on 2-cycles $\beta_i$ in CY
$q_0$ = electric D0 charges

$(i = 1, \dots, h^{1,1})$

$$Z_{BH} = \sum_q \Omega(p,q) \exp(-q_0\phi^0 - q_i\phi^i)$$

$\Omega(p,q) =$ Witten index of quantum states of the black hole

Conjecture (Strominger, Vafa + H.O.)

$$Z_{BH} = |Z^{top}(X)|^2$$

$$\left( \begin{array}{c} X^\mu = p^\mu + \frac{i}{\pi}\phi^\mu \\ \mu = 0, 1, \dots, h^{1,1} \end{array} \right)$$

Black hole partition function:

$$Z_{BH} = \sum_{q} \Omega(p,q)\, e^{-q\cdot\phi}$$

$p$ : magnetic charges of the black hole

$\phi$ : chemical potentials for electric charges

Perturbative topological string partition function:

$$Z^{top}_{pert.} = \exp\left[\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t)\right]$$

$\lambda = \frac{4\pi i}{X^0}$ : string coupling constant

$t^i = \frac{X^i}{X^0}$ : Kaehler moduli $(i = 1, \cdots, h^{1,1})$

Black Hole Charges <==> Calabi-Yau Moduli.

$$X^\mu = p^\mu + \frac{i}{\pi}\phi^\mu \quad (\mu = 0, 1, \cdots, h^{1,1})$$

$$Z_{BH}(p,\phi) = \left| Z^{top}(X) \right|^2$$

# Why?

The perturbative topological string amplitudes give low energy effective action terms.

BCOV/1994

It turns out that these are the terms that are relevant in computing perturbative string corrections to the Bekenstein-Hawking entropy formula.

Lopez-Cardoso, de Wit + Mohaupt/1998-99

Define $\mathcal{F} = \log |Z^{top}_{pert}(X)|^2$

($X^\mu$ : $\mathcal{N}=2$ chiral superfield)

Black Hole Attractor Eguations:

$$X^\mu = p^\mu + \frac{i}{\pi}\phi^\mu, \quad q_\mu = \frac{\partial}{\partial\phi^\mu}\mathcal{F}(p,\phi)$$

Black Hole Entropy (all order in string perturbation):

$$S_{BH}(p,\phi) = q_\mu \phi^\mu + \mathcal{F}(p,\phi)$$

This is the Legendre transformation:

$$q \longleftrightarrow \phi$$

We will test this conjecture for some class of Calabi-Yau's for which both $Z_{BH}$ and $Z^{top}_{pert.}$ are computable.

For these Calabi-Yau's:

$Z_{BH}$ is a partition function of the N=4 super Yang-Mills on a 4-manifold

==> 2d Yang-Mills with *q*-deformation

==> Gross-Taylor type factorization at large *N* with Omega points

$$Z_{BH} = \sum_{m=-\infty}^{\infty} \sum_{R_1, \cdots, R_{|2g-2|}} Z^{top}_{R_1 \cdots R_{|2g-2|}}(t + pm\lambda) \cdot \bar{Z}^{top}_{R_1 \cdots R_{|2g-2|}}(\bar{t} - pm\lambda)$$

representation of $SU(\infty)$

$Z^{top}_{0 \cdots 0}$ : closed topological string amplitude

$Z_{R_1 \cdots R_{|2g-2|}}$ with $R_i \neq 0$ : $CY_3$ with D branes

$$CY_3 : \mathcal{O}(-p) \oplus \mathcal{O}(p+2g-2) \to \Sigma_g$$

Two line bundles of degrees *-p* and *p+2g-2* over a genus *g* Riemann surface

The total space is a Calabi-Yau manifold.

$Z^{top}_{pert.}$ Topological string amplitudes on this CY was recently computed to all order in the perturbative expansion.

Bryan + Pandharipande/2004

$Z_{BH}$ Consider *N* D4 branes on the total space of the degree *-p* bundle over the Riemann surface.

$$\mathcal{O}(-p) \to \Sigma_g$$

$$X^0 = \frac{4\pi i}{\lambda} \quad (\text{no } D_6 \text{ charge})$$

$$X^1 = \underbrace{(p+2g-2)N}_{D_4 \text{ charge}} + \frac{i}{\pi}\phi^1$$

We want to compute the partition function of the N=4 super Yang-Mills on this 4-manifold.

## $\mathcal{N}=4$ SYM on $\mathcal{O}(-p) \to \Sigma_g$

$$S = \frac{i}{2\lambda} \int \mathrm{tr} \left( F \wedge F + 2\theta F \wedge k \right)$$

$k$ : Kähler form on $\Sigma_g$

$\Downarrow$ Equivariant reduction on the fiber with respect to the $U(1)$ rotation.

## 2d YM with $q$-deformation

$$S_{qYM} = \frac{i}{\lambda} \int_{\Sigma_g} \mathrm{tr} \left( \Phi F + \theta \Phi + i \frac{p}{2} \Phi^2 \right)$$

$q$-deformed since $\Phi$ is periodic.

( $e^{\Phi} = e^{\oint A}$ is a good variable. )

$$q = e^{-\lambda}$$

# 2d $q$-YM partition function

$$q = e^{-\lambda}$$

$$Z_{BH} = S^{2-2g} \sum_{R} (\dim_q R)^{2-2g}\, q^{\frac{p}{2} C_2} e^{i\theta C_1}$$

$R = [R_1, \cdots, R_N]$ : representation of $U(N)$

$$C_2 = \sum_i R_i (R_i - 2i + 1) + N R_i$$

$$C_1 = \sum_i R_i$$

$$\dim_q R = \prod_{i<j} \frac{[R_i - R_j - i + j]_q}{[i-j]_q}$$

$$\left(\text{where } [m]_q = q^{m/2} - q^{-m/2}\right)$$

$$S = q^{\rho^2} \prod_{i<j} [i-j]_q \quad , \quad \vec{\rho} : \text{Weyl vector}$$

The chemical potentials are given by

$$\phi^0 = \frac{4\pi^2}{\lambda} \quad , \quad \phi^1 = \frac{2\pi\theta}{\lambda}$$

The above expansion is not in the form expected for $Z_{BH}$

$Z_{BH}$ can be brought into the form expected from the black hole state counting as well as from the instanton expansion of the N=4 SYM:

$$Z_{BH} = \sum_{q} \Omega(p,q)\, e^{-q\cdot\phi}$$

by the S-duality transformation, $\lambda \to \frac{1}{\lambda}$.

The resulting expression reproduces mathematical facts about cohomologies on the moduli space of U(N) instantons on the 4-manifold: $\mathcal{O}(-p) \to \Sigma_g$

For $\mathcal{O}(-2) \to \mathbb{P}^1$,
the cohomologies of the moduli space of of U(N) instantons make representations of the affine SU(2) Lie algebra of level N. (Nakajima)

Thus, $Z_{BH}$ should be expressed in terms of the SU(2) affine Lie algebra characters and indeed it does.

$Z_{BH}$ also reproduce the blow-up formula in the case of $\mathcal{O}(-1) \to \mathbb{P}^1$. (Yoshioka)

# Large N Limit

$$Z_{BH} = S^{2-2g} \sum_{R} (\dim_q R)^{2-2g}\, q^{\frac{p}{2} C_2} e^{i\theta C_1}$$

$$\sim \sum_{m=-\infty}^{\infty} \sum_{R_i} Z_{R_1 \cdots R_{|2g-2|}} (\lambda ; t + p\lambda m)$$

$$\times \bar{Z}_{R_1 \cdots R_{|2g-2|}} (\lambda ; \bar{t} - p\lambda m)$$

$$Z_{R_1 \cdots R_{2g-2}} (\lambda ; t)$$

$g > 1$ case

$$= C(t) \sum_{R} q^{\frac{1}{2}(p+2g-2) K_R} e^{-t|R|}$$

$$\times \frac{W_{RR_1} \cdots W_{RR_{2g-2}}}{(W_{R0})^{4g-4}}$$

where

$$W_{R_1 R_2}(q) = \lim_{N \to \infty} S_{R_1 R_2}(q, N)$$

modular $S$-matrix of the WZW model

The chiral components:

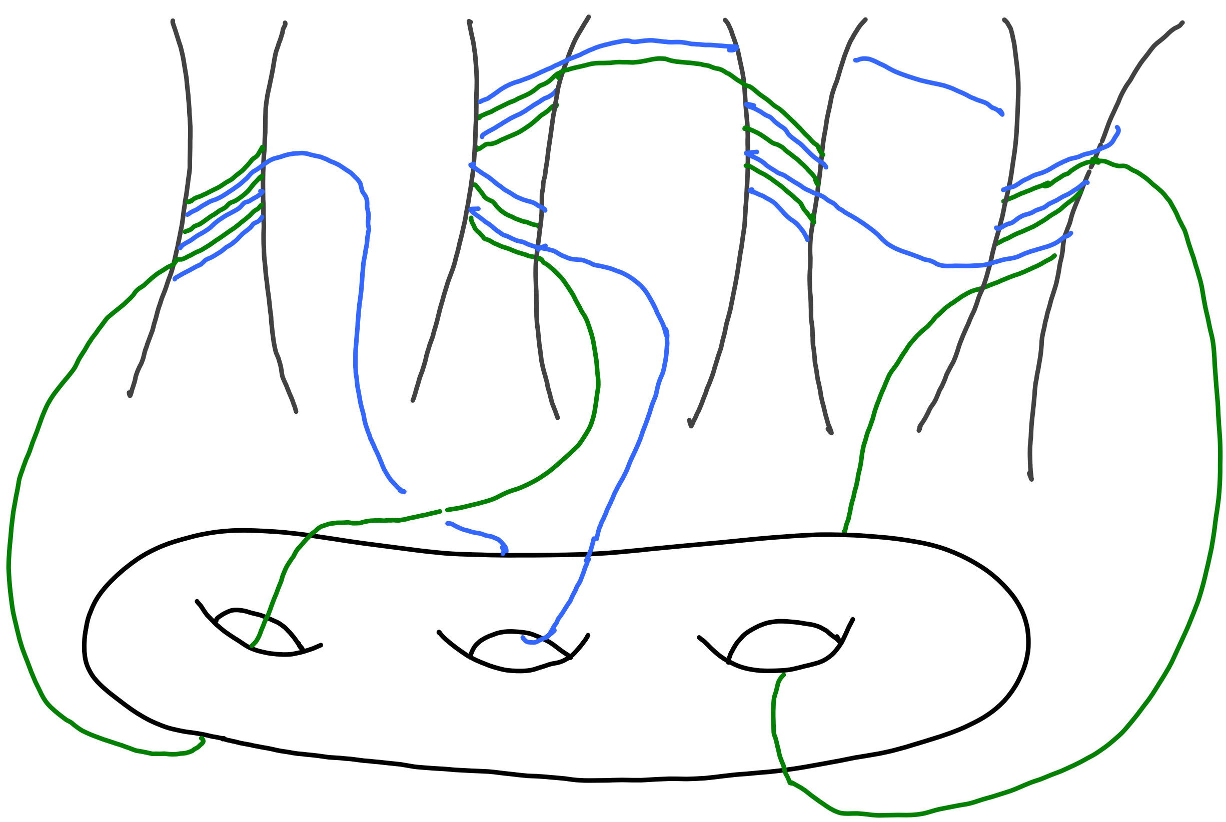
$$Z(U_1, \cdots, U_{2g-2})$$

$$= \sum_{R_1, \cdots, R_{2g-2}} Z_{R_1 \cdots R_{2g-2}} \times \mathrm{Tr}_{R_1} U_1 \cdots \mathrm{Tr}_{R_{2g-2}} U_{2g-2}$$

are topological string amplitudes for the Calabi-Yau with (2g-2) D branes wrapping Lagrangian 3-cycles.

$U_1, \cdots, U_{2g-2}$ are holonomies on the D branes.

These D branes correspond to the Omega points in the ordinary (undeformed) 2d YM.



# Summary

The mixed ensemble of BPS black holes gives a non-perturbative definition of topological string theory.

This is a large N duality relating the worldvolume theory of D6 and D4 branes on a Calabi-Yau manifold (D2 and D0 charges are allowed to fluctuate) to the closed topological string theory.

We studied the case when the Calabi-Yau manifold is the total space of the rank 2 vector bundle.

The worldvolume theory of D4 branes wrapping one of the line bundles is related to the *q*-deformed Yang-Milles theory on the base Riemann surface.

The large N limit of the gauge theory partition function is holomorphically factorized, and the chiral blocks are interepreted in terms of the perturbative topological string theory.

The Omega points are related to the presence of the hidden D branes.

*Fin*