Integrable Spin Chains in $\mathcal{N} = 4$ SYM and QCD

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`hep-th/0307015, 0307042, 0310252, 04mmnnn`.

Scaling Dimensions and AdS/CFT

Scaling dimension D of local operator $\mathcal{O}$ determines two-point function

$$\langle \mathcal{O}(x)\, \mathcal{O}(y) \rangle = \frac{1}{|x-y|^{2D(g)}}\,.$$

AdS/CFT correspondence claims equivalence of

- IIB string theory on $AdS_5 \times S^5$ and
- $\mathcal{N} = 4$ superconformal gauge theory.

Isometry/symmetry group: $\mathfrak{psu}(2,2|4)$ in both cases.

AdS/CFT predicts the agreement of

- the spectrum of energies $\{E\}$ in string theory with
- the spectrum of scaling dimensions $\{D\}$ in gauge theory.

Large Spin Limits of AdS/CFT

Tests of AdS/CFT usually prevented by strong/weak nature of the duality.

Proposal: Consider states with large spin J on S^5/of flavour $\mathfrak{so}(6)$

- BMN limit; non-planar and near $\mathcal{O}(1/J)$ extensions: [Berenstein Maldacena Nastase]

$$\mathcal{O} \sim \operatorname{Tr} \phi_1 \phi_1 \ldots \phi_2 \ldots \phi_2 \ldots \phi_2 \ldots \phi_1 \phi_1 \longleftrightarrow \text{short quantum strings.}$$

- Semiclassical spinning strings: [Frolov Tseytlin]

$$\mathcal{O} \sim \operatorname{Tr} \phi_1 \ldots \phi_1 \phi_2 \ldots \phi_2 \phi_1 \ldots \phi_1 \phi_2 \ldots \phi_2 \longleftrightarrow \text{long classical strings.}$$

Effective coupling constant $$\lambda' = \frac{\lambda}{J^2}\,.$$

- String theory: Expansion in λ' and $1/J \sim 1/\sqrt{\lambda}$,
- Gauge theory: ℓ-loop contribution suppressed by (at least) $1/J^{2\ell}$.

Expansion in λ' apparently equivalent to expansion in λ. Compare!

Three-Loop Discrepancies

BMN state with 2 excitations

$$\mathcal{O}_n \approx \sum_{p=0}^{J} \exp \frac{2\pi i n p}{J} \operatorname{Tr} \phi_1^p \phi_2 \phi_1^{J-p} \phi_2, \qquad D - J \approx 2\sqrt{1 + \frac{\lambda n^2}{J^2}}\,.$$

Gauge theory dimension in near BMN limit $\mathcal{O}(1/J)$ [NB Kristjansen Staudacher]

$$D - J = 2 + \frac{\lambda n^2}{J^2}\left(1 - \frac{2}{J}\right) - \frac{\lambda^2 n^4}{J^4}\left(\frac{1}{4} + \frac{0}{J}\right) + \frac{\lambda^3 n^6}{J^6}\left(\frac{1}{8} + \frac{1}{2J}\right) + \ldots$$

Energy of near plane-wave string [Callan, Lee, McLoughlin Schwarz, Swanson, Wu]

$$E - J = 2 + \lambda' n^2\left(1 - \frac{2}{J}\right) - \lambda'^2 n^4\left(\frac{1}{4} + \frac{0}{J}\right) + \lambda'^3 n^6\left(\frac{1}{8} + \frac{0}{J}\right) + \ldots$$

Three-loop mismatch also for 3 excitations. [Callan McLoughlin Swanson]
Similar disagreement for spinning strings. [Serban Staudacher]

Outline

- Local operators? Spin Chains?? Dilatation operator?!
  What am I talking about?
- How to obtain these results? (with as little work as possible)
- What is (higher-loop) integrability?
- How to make use of it?

Spin-offs for one-loop QCD

- The complete one-loop mixing matrix.
- Largest integrable sector and its Bethe ansatz.
- The lowest anomalous dimension:
  The spin-1 antiferromagnet and its excitations.

Local Operators

Local, gauge-invariant combination of the fields. Example:

$$\mathcal{O}^a_{\mu m \alpha}(x) = 27\, \mathrm{Tr}\, \Phi_n(x)\, \mathcal{D}^\nu \Psi^a_\alpha(x)\, \mathcal{F}_{\nu\mu}(x)\, \mathrm{Tr}\, \Phi_m(x)\, \Phi_n(x)$$
$$+ 13g\, \mathrm{Tr}\, \mathcal{D}_\mu \mathcal{D}_\nu \Phi_m(x)\, \Phi_n(x)\, \mathcal{D}^\nu \Phi_n(x)\, \Psi^a_\alpha(x) + \ldots$$

- Position-space representation of QFT.
- Do not identify $\mathcal{O}$ and descendants $\partial_\mu \mathcal{O}$, $\partial_\mu \partial_\nu \mathcal{O}$, $\ldots$.
- Drop position (x): Local operators as abstract objects.
- Building blocks $\mathcal{W}_A = \{\mathcal{D}^n \Phi, \mathcal{D}^n \Psi, \mathcal{D}^n \mathcal{F}\}$.
  Canonical gauge transformation: $\mathcal{W}_A \mapsto U \mathcal{W}_A U^{-1}$.
- Mixing problem: Lots and lots of similar operators.

Basis of Building Blocks

The building blocks are fields and their derivatives

$$\mathcal{W}_A = \{\mathcal{D}^n \Phi, \mathcal{D}^n \Psi, \mathcal{D}^n \mathcal{F}\}.$$

Consider a scalar Φ. No problem with Φ and $\mathcal{D}_\mu \Phi$, but

$$\mathcal{D}_\mu \mathcal{D}_\nu \Phi = \mathcal{D}_{(\mu} \mathcal{D}_{\nu)} \Phi + \mathcal{D}_{[\mu} \mathcal{D}_{\nu]} \Phi + \tfrac{1}{4} \eta_{\mu\nu} \mathcal{D}^2 \Phi.$$

Jacobi identity

$$\mathcal{D}_{[\mu} \mathcal{D}_{\nu]} \Phi = -ig[\mathcal{F}_{\mu\nu}, \Phi] = \mathcal{O}(g\mathcal{F}\Phi).$$

Equations of motion

$$\mathcal{D}^2 \Phi = \mathcal{O}(g\Psi^2) + \mathcal{O}(g^2 \Phi^3).$$

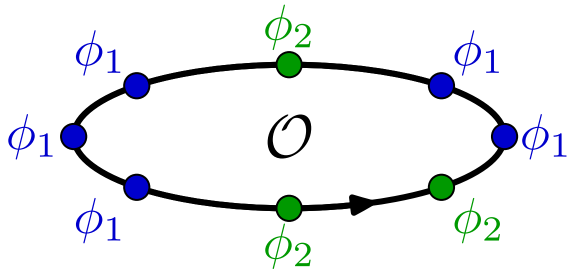
Only $\mathcal{D}_{(\mu} \mathcal{D}_{\nu)} \Phi$ is elementary; $\mathcal{D}_{[\mu} \mathcal{D}_{\nu]} \Phi$ and $\mathcal{D}^2 \Phi$ are reducible.

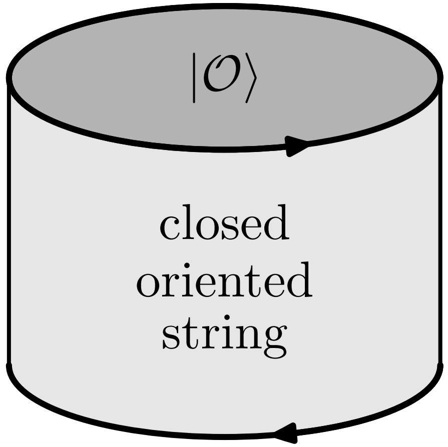
Large N_c and Spin Chains

Single trace operator, two complex scalars ϕ_1, ϕ_2 (a.k.a. $\mathcal{Z}, \phi$ or Z, X)

$$\mathcal{O} = \operatorname{Tr} \phi_1\phi_1\phi_2\phi_1\phi_1\phi_1\phi_2\phi_2$$

Length L: # of fields

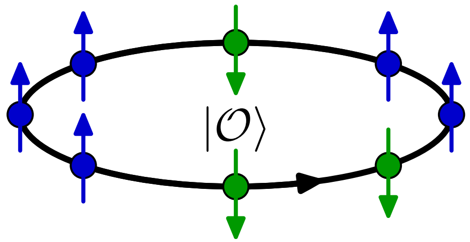




Identify $\phi_1 = |\uparrow\rangle$, $\phi_2 = |\downarrow\rangle$

$$|\mathcal{O}\rangle = |\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\rangle$$

Length L: # of sites

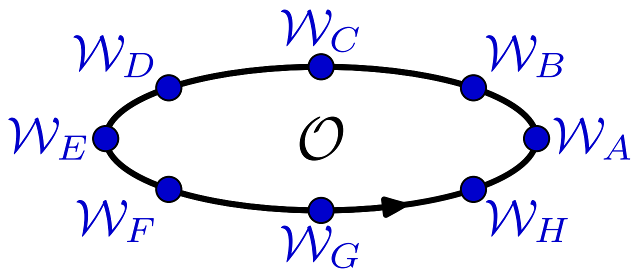


Operator mixing, quantum superposition: $|\mathcal{O}\rangle = *|\ldots\rangle + *|\ldots\rangle + \ldots$

Full $\mathcal{N} = 4$ SYM and Subsectors

Generic single trace state

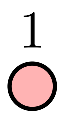
$$\mathcal{O} = \operatorname{Tr} \mathcal{W}_A \mathcal{W}_B \mathcal{W}_C \mathcal{W}_D \mathcal{W}_E \mathcal{W}_F \mathcal{W}_G \mathcal{W}_H \ldots$$



$\mathfrak{su}(2)$ Subsector

$$\mathcal{W} \in \{\phi_1, \phi_2\}$$

(fundamental $\mathbf{2}$)

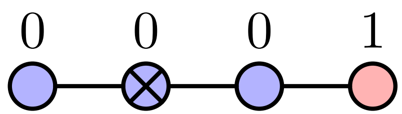


$\mathfrak{su}(2)$

$\mathfrak{su}(2|3)$ Subsector

$$\mathcal{W} \in \{\phi_{1,2,3}, \psi_{1,2}\}$$

(fundamental $\mathbf{3}|\mathbf{2}$)

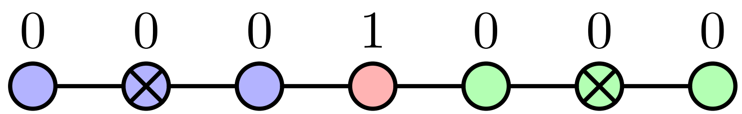


$\mathfrak{su}(2|3)$

Full $\mathcal{N} = 4$ SYM

$$\mathcal{W} \in \{\mathcal{D}^k \Phi, \mathcal{D}^k \Psi, \mathcal{D}^k \mathcal{F}\}$$

(non-compact rep.)



$\mathfrak{psu}(2,2|4)$

and many more...

[NB hep-th/0407277]

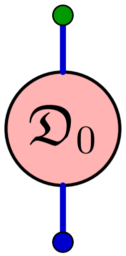
Dilatation Generator

Scaling dimensions $D_{\mathcal{O}}(g)$ as eigenvalues of the dilatation generator $\mathfrak{D}(g)$

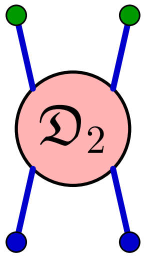
$$\mathfrak{D}(g)\,\mathcal{O} = D_{\mathcal{O}}(g)\,\mathcal{O}.$$

Quantum corrections in perturbation theory: $g \sim \sqrt{\lambda}$

$$\mathfrak{D}(g) =$$

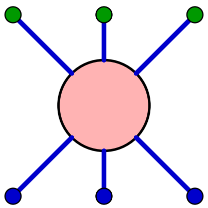


$$+\, g^2$$

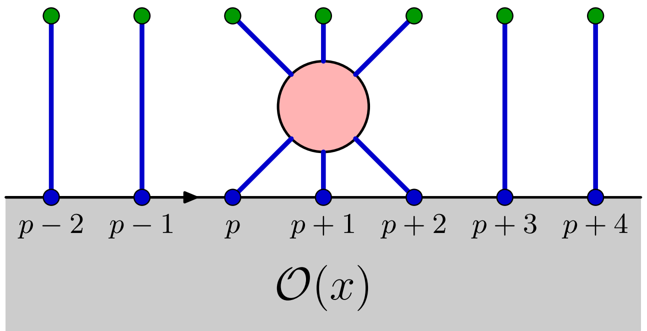


$$+\, g^3\mathfrak{D}_3 + g^4\mathfrak{D}_4 + \ldots\, .$$

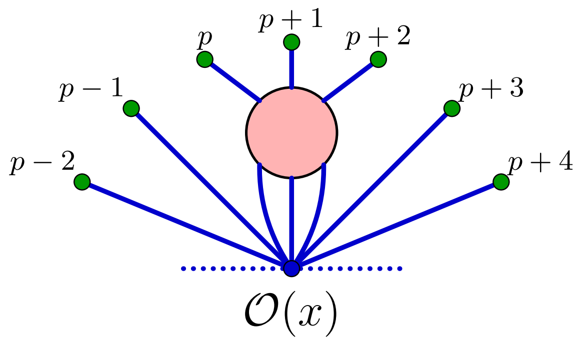
Local action along spin chain (homogeneous)



$$= \sum_{p=1}^{L}$$



$$= \sum_{p=1}^{L}$$



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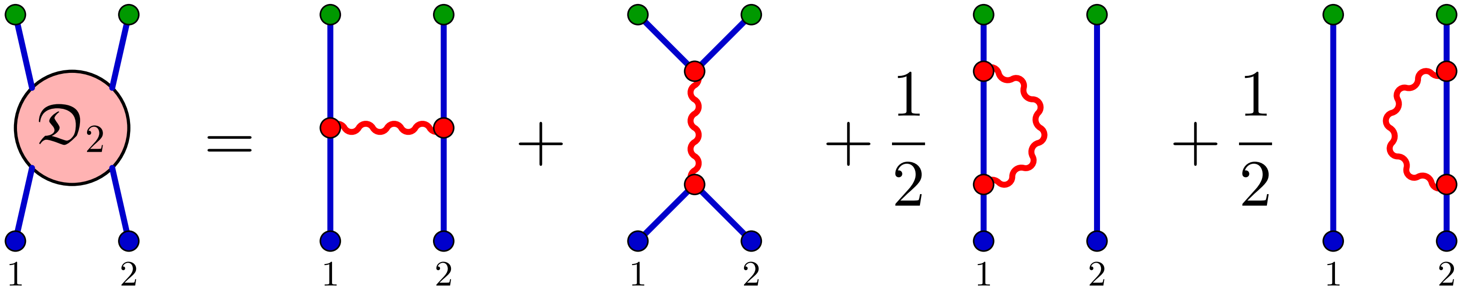
One-Loop

One-loop $\mathcal{O}(g^2)$ dilatation operator $\mathfrak{D}_2$:

$\mathfrak{D}_{2(12)} =$



Extract logarithmic piece of Feynman diagrams.

$\mathfrak{su}(2)$ Subsector

$\mathfrak{D}_{2(12)} = 1 - P_{(12)}$.

[Minahan Zarembo]

$1 = I_{(12)}$: identity

$P_{(12)}$: permutation

Heisenberg chain

$\mathfrak{su}(2|3)$ Subsector

$\mathfrak{D}_{2(12)} = 1 - SP_{(12)}$.

[NB hep-th/0310252]

SP_{12}: graded perm.

Full $\mathcal{N} = 4$ SYM

$\mathfrak{D}_{2(12)} = 2h(J_{(12)})$.

[NB hep-th/0307015]

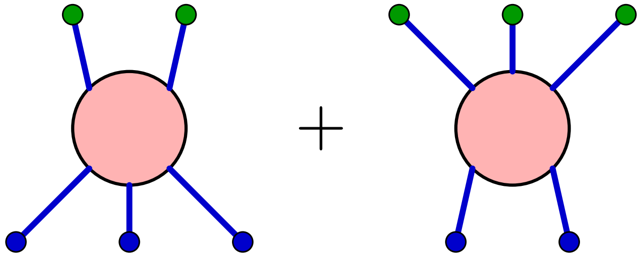
$J_{(12)}$: “total spin” op.

harmonic n. $h(s) = \sum_{k=1}^{s} \frac{1}{k}$

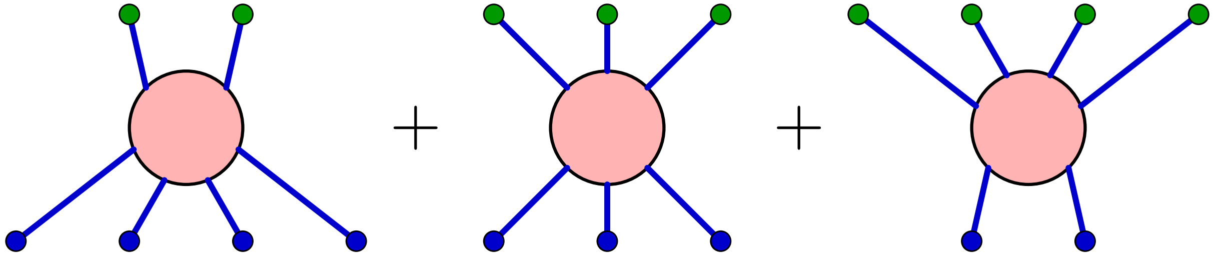
Higher-Loops

Contribution to dilatation generator at $\mathcal{O}(g^k)$ has (up to) $k+2$ legs

$\mathfrak{D}_3 =$



$\mathfrak{D}_4 =$

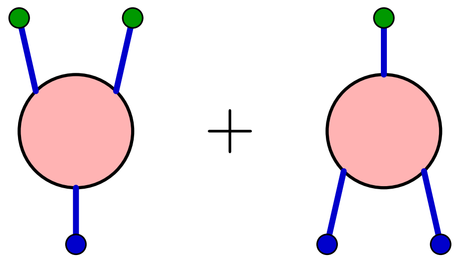


$+ \ldots$

At higher-loops: Length L fluctuates, dynamic spin chain. [NB hep-th/0310252]

Also (super)momenta $\mathfrak{Q}, \mathfrak{P}$ & (super)boosts $\mathfrak{S}, \mathfrak{K}$ are corrected, e.g.

$\mathfrak{P}_1, \mathfrak{K}_1, \mathfrak{Q}_1, \mathfrak{S}_1 =$



Algebraic Construction

Direct computation of higher-loops is extremely laborious.

Proposal: Try to reconstruct $\mathfrak{D}(g)$ from known properties.

- Consider all possible structures (# of legs)
- Assume most general form
- Demand closure of symmetry algebra, e.g.

$$[\mathfrak{D}(g), \mathfrak{P}(g)] = \mathfrak{P}(g), \quad [\mathfrak{D}(g), \mathfrak{Q}(g)] = \tfrac{1}{2}\mathfrak{Q}(g), \quad \ldots$$

To fix remaining coefficients, may use

- BMN scaling behavior: $\lambda' = \lambda/L^2$ effective coupling.
- Known results: e.g. Konishi, twist-2 operators
- Integrability (later...): $[\delta\mathfrak{D}, \mathcal{Q}_3] = 0$

Algebraic Construction: Results

Full $\mathcal{N}=4$ SYM: at one loop

- Fixed by algebra up to one overall constant (g). [NB hep-th/0407277]

$\mathfrak{su}(2|3)$ Subsector: at three loops (using BMN scaling) [NB hep-th/0310252]

- Dimension of Konishi

$$D_{\mathcal{K}} = 2 + \frac{3\lambda}{4\pi^2} - \frac{3\lambda^2}{16\pi^4} + \frac{21\lambda^3}{256\pi^6} + \ldots$$

  Confirmed by explicit computation. [Moch Vermaseren Vogt][Kotikov,Lipatov Onishchenko Velizhanin][Eden, Jarczak Sokatchev]
- Agrees with BMN matrix model. [Klose Plefka]
- Yields near-BMN result from beginning of talk.

$\mathfrak{su}(2)$ Subsector: at five loops (using BMN scaling & integrability)

- Reproduces BMN energy formula $D - J = \sum_k \sqrt{1 + \lambda' n_k^2}$. [NB, Dippel Staudacher]

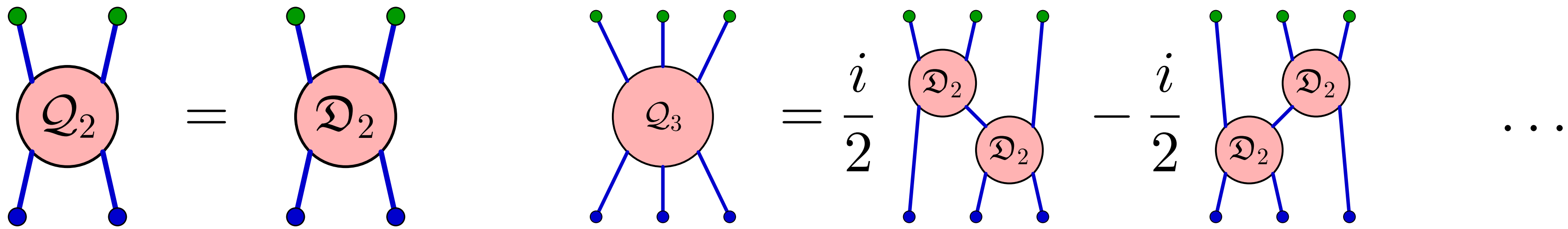
One-Loop Integrability

Only in planar limit!

Existence of higher charges $\mathcal{Q}_{2,3,4,\ldots}$ (scalar, commuting),

$$[\mathfrak{J}_0, \mathcal{Q}_r] = [\mathcal{Q}_r, \mathcal{Q}_s] = 0, \qquad \mathfrak{D}_2 = \mathcal{Q}_2.$$

Structure of charges



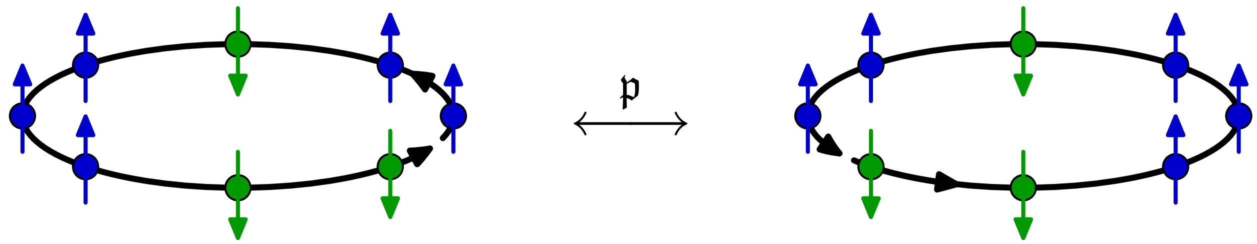
One-loop integrability found for

- $\mathfrak{so}(6)$ **Subsector** of scalars Φ_m [Minahan Zarembo]
- **Complete $\mathcal{N} = 4$ SYM:** $\mathfrak{su}(2,2|4)$ super spin chain. [NB Staudacher]
- Subsector of large N_c QCD. [Braun Derkachov Manashov][Braun, Derkachov Korchemsky Manashov][Belitsky hep-ph/9907420]

Test for Integrability

Consider parity $\mathfrak{p}$ (charge conjugation/spin chain/world sheet):

$$\operatorname{Tr} \phi_1\phi_1\phi_2\phi_1\phi_1\phi_1\phi_2\phi_2 \overset{\mathfrak{p}}{\longleftrightarrow} \operatorname{Tr} \phi_2\phi_2\phi_1\phi_1\phi_1\phi_2\phi_1\phi_1$$



Even/odd charges have even/odd parity

$$\mathfrak{p}\, \mathcal{Q}_r \mathfrak{p}^{-1} = (-1)^r\, \mathcal{Q}_r.$$

Implies degenerate pairs of opposite parity: (only in planar limit!)

$$D_+ = D_-.$$

Test for integrability. [Grabowski Mathieu]

Higher-Loop Integrability

No formalism yet (R-matrix, Yang-Baxter equation, ...).

Existence of higher charges $\mathcal{Q}_r(g)$, [NB Kristjansen Staudacher]

$$[\mathfrak{J}(g), \mathcal{Q}_r(g)] = [\mathcal{Q}_r(g), \mathcal{Q}_s(g)] = 0, \qquad \mathfrak{D}(g) = \mathfrak{D}_0 + g^2 \mathcal{Q}_2(g).$$

Test: Planar parity pairs preserved at higher-loops

$$D_+(g) = D_-(g).$$

Higher-loop integrability for

- $\mathfrak{so}(6)$ **Subsector**: Sector not closed at higher-loops due to mixing.
- $\mathfrak{su}(2|3)$ **Subsector**: Observed at three-loops (pairs). [NB hep-th/0310252]
  Even through length fluctuates.
- $\mathfrak{su}(2)$ **Subsector**: Construct five-loops dilop. via integrability. [NB, Dippel Staudacher]

Bethe Ansätze

Naive procedure to compute anomalous dimensions:

- Find a basis of operators which mix. (generically huge!)
- Evaluate matrix of anomalous dimensions. (dil.op.)
- Diagonalize the matrix.

Bethe equations to bypass the first two steps:

- Solve system of algebraic equations.

Known Bethe ansätze for $\mathcal{N} = 4$ SYM:

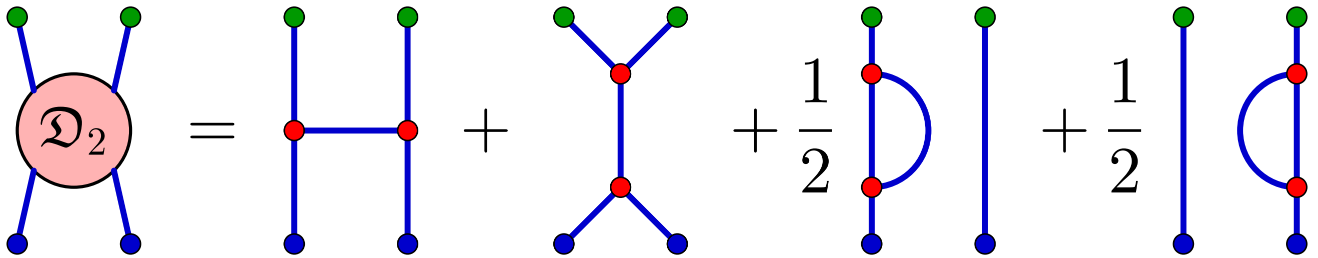
- $\mathfrak{so}(6)$ sector at one loop. [Minahan Zarembo]
- Complete at one loop: $\mathfrak{psu}(2,2|4)$ spin chain. [NB Staudacher]
- $\mathfrak{su}(2)$ sector to three loops and beyond. [Serban Staudacher][NB, Dippel Staudacher]

States with a large number of excitations (e.g. spinning strings):

- Bethe ansätze only means to obtain anomalous dimensions.
- Bethe equations approximated by integral equations.

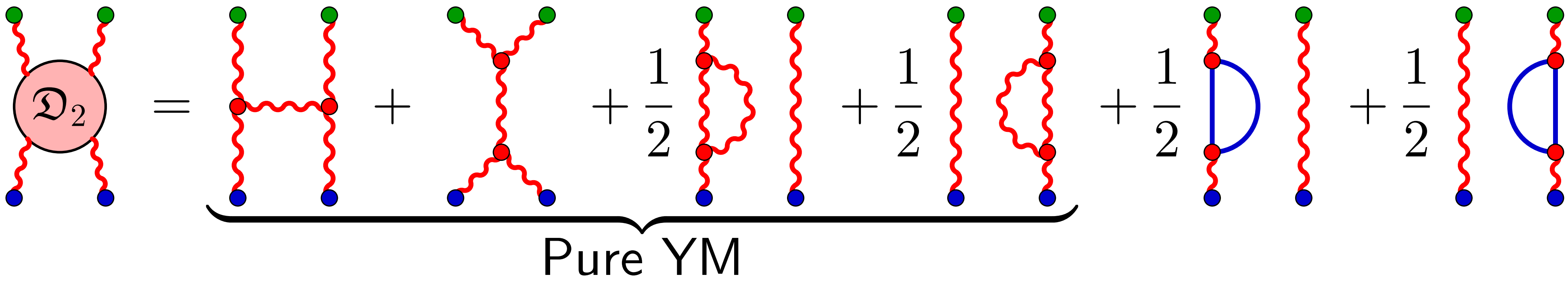
From $\mathcal{N} = 4$ SYM to Pure YM

One-loop contribution in $\mathcal{N} = 4$ SYM (generic lines):



Restrict to gluons in external lines:

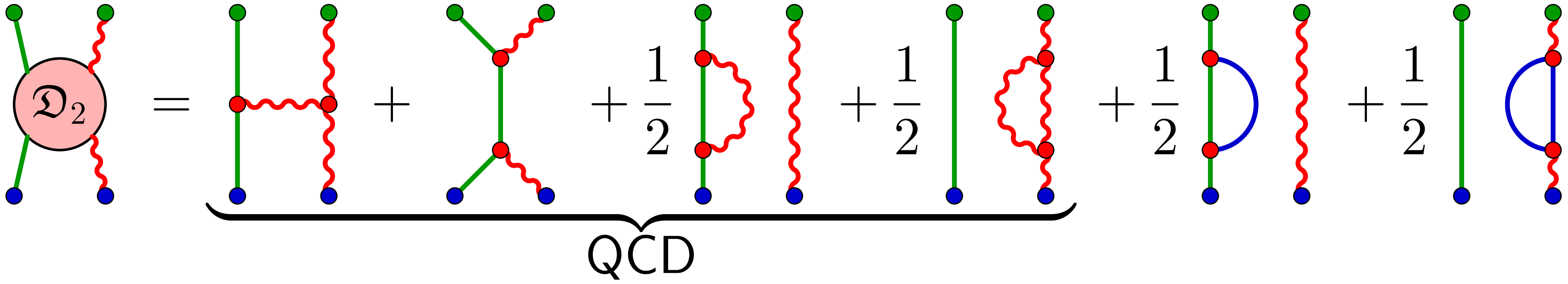
[NB, Ferretti Heise, Zarembo]



- Gluon couplings universal.
- Fermions and scalars appear only within internal loops.

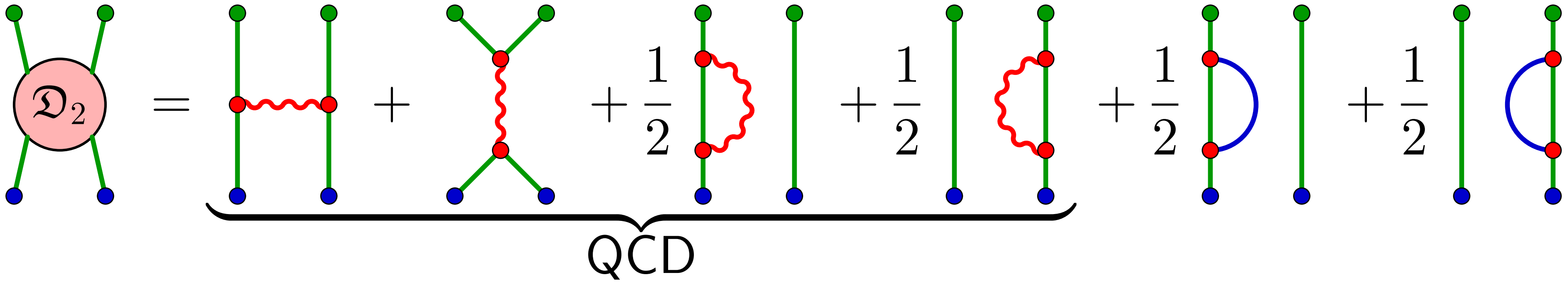
Fundamental Fermions

Restrict to one fermion and one gluon:



- Fundamental couplings can be deduced.

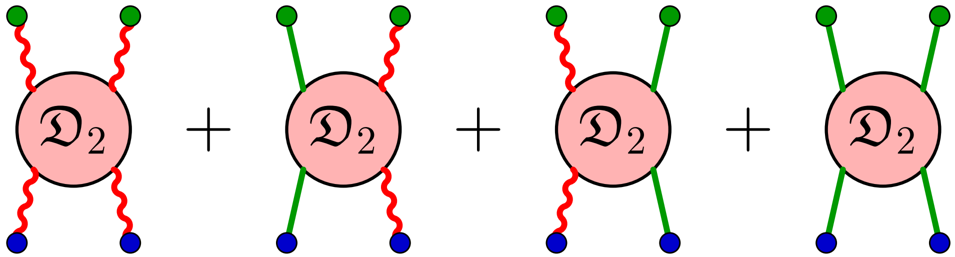
Restrict to two fermions:



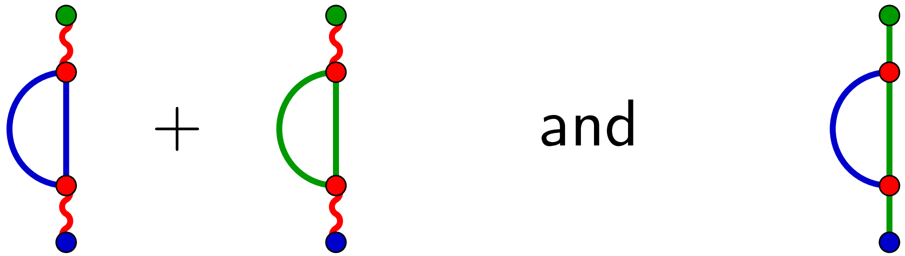
- Scalars couple only to different flavours of fermions.

Reduction of $\mathcal{N} = 4$ SYM to QCD

- Restrict $\mathfrak{D}_2$ from $\mathcal{N} = 4$ SYM to extenal gluons and fermions



- Subtract internal scalar and fermion loops.



- Account for dimensionful coupling constant (β-function)

$$\mathfrak{D} = \mathfrak{D}_0 + g^2\mathfrak{D}_2 + \ldots + \left(\frac{11g^2}{6} + \ldots\right)\frac{g\partial}{\partial g}.$$

Integrability in QCD

What is known?

- Integrability for (baryonic) light-cone operators. [Braun Derkachov Manashov][Braun, Derkachov Korchemsky Manashov]
- Integrable spin-$(-\frac{3}{2})$ chain for light-cone gluons $\{\mathcal{D}^n_{1\dot{1}}\mathcal{F}_{11}\}$. [Belitsky hep-ph/9907420]
- Integrable spin-$(+1)$ chain for chiral gluons $\{\mathcal{F}_{\alpha\beta}\}$. [Ferretti, Heise Zarembo]
- XXZ spin chain for $\{\mathcal{F}_{11}, \bar{\mathcal{F}}_{\dot{1}\dot{1}}\}$. [Belitsky, Derkachov Korchemsky, Manashov]
- Counter-examples for generic operators.

What can we learn from $\mathcal{N} = 4$ SYM? [NB, Ferretti Heise, Zarembo]

- Subsector of maximal chirality $\{\mathcal{D}^n\mathcal{F}\}$ is closed at one loop.
- Pure YM inherits dilatation operator up to global shift.
- Pure YM inherits integrability from $\mathcal{N} = 4$ SYM.
- Interactions of $\mathcal{D}^n\mathcal{F}$ and $\mathcal{D}^m\bar{\mathcal{F}}$ violate integrability.
  Mixing with scalars and fermions of $\mathcal{N} = 4$ SYM missing.

The Pure YM Integrable Spin Chain

- $\{\mathcal{D}^n\mathcal{F}\}$ transforms in a $[2,-3,0]$ representation of $\mathfrak{su}(2,2)$.
- $\mathfrak{su}(2,2)$ spin chain with spins $[2,-3,0]$.
- Algebraic Bethe ansatz to compute spectrum.
- Combines spin-$(-\frac{3}{2})$ and spin-$(+\frac{2}{2})$ chains.

Maximal integrable sector in pure YM:

- No integrability for anti-chiral insertions $\mathcal{D}^n\bar{\mathcal{F}}$.
  Exception: XXZ spin chain for $\{\mathcal{F}_{11}, \bar{\mathcal{F}}_{\dot{1}\dot{1}}\}$. Accident?
- No integrability beyond one-loop due to violation of chirality?
  Mixing of $\mathcal{D}^n\mathcal{F}$ and $\mathcal{D}^m\bar{\mathcal{F}}$ cannot be suppressed.
- Artefact of relation to $\mathcal{N}=4$ SYM?

The Bethe Ansatz for Pure YM

Standard Bethe equations for $\mathfrak{su}(2,2)$ symmetry and spins $[2,-3,0]$.

$$-\frac{(a_k+2i)^L}{(a_k-2i)^L} = \prod_{j=1}^{K_a}\frac{a_k-a_j+2i}{a_k-a_j-2i}\prod_{j=1}^{K_b}\frac{a_k-b_j-i}{a_k-b_j+i},$$

$$-\frac{(b_k-3i)^L}{(b_k+3i)^L} = \prod_{j=1}^{K_a}\frac{b_k-a_j-i}{b_k-a_j+i}\prod_{j=1}^{K_b}\frac{b_k-b_j+2i}{b_k-b_j-2i}\prod_{j=1}^{K_c}\frac{b_k-c_j-i}{b_k-c_j+i},$$

$$-1 = \prod_{j=1}^{K_b}\frac{c_k-b_j-i}{a_k-b_j+i}\prod_{j=1}^{K_c}\frac{c_k-c_j+2i}{c_k-c_j-2i}.$$

Momentum constraint and one-loop scaling dimension:

$$1=\prod_{j=1}^{K_a}\frac{a_j+2i}{a_j-2i}\prod_{j=1}^{K_b}\frac{b_j-3i}{b_j+3i},\qquad D=2L+K_b+\frac{g_{\mathrm{YM}}^2N}{8\pi^2}\left(\frac{7L}{6}-\sum_{j=1}^{K_a}\frac{8}{a_j^2+4}+\sum_{j=1}^{K_b}\frac{12}{b_j^2+9}\right).$$

Lowest Anomalous Dimension

What is the lowest non-trivial anomalous dimension for fixed length L?

Eigenvalues of two-gluon "Hamiltonian" $\mathfrak{D}_{2,(12)}$:

- $-\frac{11}{6}, +\frac{1}{6}, +\frac{7}{6}$ for $\mathcal{F}\mathcal{F}$ as singlet, spin-1, spin-2 of $\mathfrak{su}(2)$.
- $+0, +\frac{1}{2}$ for $\bar{\mathcal{F}}\mathcal{F} \pm \mathcal{F}\bar{\mathcal{F}}$.
- even more when including derivatives.

Locally, it is preferred to have singlets $\mathcal{F}\mathcal{F}$:

- Antiferromagnetic ground state.
- Not possible to have all neighbours in singlets.
- Bethe ansatz:

$$\delta D = -\frac{5 g_{\mathrm{YM}}^2 N_{\mathrm{c}} L}{48\pi^2} + \mathcal{O}(L^0).$$

Excitations of the Ground State

Three types of excitations:

- Anti-chiral insertions $\bar{\mathcal{F}}$.
  Dispersion relation: $e \sim |p|^0 \sim 1/L^0$.
- Spinons: (fractional) spin flips.
  Fractional spin: $(\frac{1}{2}, 0)$.
  Dispersion relation: $e \sim |p|^1 \sim 1/L^1$.
- Derivative insertions $\mathcal{D}_{\alpha\dot{\alpha}}$.
  Spin separation $(0, \frac{1}{2})$, left-spin dissolves in sea of $\mathcal{F}_{\alpha\beta}$.
  Dispersion relation: $e \sim |p|^2 \sim 1/L^2$.

Non-trivial exchange statistics of excitations.
New physicality constraint: Integrality of spins. (alike momentum constr.)
Fractional spins $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$ not observable.
Number of independent momenta enhanced.

Conclusions & Outlook

⋆ **Dilatation Operator in a conformal gauge theory**

- Constructive methods for scaling dimensions.
- Higher-loop scaling dimensions without field theory computation.

⋆ **Integrability in $\mathcal{N} = 4$ SYM**

- Integrable $\mathfrak{su}(2,2|4)$ spin chain.
- Higher-loop integrability at least in subsectors.
- Full planar $\mathcal{N} = 4$ integrable? Support by AdS/CFT conjecture.

⋆ **Integrability in Pure YM**

- Largest integrable sector identified: $\mathfrak{su}(2,2)$ spin chain at one loop.
- $\mathcal{N} = 4$ SYM as the origin?
- Antiferromagnetic ground state.
- Excitations of the ground state.
- Fundamental fermions & QCD: $\mathfrak{su}(2,2)$ open spin chain integrable?