3 DIM. YANG-MILLS : MASS GAP, VACUUM, STRING TENSION

WHY IS THIS INTERESTING?

1. INTERESTING IN ITS OWN RIGHT

2 DIM	<	3 DIM	<	4 DIM
NO PROPAGATING DEGREES OF FREEDOM		PROPAGATING D.O.F		DIFFICULT
EXACTLY SOLVABLE		NON TRIVIAL		

DIMENSIONAL COUPLING

SUPER-RENORMALIZABLE

2. A REAL PHYSICAL CONTEXT FOR 3 DIM YM.

→ HIGH T QCD

$$A(\vec{x}, x^0) = \sum A_n(\vec{x})\, e^{2\pi i n T x^0}$$

$$\langle A\ A \rangle \sim \frac{1}{\vec{k}^2 + (2\pi n T)^2}$$

HIGH T → $n = 0$ IS THE IMPORTANT MODE

→ 3 DIM YM.

IR DIVERGENCES IN 4 DIM QCD AT $T \gg T_c$

MAGNETIC TYPE DIVERGENCES

(GAUGE BOSON PROPAGATION, PERTURBATIVE EXPANSION FOR THERMODYNAMIC QUANTITIES, ...)

→ MASS GAP OF 3 DIM YM WILL CURE THESE.

(GROSS, PISARSKI, YAFFE LINDE)

CAN WE DEFINE THERMAL PERTURBATION THEORY FOR 4 DIM QCD ?

HAMILTONIAN ANALYSIS OF (2+1) YM

JOINT WORK WITH
D. KARABALI
CHANJU KIM

CHOOSE $A_0 = 0$, A_i

$$A_i^g = g^{-1} A_i\, g + g^{-1} \partial_i g$$

WAVE FUNCTIONS ARE GAUGE INVARIANT

THREE STEPS

1. INNER PRODUCT ON A_i's MODULO GAUGE TRANSFORMATIONS
2. HAMILTONIAN OPERATOR
3. SCHRODINGER EQUATION

OUTLINE / SUMMARY

1. MATRIX VARIABLES

$$A_z = -\partial M M^{-1}$$

$$M^3 = g M \qquad H = M^\dagger M \quad \text{GAUGE INVARIANT}$$

2. VOLUME OF GAUGE-INVARIANT CONFIGURATIONS

$$d\mu(\mathcal{A}/\mathcal{G}) = e^{2 C_A S(H)} d\mu(H)$$

(EXACT)

3. INTUITIVE ARGUMENT FOR MASS GAP

4. CHOICE OF "CORRECT" VARIABLES

CFT ARGUMENT $\qquad J = \frac{C_A}{\pi} \partial H H^{-1}$

5. HAMILTONIAN

$$H = m\left[\int J \frac{\delta}{\delta J} + \Omega_{ab}(u,v) \frac{\delta}{\delta J^a(u)} \frac{\delta}{\delta J^b(v)}\right] + \frac{2\pi^2}{e^2 C_A^2} \int \delta J \, \delta J$$

6. SCHRODINGER EQUATION

$\rightarrow \Psi_0$

$$\langle W_F(C) \rangle \sim e^{-\sigma \mathcal{A}(C)}$$

$$\sqrt{\sigma} = e^2 \sqrt{\frac{N^2-1}{8\pi}}$$

COMPARE WITH LATTICE $\qquad$ discrepancy $\lesssim 3\%$

7. BACK TO MAGNETIC MASS

$$m = \frac{e^2 N}{2\pi} = \frac{g^2 T N}{2\pi}$$

COMPARE WITH LATTICE

8. GLUEBALLS

9. YANG-MILLS + CHERN-SIMONS

SCREENING, HOW DOES THIS ARISE ?

MATRIX VARIABLES & VOLUME ELEMENT

$z = x_1 - i x_2 \qquad \bar{z} = x_1 + i x_2$

$A = A_z = \frac{A_1 + i A_2}{2} \qquad \bar{A} = \frac{A_1 - i A_2}{2}$

a) PARAMETRIZE A AS

$$A_z = -\partial_z M M^{-1}$$

$$\bar{A} = M^{\dagger -1} \bar{\partial} M^\dagger$$

$G = SU(N) \rightarrow M \in SL(N, \mathbb{C}) = SU(N)^{\mathbb{C}} \qquad (G \rightarrow G^{\mathbb{C}})$

b) WHY IS THIS POSSIBLE?

$$\partial_z \frac{1}{\pi(\bar{z} - \bar{w})} = \delta^{(2)}(z - w)$$

$$\partial_z \bar{A} - \bar{\partial} A + [A, \bar{A}] = 0$$

$$M(x) = P \exp - \int_{0,c}^{x} (A_z dz + \bar{A} d\bar{z})$$

$$\rightarrow \quad A = -\partial M M^{-1}$$

c) M & $M V(\bar{z}) \Rightarrow$ SAME A

(COME BACK TO THIS)

d) GAUGE TRANSFORMATION

$$M^g = g M \qquad A_i^g = g A_i g^{-1} - \partial_i g g^{-1}$$

$H = M^\dagger M$ IS GAUGE INVARIANT

e) VOLUME ELEMENT

$$\delta A = -\partial(\delta M M^{-1}) + [\partial M M^{-1}, \delta M M^{-1}]$$

$$= -D(\delta M M^{-1})$$

$$\delta \bar{A} = \bar{D}(M^{\dagger -1} \delta M^\dagger)$$

$$ds^2_{\mathcal{A}} = \int \mathrm{Tr}(\delta A \, \delta \bar{A}) \, d^2x$$

$$= \int \mathrm{Tr}(M^{\dagger -1} \delta M^\dagger)(-\bar{D} D)(\delta M M^{-1})$$

$$ds^2_{SL(N,\mathbb{C})} = \int \mathrm{Tr}(M^{\dagger -1} \delta M^\dagger \; \delta M M^{-1})$$

Volume

$$d\mu_A = \det(-D\bar{D})\ \underbrace{d\mu(M,M^\dagger)}_{\text{HAAR MEASURE FOR } SL(N,\mathbb{C})}$$

$$M = Up \quad \rightarrow \quad \delta M M^{-1} = U(\delta p\, p^{-1} + u^{-1}\delta u)\, u^{-1}$$

→ TAKE WEDGE PRODUCTS

$$\rightarrow \quad d\mu(M,M^\dagger) = \underbrace{d\mu(U)}_{\text{HAAR FOR } SU(N)} \cdot \underbrace{d\mu(H)}_{SL(N,\mathbb{C})/SU(N)} \qquad H = M^\dagger M$$

$$H^{-1}\delta H = \delta\varphi^a R_{ab}(\varphi)\, t^b$$

$$H = e^{t_a\varphi_a} \quad \rightarrow \quad d\mu(H) = [d\varphi](\det R)$$

$$d\mu_A = \det(-D\bar{D})\, d\mu(H)\, d\mu(u)$$

$$\rightarrow \quad d\mu(A/\mathcal{G}) = \det(-D\bar{D})\, d\mu(H)$$

$$= \exp[2C_A S(H)]\ d\mu(H)$$

$$C_A\delta_{ab} = f_{amn} f_{bmn} = N\delta_{ab} \quad \text{for } SU(N)$$

$$S(H) = \frac{1}{2\pi}\int \mathrm{Tr}(\partial H\, \bar{\partial} H^{-1}) - \frac{i}{12\pi}\int \mathrm{Tr}(H^{-1}dH)^3$$

(WZW ACTION FOR H)

f) INNER PRODUCT

$$\langle 1|2\rangle = \int d\mu(H)\, e^{2C_A S(H)}\, \psi_1^*(H)\, \psi_2(H) \qquad (*)$$

TWO REMARKS

1. 2+1 YM HAS GRIBOV PROBLEM. BUT (*) HAS NO DIFFICULTY DUE TO THIS (*) IS EXACT

2. MATRIX ELEMENTS IN (2+1) YM = CORRELATORS OF A HERMITIAN WZW MODEL

UNITARY MODEL

LEVEL k

$e^{k\,S(U)}$

$\kappa = k + C_A$

INTEGRABLE REP.'S

SPIN $\leq k$

NON INTEGRABLE

$\rightarrow$ CORRELATORS $= 0$

HERMITIAN MODEL

LEVEL $k + 2C_A$

$e^{(k+2C_A)\,S(H)}$

$\kappa = -(k + C_A)$

NON INTEGRABLE

$\rightarrow$ CORRELATORS $= \infty$

$\kappa \longleftrightarrow -\kappa$

$\rightarrow$ "FINITE NORM"

$\Rightarrow$ INTEGRABLE REP.'S

$\kappa = 0$ FOR US

$\rightarrow$ ONLY J FOR $e^{2C_A\,S(H)}$

$J = \frac{C_A}{\pi}\,\partial H H^{-1}$

$$W(C) = \mathrm{Tr}\,P e^{-\oint A} = \mathrm{Tr}\,P e^{\oint J\,(\pi/C_A)}$$

$\underline{\det(-\partial\bar{D})}$

$$\Gamma = \mathrm{Tr}\,\log(-\partial\bar{D})$$

$$\frac{\delta\Gamma}{\delta \bar{A}^a(x)} = -i\,\mathrm{Tr}\left[\bar{D}^{-1}_{Reg}(x,y)\,t^a\right]\Big|_{y\to x}$$

$$= -i\,\mathrm{Tr}\left[(A - M^{+-1}\partial M^{+})\,t^a\right]$$

RE INTEGRATE

$\uparrow$ $\Gamma = 2C_A\,S(H)$

AN INTUITIVE ARGUMENT FOR GAP

$$\mathcal{H} = \int \frac{1}{2}\left(e^2 E^2 + \frac{B^2}{e^2}\right)$$

$$[E, B] \sim p \qquad \Delta E \, \Delta B \sim p$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{e^2 p^2}{\Delta B^2} + \frac{\Delta B^2}{e^2}\right)$$

MINIMIZE $\rightarrow$ $\Delta B^2 \sim e^2 p$ $\qquad \mathcal{E} \sim p$

(PHOTON)

$$\langle \mathcal{H} \rangle \sim \int d\mu(H)\, e^{2 C_A S}\, \mathcal{H}$$

$$S(H) \sim -\frac{C_A}{2\pi} \int B \frac{1}{p^2} B + \cdots$$

$$\rightarrow \Delta B^2 \sim p^2 \frac{\pi}{C_A} \qquad \rightarrow \qquad \mathcal{E} \sim \frac{e^2 C_A}{2\pi} + G(p^2)$$

HAMILTONIAN

$$\mathcal{H} = \frac{1}{2}\int\left(e^2 E^2 + \frac{B^2}{e^2}\right) = T + V$$

$$V = \frac{1}{2e^2}\int B^2 = \frac{2\pi^2}{e^2 C_A^2}\int :\bar{\delta} J^a \, \bar{\delta} J^a:$$

$$T\,\Psi(J) = -\frac{e^2}{2}\int \frac{\delta^2 \Psi}{\delta A(x)\, \delta \bar{A}(x)}$$

$$= -\frac{e^2}{2}\left[\int \underbrace{\frac{\delta J(u)}{\delta A(x)} \frac{\delta J(v)}{\delta \bar{A}(x)}}_{\Omega} \frac{\delta^2 \Psi}{\delta J(u)\, \delta J(v)} + \int \underbrace{\frac{\delta^2 J(u)}{\delta A(x)\, \delta \bar{A}(x)}}_{\omega} \frac{\delta \Psi}{\delta J(u)}\right]$$

$$= \int \omega^a(x) \frac{\delta \Psi}{\delta J^a(x)} + \int \Omega_{ab}(u,v) \frac{\delta^2 \Psi}{\delta J^a(u)\, \delta J^b(v)}$$

$$\omega^a(u) = -\frac{e^2}{2}\int \frac{\delta^2 J^a(u)}{\delta A^b(x)\, \delta \bar{A}^b(x)}\, d^2x$$

SAME AS BEFORE

$$= \frac{e^2 C_A}{2\pi} M^{\dagger}_{am}(u)\, \mathrm{Tr}\left[t^m \bar{D}^{-1}(y,x)\right]_{y \to x}$$

$$\omega^a = m J^a \qquad m = \frac{e^2 C_A}{2\pi}$$

$$T = m\left[\int J^a \frac{\delta}{\delta J^a} + \int \Omega_{ab}(u,v) \frac{\delta}{\delta J_a(u)} \frac{\delta}{\delta J_b(v)}\right]$$

$$\Omega_{ab}(u,v) = \frac{C_A}{\pi} \frac{\delta_{ab}}{(u-v)^2} - \frac{i f_{abc} J_c(v)}{u-v}$$

FURTHER CHECK: WE NEED $J.\frac{\delta}{\delta J}$ - TERM FOR SELF-ADJOINTNESS WITH INNER PRODUCT

$$\int d\mu(H)\, e^{2C_A S}\, \psi_1^* \psi_2.$$

REGULARIZATION

$$\bar{\mathcal{G}}_{ma}(x,y) = \frac{1}{\pi(x-y)}\left[\delta_{ma} - e^{-|x-y|^2/\epsilon}\{H(x,\bar{y})H^{-1}(y,\bar{y})\}_{ma}\right]$$

(REGULATED FORM OF $\frac{1}{\pi(x-y)}$)

ALL RESULTS CHECKED WITH REGULARIZATION.

SCHRODINGER EQUATION

$$\mathcal{H}\psi_0 = 0 \qquad \text{VACUUM STATE}$$

IGNORE V FOR THE MOMENT.

$$T\psi_0 = 0 \quad \rightarrow \quad \underbrace{\psi_0 = 1.}_{\text{NORMALIZABLE}}$$

$$\int \psi_0^* \psi_0 = \int d\mu(H) e^{2C_A S} < \infty$$

POTENTIAL ENERGY

$$V = \frac{\pi}{m C_A}\left[\int_{x,y} \sigma(x,y,\epsilon)\, \bar{\partial} J^a(x) [H(x,\bar{y})\, H^{-1}(y,\bar{y})]_{ab} \bar{\partial} J_b(y) - \frac{C_A \dim G}{4\pi^2 \epsilon^2}\right]$$

IMPORTANT RESULT

$$T \cdot V = 2m\, V$$

ANSATZ FOR ψ_0 : $= \exp P$

$$P = \int \bar{\partial} J(x) K(x,y) \bar{\partial} J(y) + \int J(x) J(y) J(z) f(x,y,z) + \cdots$$

SUBSTITUTE & SOLVE.

$$P = -\frac{2}{e^2}\left[\frac{\pi^2}{c_A^2}\,\bar{\partial} J_a \frac{1}{m+\sqrt{m^2-\nabla^2}}\,\bar{\partial} J_a + \int f_{abc}\, J_a(x)\, J_b(y)\, J_c(z)\, f(x,y,z) + \cdots\right]$$

$$f(x,y,z) = \int (2\pi)^2\,\delta(k+p+q)\; e^{i(kx+py+qz)} \times \left(\frac{\pi}{2c_A}\right)^3 \frac{(\sqrt{k^2+m^2}-m)(\sqrt{p^2+m^2}-m)}{\sqrt{k^2+m^2}+\sqrt{p^2+m^2}+\sqrt{q^2+m^2}} \times \left(\frac{\bar{k}-\bar{p}}{kp}\right)$$

SECOND WAY: START WITH $\Psi_0 = 1$ & INCLUDE V PERTURBATIVELY AND SUM.

$\rightarrow$ SAME RESULT

$$\Psi_0 = e^{P}$$

$$\approx \exp\left[-\frac{1}{2e^2}\int B\,\frac{1}{\sqrt{-\nabla^2}}\,B\right] \qquad \frac{k}{m} \gg 1$$

HIGH MOMENTUM

$$\approx \exp\left[-\frac{1}{4e^2 m}\int B^2\right] \qquad \frac{k}{m} \ll 1$$

$G(J^3, J^4 \ldots)$ TERMS ARE SMALL AT $k \gg e^2$, $k \ll e^2$

$$W_F(C) = \mathrm{Tr}_F\, P\, e^{\frac{\pi}{c_A}\oint_C J}$$

$$\langle W_F(C)\rangle = (\text{constant})\; e^{-\sigma\, \mathcal{A}(C)}$$

$$\sqrt{\sigma} = e^2\sqrt{\frac{N^2-1}{8\pi}}$$

LARGE N ✓

EXACT ?

	SU(2)	SU(3)	SU(4)	SU(5)
$\frac{\sqrt{\sigma}}{e^2}$	0.345	0.564	0.772	0.977
(LATTICE: TEPER)	0.335	0.553	0.758	0.966

DIFFERENCE $\lesssim 3\%$

MASS GAP, "RESUMMED" PERTURBATION THEORY

$$T = m\left[\int J \frac{\delta}{\delta J} + \Omega \frac{\delta}{\delta J}\frac{\delta}{\delta J}\right] \qquad m = \frac{e^2 c_A}{2\pi}$$

$$T\, J^a = m\, J^a \qquad \text{(EXACT)}$$

$$(T+V)\, J^a = \sqrt{k^2+m^2}\; J^a + \cdots$$

$$\Psi = e^{-c_A S(A)}\,\Phi \qquad \langle 1|2\rangle = \int d\mu(A)\, \Phi_1^* \Phi_2$$

$$\mathcal{H} \to e^{-c_A S}\, \mathcal{H}\, e^{-c_A S}$$

$$H = M^\dagger M = e^{t^a \varphi^a} \qquad \phi = \sqrt{-\partial\bar\partial}\;\varphi$$

$$\mathcal{H} = \int \frac{1}{2}\left[-\frac{\delta^2}{\delta\phi^a\delta\phi^a} + \phi^a(-\nabla^2\phi^a) + m^2\phi^a\phi^a + \cdots\right]$$

PROPAGATOR MASS FOR GAUGE PARTICLES (MAGNETIC MASS)

$$m \simeq \frac{e^2 c_A}{2\pi}$$

$$\simeq 0.32\, e^2 \qquad \text{FOR } SU(2)$$

LATTICE VALUES

$\frac{m}{e^2}$ =	
0.35	COMMON FACTOR FROM GLUEBALL MASSES PHILIPSEN
0.51	MAX. ABELIAN GAUGE (KARSEN ET AL)
0.52	LANDAU "
0.44	$\lambda_3 = 2$ "
0.38	RESUMMATION OF PERTURBATION THEORY A.N.
0.28	B.P. & J.P.
0.37	PHILIPSEN GAUGE-INVARIANT LATTICE DEFINITION

GLUEBALLS

$$T\, J_a = m\, J_a$$

J_a IS NOT A GOOD STATE. WE NEED HOLOMORPHIC INVARIANCE.

$$A = -\partial M M^{-1} \qquad M,\ MV(\bar z) \longrightarrow \text{SAME } A.$$

$$J \to V J V^{-1} - \partial V V^{-1}$$

$$\bar\partial J \to V\, \bar\partial J\, V^{-1}$$

A ∂J- STATE (0^{++})

$$\Psi_2 = \int f(x,y)\, :\bar\partial J_a(x)\, [H(x,\bar y)\, H^{-1}(y,\bar y)]_{ab}\, \bar\partial J_b(y):$$

TAKE SAME $x, y \to f(x,y) = \sigma(x,y,\epsilon)$

$$T\, \Psi_2 = 2m\, \Psi_2.$$

HIGHER NUMBER OF J's

$$\Psi^n : \bar\partial J^{a_1} \bar\partial J^{a_2} \cdots \bar\partial J^{a_n} : \quad \underbrace{w_{a_1 \cdots a_n}}_{\text{INVARIANT TENSOR OF SU(N)}}$$

$$T\, \Psi^n = m_n\, \Psi^n$$

ONE CAN INCLUDE C.M. MOTION FROM $\int B^2/2e^2$.

RELATIVE MOTION $\uparrow$ HIGHER "RADIAL" EXCITATIONS

REGGE TRAJECTORY

BACK TO Ψ_2:

$$\left\{ 2m + \left(-\frac{\nabla_1^2}{2m} + \frac{\nabla_2^2}{2m}\right) + m \log \frac{|x-y|}{2e} \right\} f(x,y) = E\, f(x,y)$$

YANG-MILLS + CHERN-SIMONS

$$\Psi = \exp\left[k\, S(M^\dagger) - \frac{k}{4\pi}\int A^a \bar{A}^a\right] \Phi(H)$$

$$\langle 1|2\rangle = \int d\mu(H)\, e^{(k+2c_A)\, S}\, \Phi_1^* \Phi_2$$

$$T = \frac{e^2}{4\pi}(k+2c_A)\int J\cdot\frac{\delta}{\delta J} + \frac{e^2 c_A}{2\pi}\int \Omega \frac{\delta}{\delta J}\frac{\delta}{\delta J}$$

$$\Phi_0(H) = \exp\left[\frac{-\pi}{m c_A}\int \bar{\partial} J \frac{1}{\tilde{m} + \sqrt{\tilde{m}^2 - \nabla^2}}\, \bar{\partial} J\right]$$

$$\tilde{m} = \frac{e^2}{4\pi}(k+2c_A)$$

→ NEW INTEGRABLE OPERATORS FROM CFT

→ SCREENING OF $W_F(C)$

→ A NUMBER OF EIGENSTATES OF T CAN BE CONSTRUCTED

→ LARGE k, STANDARD PERTURBATION THEORY