Hydrodynamics from gauge/gravity duality

Dam Thanh Son

INT, University of Washington
Plan

- Hydrodynamics
  - Relation to finite-temperature field theory
  - First-order hydrodynamics
  - Second-order hydrodynamics

- Gauge/gravity duality
  - Hydrodynamics as low-energy dynamics of black-brane horizons
  - Second-order transport coefficients from gauge/gravity duality

Refs: R. Baier, P. Romatschke, DTS, A. Starinets, M. Stephanov, arxiv:0712.2451
related work: Bhattacharyya, Hubeny, Minwalla, Rangamani, arxiv:0712.2456;
Loganayaram, arxiv:0801.3701
Why hydrodynamics

- Applications, e.g., elliptic flow in heavy ion collisions
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- Conceptually a much simpler theory than QFT:
  - Few d.o.f.
  - Classical: bosonic modes at $\omega \ll T$
  - “Every cook has learned how to use hydrodynamics”
Why gauge/gravity duality

Practical consideration:

- Strong coupling, not treatable by other methods
- Simple calculations

Conceptual consideration:

- Deep connection between QFT and black-hole physics
- Sharp contrast to weak coupling:
  weak coupling: QFT $\rightarrow$ kinetic theory $\rightarrow$ hydro
  strong coupling: QFT $\rightarrow$ hydro

Hope: more to be discovered, e.g., for second-order hydrodynamics.
Hydrodynamics as effective theory

Consider a finite-temperature interacting QFT
Real-time: close-time-path formalism

\[ Z[g^{1}_{\mu\nu}, g^{2}_{\mu\nu}] = \int D\psi_1 D\psi_2 \exp(iS[\psi_1, g^{(1)}_{\mu\nu}] - iS[\psi_2, g^{(2)}_{\mu\nu}]) \]

\[ \langle T^{\mu\nu}(x)T^{\alpha\beta}(y) \rangle = \frac{\delta^2 \ln Z}{\delta g_{\mu\nu}(x) \delta g_{\alpha\beta}(y)} \]

We want an effective field theory that gives correlators of \( T^{\mu\nu} \) at low momenta.

Hydrodynamics: gives \( \langle T^{\mu\nu} \rangle \) for any given smooth source \( g^{1}_{\mu\nu} = g^{2}_{\mu\nu} \).

Validity: length scales \( \gg \) mean free path
Degrees of freedom

- Chiral perturbation theory: d.o.f. = Goldstone modes
- Hydrodynamics: d.o.f = “collective coordinates” of thermal ensemble
  For a plasma with no conserved charge:
  - Temperature $T(x)$
  - Velocity (boost) $u^\mu(x)$, $u^2 = -1$

Optional:
- $\mu(x)$ for each conserved charge
- Phases of condensate (superfluid hydrodynamics)
- U(1) magnetic fields (magnetohydrodynamics)
Ideal and first-order

Ideal (zeroth order) hydrodynamics

\[ \nabla_\mu T^{\mu\nu} = 0 \quad T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu} \]

First-order hydrodynamics (relativistic Navier-Stokes)

\[ T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + \Pi^{\mu\nu} \]

Ambiguity of defining \( u^\mu \) beyond leading order: fixed by \( u_\mu \Pi^{\mu\nu} = 0 \)

\[ \Pi^{\mu\nu} = -\eta \nabla^{\langle\mu} u^{\nu\rangle} - \zeta P^{\mu\nu} (\nabla \cdot u) \]

\[ P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \]

\[ A^{\langle\mu\nu\rangle} = \frac{1}{2} P^{\mu\alpha} P^{\nu\beta} (A_{\alpha\beta} + A_{\beta\alpha}) - \frac{1}{3} P^{\mu\nu} P^{\alpha\beta} A_{\alpha\beta} \]

Shear viscosity \( \eta \) and bulk viscosity \( \zeta \). Affect damping of shear and sound modes.
Second order: Müller-Israel-Stewart

Modified relationship between $\Pi^{\mu\nu}$ and $\nabla^\mu u^\nu$.

$$(\tau_\pi u^\lambda \nabla_\lambda + 1)\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}$$

$\tau_\pi D \ll 1$: equivalent to keeping one next term in derivative expansion

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \eta\tau_\pi (u \cdot \nabla)\sigma^{\mu\nu},$$

Matching with AdS/CFT, $\mathcal{N} = 4$ SYM

Bjorken flow: $\tau_\pi = \frac{1 - \ln 2}{6\pi T}$

sound: $\tau_\pi = \frac{2 - \ln 2}{2\pi T}$

This indicates not all second-order terms are taken into account.

Need to include all second-derivative terms consistent with symmetry. After eliminating redundant ones: 16 independent terms
Conformal invariance

Assume fundamental theory is a CFT,

\[ T^\mu_\mu = 0 \quad \text{in flat space} \]

In curved space: Weyl anomaly

\[ g_{\mu\nu} T^{\mu\nu} \sim R^2_{\mu\nu\alpha\beta} \quad \text{in curved space} \]

But \( R \sim \partial^2 g_{\mu\nu} \): Weyl anomaly reproduced in hydrodynamics only at fourth order in derivatives.

\[ \Rightarrow g_{\mu\nu} T^{\mu\nu} = 0 \quad \text{for our purposes} \]

First order: \( \zeta = 0 \),

Second order: 8 possible structures in \( \Pi^{\mu\nu} \)
Conformal invariance (II)

Further constraint: $T^{\mu\nu}$ transforms simply under Weyl transformation

$$g_{\mu\nu} \rightarrow e^{2\omega} g_{\mu\nu}, \quad T_{\mu\nu} \rightarrow e^{6\omega} T_{\mu\nu}$$

8 → 5 possible structures in $\Pi^{\mu\nu}$

$$\Pi^{\mu\nu}_{2nd\ order} = \eta^{T_\pi} \left[ \langle D\sigma^{\mu\nu} \rangle + \frac{1}{3} \sigma^{\mu\nu} (\nabla \cdot u) \right] + \kappa \left[ R^{\langle \mu\nu \rangle} - 2u_\alpha R^{\alpha\langle \mu\nu \rangle \beta} u_\beta \right]$$

$$+ \lambda_1 \sigma^{\langle \mu \lambda \sigma^{\nu} \rangle \lambda} + \lambda_2 \sigma^{\langle \mu \lambda \Omega^{\nu} \rangle \lambda} + \lambda_3 \Omega^{\langle \mu \lambda \Omega^{\nu} \rangle \lambda}$$

$$D \equiv u^\mu \nabla_\mu$$

$$\sigma^{\mu\nu} = 2\nabla^{\langle \mu} u^{\nu \rangle}$$

$$\Omega^{\mu\nu} = \frac{1}{2} (\nabla^{\langle \mu} u^{\nu \rangle} - \nabla^{\nu} u^{\mu \rangle}) \quad \text{vorticity}$$

$\kappa$ only in curved space, but affects 2-point function of $T^{\mu\nu}$

$\lambda_i$ nonlinear response
Hydrodynamics from AdS/CFT

Main philosophy:

Finite-$T$ field theory in flat space $\Leftrightarrow$ black hole with flat horizon

Example: nonextremal D3 metric

$$ds^2 = \frac{r^2}{R^2}(-f dt^2 + d\vec{x}^2) + \frac{R^2}{r^2 f} dr^2, \quad f = 1 - \frac{r^4}{r_0^4}$$

Construct a family of configurations by changing $T \sim r_0/R^2$ and boosting along $\vec{x}$ directions by velocity $\vec{u}$

$$g_{\mu\nu} = g_{\mu\nu}(z; T, u^\mu)$$

Promote $T$ and $\vec{u}$ into fields.

Require regularity away from $r = 0 \Rightarrow$ hydrodynamic equations

Concretely realized by

Janik, Peschanski, Heller
Bhattacharyya, Hubeny, Minwalla, Rangamani
Dynamics of the horizon

$$T \sim r_0 = r_0(\vec{x})$$

Generalizing black hole thermodynamics $M, Q,...$ to black brane hydrodynamics

$$T = T_H(\vec{x}), \quad \mu = \mu(\vec{x})$$

Dissipation in QFT $\Leftrightarrow$ dissipative behavior of black hole horizon $\sim$ “black-hole membrane paradigm” Damour; Thorne, Price, McDonald
Kinetic coefficients from AdS/CFT

One strategy to find $\tau_\pi$ and $\kappa$:

- Within hydro: compute some $\langle T^{\mu\nu}T^{\alpha\beta}\rangle$ from linear response theory: response to gravitational perturbations $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$
- Compare with AdS/CFT calculations

Example: for momentum $\omega$, $\vec{k} = (0, 0, k)$

$$\langle T^{xy}T^{xy}\rangle(\omega, k) = P - i\eta\omega + \eta\tau_\pi\omega^2 - \frac{\kappa}{2}(\omega^2 + k^2)$$

from that

$$\eta = \frac{s}{4\pi}, \quad \text{universal result}$$

$$\tau_\pi = \frac{2 - \ln 2}{2\pi T}, \quad \kappa = \frac{\eta}{\pi T}$$

Sound-wave dispersion: $\text{Re}\omega = c_s k + \#k^3 \Rightarrow$ the same value for $\tau_\pi$
Nonlinear coefficients $\lambda_{1,2,3}$

One needs to look beyond small perturbations around thermal equilibrium. $\lambda_1$: can be found from long-time tail of a boost-invariant solution (Janik, Peschanski, Heller):

$$\epsilon(\tau) \sim \frac{1}{\tau^{4/3}} - \frac{2\eta}{\tau^2} + \frac{\#}{\tau^{8/3}}$$

(0)

Maching the coefficient of $\tau^{-8/3}$ term:

$$\lambda_1 = \frac{\eta}{2\pi T}$$

Bhattacheraya et al. also found

$$\lambda_1 = \frac{\eta}{2\pi T}, \quad \lambda_2 = -\frac{2\ln 2}{2\pi T} \eta, \quad \lambda_3 = 0$$
Comparision with Israel-Stewart formalism

- Israel-Stewart equation valid only in hydro regime.
- Frequently terms required by Weyl invariance are thrown away,

\[ \langle D\Pi^{\mu\nu} \rangle + \frac{4}{3}\Pi^{\mu\nu}(\nabla \cdot u) \]

(equivalent to ones used by Romatschke & Romatschke). Such terms may be numerically important (U. Heinz’s talk)

- In addition, \( \lambda_1 = \lambda_3 = 0 \) in IS theory; in \( \mathcal{N} = 4 \) SYM \( \lambda_1 \neq 0 \) (but \( \lambda_3 = 0 \)).
- Additional terms nonlinear: not important for sound wave propagation, but important for Bjorken expansion
Entropy current

Loganayaram

One is unable to force the IS Ansatz $s^\mu = u^\mu + (s + \#\Pi_{\alpha\beta}\Pi^{\alpha\beta})$ to have explicitly positive derivative $\partial_\mu s^\mu \sim \Pi^2$.

More generally, $s^\mu$ has to be expressed in terms of $u^\mu$ and its derivatives,

$$s^\mu = su^\mu + \#u^\mu \Pi^2 + u^\mu \omega^2 + O(u\nabla^2 u)$$

One can construct a current so that

$$\partial_\mu s^\mu = \frac{\eta}{2T} \sigma^{\mu\nu} \sigma_{\mu\nu} + \frac{1}{4}(\kappa - 2\lambda_1)\sigma^\mu_\nu \sigma^\nu_\lambda \sigma^\lambda_\mu$$

Generally not explicitly positive, but positive in the hydrodynamic regime: $\sigma^3 \ll \sigma^2$. 
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\]

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Suprise: in \( \mathcal{N} = 4 \text{ SYM} \) \( \kappa = 2\lambda_1 \)!
Conclusion

- Hydrodynamic behavior of QFT appears naturally from black hole dynamics in a manner much simpler than at weak coupling.
- Second order hydrodynamics: 5 additional coefficients in conformal theories (4 in flat space).
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Second order hydrodynamics: 5 additional coefficients in conformal theories (4 in flat space)

To be done:

- Gravitational loop effects ($1/N_c$): thermal noise + nonlinearity of hydro equation
  - Hydrodynamic long-time tail as quantum gravity Kovtun, Yaffe
- Breaking conformal invariance Buchel
- Second-order transport coefficient at weak coupling, large-$N_c$ QCD, $\mathcal{N} = 4$ SYM can one use some kinetic theory?
- Implications for elliptic flow in heavy ion collisions