Comparison of Boltzmann Kinetics with Quantum Dynamics for Relativistic Quantum Fields

Markus Michael Müller

Nonequilibrium Phenomena in Cosmology and Particle Physics
Kavli Institute for Theoretical Physics, UCSB
St. Barbara CA
February 28, 2008
Motivations for going into the subject

The situation

- Many interesting phenomena in particle physics and cosmology require the description of systems out of thermal equilibrium.

- Very often, such nonequilibrium situations are treated by means of (approximations to) Boltzmann equations.

- However, Boltzmann equations are only a classical approximation to the quantum thermalization process described by Kadanoff-Baym equations.

An obvious question

How reliable are Boltzmann equations as compared to Kadanoff-Baym equations?
Boltzmann Equation

for a **spatially homogeneous** system in the framework of a **real scalar $\Phi^4$** quantum field theory:

$$
\partial_t n(t, \mathbf{k}) = \frac{\lambda^2 \pi}{48} \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \int d^3 r \left[ \frac{1}{E_k E_p E_q E_r} \right. \\
\times \delta (\mathbf{k} + \mathbf{p} - \mathbf{q} - \mathbf{r}) \delta (E_k + E_p - E_q - E_r) \left. - n_k n_p (1 + n_q) (1 + n_r) \right]
$$

**Momentum conservation**

**Energy conservation**

**Isotropy**: 9 dimensional integral $\Rightarrow$ 2 dimensional integral.

Important for numerics! [Dolgov, Hansen, Semikoz (1997)]
Complete Schwinger-Keldysh Propagator

**Definition**

\[ G(x, y) = \langle T_C \{\Phi(x)\Phi(y)\} \rangle \]

The index \( C \) denotes time ordering along the closed Schwinger-Keldysh real-time contour.

**Decomposition** [Aarts, Berges (2001)]

\[ G(x, y) = G_F(x, y) - \frac{i}{2} \text{sign}_C (x^0 - y^0) G_\varrho(x, y) \]

- **Statistical propagator** \( \rightarrow \) effective particle number
- **Spectral function** \( \rightarrow \) thermal mass, decay width
Effective Energy and Particle Number Densities

**Free-field ansatz** [Berges (2002)]

Effective kinetic energy density:

\[
\omega^2 (t, k) = \left( \frac{\partial x^0 \partial y^0 G_F (x^0, y^0, k)}{G_F (x^0, y^0, k)} \right)_{x^0 = y^0 = t}
\]

Effective particle number density:

\[
n(t, k) = \omega(t, k) G_F (t, t, k) - \frac{1}{2}
\]

**Advantages of these definitions**

- They furnish a particle number density which thermalizes.
- They do not rely on any quasi-particle assumption.
- They comprise conserved charges, if present in the theory.
Kadanoff-Baym Equations

for a **spatially homogeneous and isotropic** system in the framework of a **real scalar \( \Phi^4 \) quantum field theory**:

\[
\left[ \partial_{x^0}^2 + k^2 + M^2 \left( x^0 \right) \right] G_F \left( x^0, y^0, k \right) \\
= \int_0^{y^0} dz^0 \Pi_F \left( x^0, z^0, k \right) G_\varrho \left( z^0, y^0, k \right) \\
- \int_0^{x^0} dz^0 \Pi_\varrho \left( x^0, z^0, k \right) G_F \left( z^0, y^0, k \right)
\]

Effective mass: \( M^2 \left( x^0 \right) = m^2 + \) 

Nonlocal self-energy: \( \Pi \left( x^0, z^0, k \right) = \) 

Internal lines represent the complete Schwinger-Keldysh propagator!
Initial Conditions

- All initial conditions correspond to the same (conserved) average energy density.
- The initial conditions IC1 and IC2 correspond to the same initial total particle number.

[Manfred Lindner, MMM (2006)]
Universality

Kadanoff-Baym

Evolution of the particle number densities

Boltzmann

Evolution of the total particle numbers

Full Universality

Restricted Universality

[Manfred Lindner, MMM (2006)]
Chemical Equilibration

Kadanoff-Baym

Equilibrium particle number densities

Boltzmann

Full Universality
- Chemical Equilibration

Restricted Universality
- No Chemical Equilibration

[Manfred Lindner, MMM (2006)]
Separation of Time Scales

**Kadanoff-Baym**

- $t m_R = 0.0$
- $t m_R = 1.6$
- $t m_R = 4.3$
- $t m_R = 9.7$
- $t m_R = 42.4$

**Boltzmann**

- $t m_R = 0.0$
- $t m_R = 1.6$
- $t m_R = 23.3$
- $t m_R = 57.3$
- $t m_R = 113.9$
- $t m_R = 210.0$

[Manfred Lindner, MMM (2006)]
Generalization to fermionic theories

\( SU(2)_L \times SU(2)_R \times U(1)_{B-L} \) symmetric Yukawa model

\[
\lambda (\phi_a \phi_a)^2 + i \eta \bar{\psi} \phi_a \left( \sigma_a P_R - \sigma^+_a P_L \right) \psi
\]

Effective scalar mass:

\[
M^2 \left( x^0 \right) = m^2 + \phantom{\text{ extra diagram here }}
\]

Nonlocal Self Energies:

- Scalars:
  \[
  \Pi \left( x^0, z^0, k \right) = \phantom{\text{ extra diagram here }}
  \]

- Fermions:
  \[
  \Sigma \left( x^0, z^0, k \right) = \phantom{\text{ extra diagram here }}
  \]
Generalization to fermionic theories
Cont.

Kadanoff-Baym Equations [Manfred Lindner, MMM (2008)]
- Full universality [Berges et al. (2003)]
- Quantum-chemical equilibration [Berges et al. (2003)]
- Prethermalization [Berges et al. (2004)]

Boltzmann equations [Manfred Lindner, MMM (2008); MMM (2006)]
- Restricted universality
- Classical, but no quantum-chemical equilibration
- No separation of time scales
Conclusions

Quantum Dynamics (Kadanoff-Baym equations)

- take memory and off-shell effects into account.
- respect full universality.
- include chemical equilibration.
- separate time scales between kinetic and chemical equilibration.

Classical Kinetics (Standard Boltzmann equations)

- do not take memory and off-shell effects into account (molecular chaos for quasi-particles).
- comprise fake constants of motion.
- respect only a restricted universality.
- do not include quantum chemical equilibration, and therefore cannot separate time scales between kinetic and chemical equilibration.
Outlook
Renormalization of the 2PI effective action for a real scalar $\lambda \Phi^4/4!$ theory at three-loop order

### Standard approximate perturbative renormalization

- **A18**: $\lambda = 18$, $m_B^2 = -6.87 \, m_R^2$
- **A24**: $\lambda = 24$, $m_B^2 = -9.49 \, m_R^2$

### Exact nonperturbative renormalization at zero temperature

- **E18**: $\lambda_R = 18$, $\lambda_B = 37.18$, $m_B^2 = -14.39 \, m_R^2$
- **E24**: $\lambda_R = 24$, $\lambda_B = 63.43$, $m_B^2 = -25.14 \, m_R^2$

![Graph showing $G_F(t,t,k=0)$ vs. $|m_R|$ for different renormalization schemes](image_url)