Nonequilibrium Quantum Field Theory

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What is nonequilibrium?

- e.g. nonthermal occupation number

Initial conditions

\[ n(\omega) \]

Early time


Intermediate time


Late time

Bose–Einstein/Fermi–Dirac distribution

Effective loss of initial conditions

Initial conditions

\[ \omega \]

\[ \omega \]

\[ t_0 \]

Instability

Turbulence

Berges et al 2000–..

Khlebnikov 1996

Tkachev, Misha 2004,... Arnold, Moore 2006
An other setting: defects

Strings appear as topological defects in field theory. Nonequilibrium QFT accounts for their decay.

$\xi \gg d$

see Mark Hindmarsh’ talk
The classical approach

Define a lattice field theory, Solve the 2nd order Klein-Gordon (or Maxwell) equations

\[ \Phi_n(t + a_t) + \Phi_n(t - a_t) - 2\Phi_n(t) - \frac{a_t^2}{a^2} \sum_i (\Phi_{n+i}(t) + \Phi_{n-i}(t) - 2\Phi_n(t)) + a_t^2(-\Phi_n + \Phi_n^3 - h) = 0. \]

Initial condition: the field value is sampled from an ensemble that reproduces the n-point functions. Evolution of the ensemble gives the n-point functions at a later time. This evolution is NONPERTURBATIVE!

- continuum limit
- Bose-Einstein
- physical cutoff
- fermions

see Jan Smit’s talk
The kinetic approach

Particles (balls) collide and interact with a precalculated cross section.

Example: Parton thermalisation
with $gg \rightarrow ggg$

with $gg \rightarrow gg$ only

Coherence is lost between collisions.
Gradient expansion has been used. What does justify it?

see also MM Müller’s talk
Initial value problem in QFT

Define path integral along the \textit{closed time path} contour

\[
\langle \hat{X} \rangle (t) = \text{Tr} \hat{\rho}(t) \hat{X}(t_0) = \text{Tr} \hat{U}(t, t_0) \hat{\rho}(t_0) \hat{U}^{-1}(t, t_0) \hat{X}(t_0)
\]

\[
\hat{U}(t, t') = \exp \left[ -i \int_{t'}^{t} \hat{H}(t'') dt'' \right]
\]

\[
Z[J] = \int \mathcal{D}\phi e^{i \int dx [\mathcal{L}(x) + J(x) \phi(x)]}
\]

Propagators:

\[
G^>_i(x, y) = \langle \varphi_i(x) \varphi_j(y) \rangle
\]

\[
G^<_i(x, y) = \langle \varphi_j(y) \varphi_i(x) \rangle
\]

\[
G_{ij}(x, y) = \langle \mathcal{T}_c \varphi_i(x) \varphi_j(y) \rangle
\]

\[
iG_0 = (\partial^2 + m^2)^{-1}
\]

\[
F_{ij}(x, y) = \frac{1}{2} \left( G^>_i(x, y) + G^<_i(x, y) \right)
\]

\[
\rho_{ij}(x, y) = i \left( G^>_i(x, y) - G^<_i(x, y) \right)
\]

\[
G_{ij}(x, y) = F_{ij}(x, y) - \frac{i}{2} \rho_{ij}(x, y) \text{sgn}_C(x_0, y_0)
\]

\text{Aarts, Berges 2001}
Is the dynamics irreversible?

Thermal equilibrium:

\[ \hat{\rho} = \frac{e^{-\beta \hat{H}}}{\text{Tr}e^{-\beta \hat{H}}} \quad \left\langle \hat{X} \right\rangle = \text{Tr}\hat{X}\hat{\rho} \]

Is thermalization possible in closed nonlinear system?

- Equilibrium is a fixed point of the evolution
- \[ \rho \rightarrow e^{-\beta \hat{H}} / \text{Tr}e^{-\beta \hat{H}} \] Unitarity!
- \[ \langle \hat{H} \rangle = \text{const.} \] uniquely determines the equilibrium ensemble.

But: \[ \langle \hat{H}^2 \rangle, \langle \hat{H}^3 \rangle, \ldots \] conserved (initial conditions)

- The quantum ensemble cannot converge to equilibrium!
- Still, the quantum average of some selected observables may converge to the equilibrium value:

\[ \langle \Phi(x)\Phi(y) \rangle_{\text{noneq}} \rightarrow \langle \Phi(x)\Phi(y) \rangle_{\text{thermal}}, \text{ as } x_0, y_0 \rightarrow \infty \]
Perturbation theory fails

Example: a damped oscillator

1st perturbative order:

\[ \ddot{x}(t) + 2\gamma \dot{x}(t) + m^2 x(t) = 0 \]

\[ x(t) = A \sin(t \sqrt{m^2 - \gamma^2}) e^{-t\gamma} \]

\[ x(t) = A \cos(tm)(1 - t\gamma) \]

Secular behaviour: the time is part of the expansion parameter!
A nonperturbative approach: Let’s simulate on a lattice!

Euclidean Langevin equation:

\[
\partial_\vartheta \phi(x, \vartheta) = -\frac{\delta S_E[\phi]}{\delta \phi(x, \vartheta)} + \eta(x, \vartheta)
\]

\[
\langle \eta(x_1, \vartheta_1) \eta(x_2, \vartheta_2) \rangle = 2\delta(\vartheta_1 - \vartheta_2)\delta(4)(x_1 - x_2)
\]

Parisi, Wu 1981

Langevin equation on the closed time path contour

\[
\frac{\partial \phi(C_j)}{\partial \vartheta} = i \frac{\partial S}{\partial \phi(C_j)} + \eta_j(\vartheta)
\]

see Dénes Sexty’s talk

Contour points: \(C_j\)

In this algorithm probabilities are never used.

Reproduces the hierarchy of SD equations.
It really does converge in real time, too!

Toy model: anharmonic oscillator

Use the action with complex $\Delta t$, with two branches and with $\hat{\rho}$ being part of the action.

Comparison with Schrödinger’s equation:

$\langle x(t) \rangle$

$\langle x(t)x(t) \rangle_c$

Challenge: Is the solution unique?

Berges, S.B, Stamatescu, Sexty 2005-...
A diagrammatic approach: the 2PI resummation

\[ \Sigma = G_0 + \cdots \]

\[ G = \cdots \]

\[ = \cdots + \cdots \]

\[ \text{time evolution} \]
... or in a Yukawa theory:

late time behaviour

quarks, p, n
pions
The 2PI effective action

\[ Z [J, K] = \int \prod_{c=1}^{N} \mathcal{D}\phi_c (x) \exp \left( i \int d^4x [\mathcal{L} (x) + J_a (x) \phi_a (x)] + \frac{i}{2} \int d^4x \int d^4y [\phi_a (x) K_{ab} (x, y) \phi_b (y)] \right), \]

1st Legendre transform: effective action (1PI diagrams)
2nd Legendre transform: 2PI effective action (2PI diagrams)

\[ W [J, K] = -i \log (Z [J, K]) \quad \delta W [J, K]/\delta J = \phi \quad \delta W [J, K]/\delta K = (\phi^2 - G)/2 \]
\[ \Gamma [\phi, G] = W [J, K] - \int d^4x \left[ J_a (x) \phi_a (x) \right] \]
\[ -\frac{1}{2} \int d^4x \int d^4y \left[ G_{ab} (x, y) K_{ab} (x, y) + \phi_a (x) K_{ab} (x, y) \phi_b (y) \right] \]

Result: ladder resummation, no overcounting

Cooper at al (2PI, BVA) 2000
Equations of Motion

are the stationarity conditions:

\[(a) \quad \frac{\delta \Gamma[\phi,G]}{\delta \phi_a(x)} = -J_a(x) - \int \mathcal{C} d^4 y \left[ K_{ab}(x,y) \phi_b(y) \right] \uparrow = 0\]

\[(b) \quad \frac{\delta \Gamma[\phi,G]}{\delta G_{ab}(x,y)} = -\frac{1}{2} K_{ab}(x,y) \uparrow = 0 \quad \rightarrow \quad G_{ab}(x,y;\phi) = \langle \mathcal{T}_C \hat{\phi}(x)\hat{\phi}(y) \rangle_c\]

Decomposition:

\[\Gamma_b[\phi,G] = S[\phi] + \frac{i}{2} \text{tr}_C \left[ \log \left[ G^{-1} \right] \right] + \frac{i}{2} \text{tr}_C \left[ G_0^{-1} G \right] + \Gamma_{\text{int}}[\phi,G] + \text{const}\]

\[\Gamma_f[\psi,D] = S[\psi] - i \text{tr}_C \left[ \log \left[ D^{-1} \right] \right] - i \text{tr}_C \left[ D_0^{-1} D \right] + \Gamma_{\text{int}}[\psi,D] + \text{const}\]

With

\[\Sigma_f(x,y) \equiv 2i \frac{\delta \Gamma_{\text{int}}[G]}{\delta G(y,x)} \quad \Sigma_s(x,y) \equiv -i \frac{\delta \Gamma_{\text{int}}[D]}{\delta D(y,x)}\]

\[(\partial_x^2 + m^2)G_{ab}(x,y) = \int \mathcal{C} d^4 z \Sigma_{ab}(x,z;G,D) G_{bc}(z,y) + \delta_C(x,y) \delta_{ab},\]

\[(\partial_x + i m_f)D_{ij}(x,y) = \int \mathcal{C} d^4 z \Sigma_{ik}(x,z;G,D) D_{kj}(z,y) + \delta^4_C(x,y) \delta_{ij}\]

← equivalent to Kadanoff–Baym equations
EoM: in terms of real time propagators:

\[
F_{ij}(x, y) = \frac{1}{2} \left( G^{>}_{ij}(x, y) + G^{<}_{ij}(x, y) \right)
\]

\[
\rho_{ij}(x, y) = i \left( G^{>}_{ij}(x, y) - G^{<}_{ij}(x, y) \right),
\]

(or with opposite signs for the fermions)

For fermionic fields:

\[
(i \partial - m - \Sigma_0) F(x, y) = \int_{x_0}^{y_0} dz \Sigma^\rho(x, z) F(z, y) - \int_{x_0}^{y_0} dz \Sigma^F(x, z) \rho(z, y)
\]

\[
(i \partial - m - \Sigma_0) \rho(x, y) = \int_{y_0}^{x_0} dz \Sigma^\rho(x, z) \rho(z, y)
\]

For scalar fields:

\[
\left( \partial_x^2 + m^2 + \Sigma_{0,i}(x) \right) F_{ij}(x, y) = \int_{y_0}^{x_0} dz \Sigma^F_{ik}(x, z) \rho_{kj}(z, y) - \int_{x_0}^{y_0} dz \Sigma^\rho_{ik}(x, z) F_{kj}(z, y)
\]

\[
\left( \partial_x^2 + m^2 + \Sigma_{0,i}(x) \right) \rho_{ij}(x, y) = \int_{x_0}^{y_0} dz \Sigma^\rho_{ik}(x, z) \rho_{kj}(z, y)
\]
Timescales of losing information

Prethermalisation  Damping, isotropisation  Equilibration

Initial fermion distribution

Equation of state $w = \rho / \rho_0.4$ to $\rho_{0.6}$

Initial fermion distribution

$p = 0.32\text{ m}$  $p = 0.78\text{ m}$  $p = 1.38\text{ m}$

Berges, SB, Serreau, Wetterich 2003/4
evolution of the spectrum:

**Fermions**

- Linear: Fermi-Dirac statistics

**Scalars**

- Linear: Bose-Einstein statistics
A growing number of studies...

**Scalars:**
1+1dim: Cox&Berges 2000, Aarts&Berges 2001, 2002 (vs exact, vs classical)
  - Blagoev&Cooper&Dawson&Mihaila 2001 (BVA)
  - Berges 2002 (O(N) resummation)
  - Gasenzer&Pawlowski 2007 (an RG approach)
2+1dim: Juhem&Cassing&Greinen 2001 (vs transport)
3+1dim: Danielewicz 1984 (nonrelativistic, vs. kinetic theory)
  - Berges&Borsanyi 2005 (isotropisation, vs. transport theory)
  - Muller&Lindner 2005 (vs. kinetic theory)
  - Berges&Serreau 2002 (parametric resonance)
  - Tranberg&Arrizabalaga&Smit 2004, 2005 (bg field, tachionic instability)
  - Tranberg&Rajantie 2006 (looking for defects)
  - Aarts&Tranberg 2008 (inflationary)

**Yukawa:**
3+1dim: Berges&Borsanyi&Serreau/Wettlerich 2003
  - Muller&Lindner 2007 (vs. kinetic theory)

**Cold atoms:**
1+1dim: Berges&Gasenzer(&Seco&Schmidt) 2005, 2007 (vs classical)
  - Gasenzer&Temme 2008 (inhomogeneous)
  - Braunschadel&Gasenzer 2008 (vs. transport)
The final state

The propagators become stationary, and the KMS condition becomes valid.

KMS condition:

\[ G^> (\omega) = e^{\beta \omega} G^< (\omega) \]

\[ F(\omega) = -i \left( \frac{1}{2} + n_B(\omega) \right) \rho(\omega) \]

Boltzmann equation (if \( n \) is time dependent)

Before equilibration: \( F \) and \( \rho \) are related through \( n(t,\omega) \).

Berges, SB, Serreau 2003, SB 2004
What is the stationary solution? from analytics

\[
(-p_0^2 + \omega_p^2) \tilde{F}(p) = \int \frac{d\omega}{2\pi} \left[ \frac{\tilde{\Sigma}^F(p) \tilde{\rho}(\omega; \tilde{p})}{i(p_0 - \omega - i\epsilon)} + \frac{\tilde{\Sigma}^\rho(\omega; \tilde{p}) \tilde{F}(p)}{i(p_0 - \omega + i\epsilon)} \right]
\]

\[
(-p_0^2 + \omega_p^2) \tilde{\rho}(p) = \int \frac{d\omega}{2\pi} \left[ \frac{\tilde{\Sigma}^\rho(p) \tilde{\rho}(\omega; \tilde{p})}{i(p_0 - \omega - i\epsilon)} + \frac{\tilde{\Sigma}^\rho(\omega; \tilde{p}) \tilde{\rho}(p)}{i(p_0 - \omega + i\epsilon)} \right]
\]

If \( \tilde{F}(\omega) = -i \left( \frac{1}{2} + n_{BE}(\omega) \right) \tilde{\rho}(\omega) \rightarrow \tilde{\Sigma}^F(\omega) = -i \left( \frac{1}{2} + n_{BE}(\omega) \right) \tilde{\Sigma}^\rho(\omega) \)

then the two equations are equivalent.

There is a stationary solution that satisfies KMS.

This means late time thermalisation (if \( \Sigma^{F/\rho} \neq 0 \))
The sunset diagram \((2 \rightarrow 2)\)

\[
\Sigma^<(\omega, \bar{x}) = -\frac{\lambda^2}{6} \int \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \frac{d\omega_3}{2\pi} \delta(\omega - \omega_1 - \omega_2 - \omega_3) G^<(\omega_1, \bar{x}) G^<(\omega_2, \bar{x}) G^<(\omega_3, \bar{x})
\]

\[
= -\frac{\lambda^2}{6} \int \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \frac{d\omega_3}{2\pi} \delta(\omega - \omega_1 - \omega_2 - \omega_3) e^{-\beta \omega_1 - \beta \omega_2 - \beta \omega_3} G^<(-\omega_1, \bar{x}) G^<(-\omega_2, \bar{x}) G^<(-\omega_3, \bar{x})
\]

\[
= -\frac{\lambda^2}{6} \int \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \frac{d\omega_3}{2\pi} \delta(\omega - \omega_1 - \omega_2 - \omega_3) e^{-\beta \omega} G^<(-\omega_1, \bar{x}) G^<(-\omega_2, \bar{x}) G^<(-\omega_3, \bar{x})
\]

\[
= -\frac{\lambda^2}{6} \int \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \frac{d\omega_3}{2\pi} \delta(-\omega - \omega_1 - \omega_2 - \omega_3) e^{-\beta \omega} G^<(\omega_1, \bar{x}) G^<(\omega_2, \bar{x}) G^<(\omega_3, \bar{x})
\]

\[
= \Sigma^<(-\omega, \bar{x}) e^{-\beta \omega} \quad \text{(similar argument for any two-loop diagram)}
\]

The self energy inherits the KMS condition from \(G\).

*What we see in numerics is a genuine thermalisation.*
Late time is equilibrium

\[
G_E(\tau; \vec{p}) = \int \frac{d\omega}{2\pi} \tilde{G}^<(\omega, \vec{p}) e^{\tau \omega}
\]

\[
\Sigma_E(\tau; \vec{p}) = \int \frac{d\omega}{2\pi} \tilde{\Sigma}^<(\omega, \vec{p}) e^{\tau \omega}.
\]

and

\[
(-\partial_\tau^2 + \omega_p^2)G_E(\tau) - \delta(\tau) = -\int d\tau' \Sigma_E(\tau - \tau')G_E(\tau').
\]

Euclidean

is equivalent to

\[
\tilde{F}(\omega) = -i \left( \frac{1}{2} + n_{BE}(\omega) \right) \tilde{\rho}(\omega)
\]

\[
\frac{d\rho(t)}{dt} \bigg|_{t=0} = 1
\]

and

\[
(\partial_x^2 + m^2 + \Sigma_0) \rho(x) = -\int_0^{x_0} dz^4 \Sigma^\rho(x - z)\rho(z)
\]

late time

\[
\text{Late time is equilibrium}
\]

\[
\text{Euclidean}
\]

is equivalent to

\[
\text{late time}
\]
Should we believe the dynamics?

Classical 2PI vs classical simulation.

Yes. In most cases. 
(Small enough expansion parameter => exact dynamics)
Topological defects: counterexample!

Aarts, Berges 2001

Rajantie & Tranberg 2006
Suppose you buy 2PI...

What should we think about other approaches?

Classical statistical field theory

Most modelling of Early Universe fields is based on classical methods: preheating, defects

Do we have a classical - quantum comparison?
Classical vs quantum

High occupancy

O(N) NLO

Tachyonic preheating

Aarts, Berges 2001

Defects, low occupancy

Z₂, LO

Arrizabalaga, Smit, Tranberg 2004

SB, Hindmarsh 2007
Suppose you buy 2PI...
What should we think about other approaches?

Classical statistical field theory
Most modelling of Early Universe fields is based on classical methods: preheating, defects

Do we have a classical - quantum comparison?

Transport theory
Boltzmann eq does the same resummations as 2PI.

\[ 2p^\mu \partial_\mu \bar{G} \leq - \{ \bar{\Sigma}^\delta + \text{Re} \, \bar{\Sigma}^R, i \bar{G} \leq \} - \{ i \bar{\Sigma} \leq, \text{Re} \, \bar{G}^R \} = i \bar{\Sigma} \leq \bar{G} \leq - i \bar{\Sigma} \leq \bar{G} \leq \]

Lowest order: 2-to-2 scattering
Particle number conservation

NLO

Lowest Order

Least order: 2-to-2 scattering
Particle number conservation

2PI equations

Transport eq.

Calzetta, Hu 1988

Muller, Lindner 2005
Similarly, for the case of the nonlinear optical (NLO) transport description, we consider the following equations:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{j}_\text{tot}) = S \]

where \( \rho \) is the particle density, \( \mathbf{J}_{\text{tot}} \) is the total current, and \( S \) is the source term. The current is given by the sum of the hadronic and electromagnetic contributions:

\[ \mathbf{J}_{\text{tot}} = \mathbf{J}_h + \mathbf{J}_e \]

where \( \mathbf{J}_h \) is the hadronic current and \( \mathbf{J}_e \) is the electromagnetic current. The hadronic current is given by the convolution of the hadronic density matrix with the hadronic flow:

\[ \mathbf{J}_h = \mathbf{S}_h * \mathbf{f}_{\text{had}} \]

where \( \mathbf{S}_h \) is the hadronic source term and \( \mathbf{f}_{\text{had}} \) is the hadronic flow. The electromagnetic current is given by the explicit expression:

\[ \mathbf{J}_e = e \mathbf{E} \times \mathbf{B} \]

where \( e \) is the charge, \( \mathbf{E} \) is the electric field, and \( \mathbf{B} \) is the magnetic field. The results for the case of the NLO transport description are compared with those from the two-particle (2Pi) approach. For a detailed discussion, see the references.

**Figure 3.** Upper part: Evolution of several momentum modes | Lower part: Time evolution of the correlator.

- 2+1 d Transport
- 3+1 d isotropisation
- 2Pi
- 3+1 d thermalisation

References:

- Juchem, Cassing, Greiner (2004)
- Berges, SB (2005)
How to renormalize?

\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \frac{1}{2} \delta \lambda_2 \]

\[ \delta \lambda_{2,\text{BPZ}} \quad \Delta \lambda_2 \]

References:
- van Hees, Knoll 2001
- Blaizot, Iancu, Reinosa 2003
- Berges, SB, Reinosa, Serreau 2004, 2005
- Cooper, Mihaila, Dawson 2004, 2006
the Bethe-Salpeter equation

\[ \Lambda = 4 \frac{\delta^2 \Gamma_2[G, \Phi]}{\delta G \delta G} \]

implements a one-channel resummation of the four-point function.

This is the same resummation as in the 2-point equation.

Renormalisation of this 4-point equation removes all sub-divergences from the 2-point equation.
Renormalization: the lazy way

Renormalize at $T_1$ and $T_2$ independently

$$G^{-1} = G_0^{-1} - \Sigma$$

The renormalization condition for $\Sigma$ fixes $\delta m^2 + \delta \lambda$, but not $\delta m^2$ and $\delta \lambda$ individually.

Keep $\delta \lambda_1 = 0$, $\delta \lambda_2 = 0$ and obtain $\delta m_1$, $\delta m_2$ and the divergent tadpoles.

$$\delta m_1^2 = m_1^2(T_1) + \delta m^2 + \delta \lambda_1$$
$$\delta m_2^2 = m_2^2(T_2) + \delta m^2 + \delta \lambda_2$$

Matching:

$$m_1^2(T) \sim \lambda_R T^2$$

Perturbative input: this defines the renormalized coupling.
Finiteness is not spoiled by the use of non-resummed input!

This realizes a renormalization condition like:

$$V|_{k^*} = \lambda_R + O(\lambda_R^2)$$

(at leading order)

Instead of this one:

$$V|_{k^*} = \lambda_R$$

Proof of these statements: follows from the Bethe-Salpeter machinery.
The 2PI propagator

The 2PI variational propagator: \[
\frac{\delta \Gamma_{2PI}[\Phi, G]}{\delta G(x,y)} = 0
\]

\[
G_{2PI}^{-1}[\Phi] = G_0^{-1}[\Phi] - \Sigma[\Phi, G]
\]

Without truncation \( G_{2PI} \) is the full propagator.
If we do truncate at some order:

In the O(N) model \( G_{2PI} \) is gapless to given order only.
In QED \( G_{2PI} \) is not transversal to given order only.

(This symmetry breaking effect appears at orders higher than the truncation of \( \Gamma_{2PI} \).)

Reason for the apparent failure: only the s-channel was resummed.
or from the

**standard effective action**

At vanishing sources:

This is the resummed effective action (non-polynomial)

An alternative definition of the propagator:

\[
G_{\text{1PI}}^{-1} = \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi \delta \Phi} = \frac{\delta^2 \Gamma_{\text{2PI}}[\Phi, G_{\text{2PI}}[\Phi]]}{\delta \Phi^2}
\]

\[
G_{\text{proper}}^{-1} = \frac{\delta^2 \Gamma_{\text{2PI}}[\Phi, G_{\text{2PI}}[\Phi]]}{\delta \Phi_x \delta \Phi_y} = G_0^{-1} - 2 \frac{\delta^2 \Gamma_2}{\delta \Phi_x \delta \Phi_y} + \Sigma'
\]

\[
\Sigma' = \frac{\delta \Sigma}{\delta \Phi}
\]

\[
\Lambda = 4 \frac{\delta^2 \Gamma_2[G, \Phi]}{\delta G \delta G}
\]

Bethe-Salpeter equation appears here naturally
**Four point function from 2PI**

\[ i \Gamma_{1234}^{(4)} = iV + \frac{1}{2}iV \]

\[ = iV + \frac{1}{2}iV \]

\[ = \frac{1}{2}iV + \frac{1}{2}iV \]

All three channels are present

**Bethe-Salpeter equation**
resummation in one channel only

Berges, SB, Reinosa, Serreau 2004
restoration of the Goldstone theorem

\[ G_{2\Pi I} : \]

\[ G_{1\Pi I} : \]

s channel only

s+t+u channels

van Hees, Knoll 2001
Berges, SB, Reinosa, Serreau 2004
restoration of the Ward identities

$G_{2\Pi}$ : s channel only

$G_{1\Pi}$ : s+t+u channels

2PI effective action is just a means to ladder-resum the standard effective action

Reinosa, Serreau 2006-7
Carrington, Kovalchuk 2007
Can we do gauge fields?

2-loop order

resummed: 3-loop order

Broken gauge invariance: new counterterms appear.

Gauge fixing: Covariant gauge: $\lambda = 1/\xi$

Usual counterterms:

$$ (e^2 \log a) \sim \delta Z_3, \quad \delta Z_2 $$

New counterterms:

$$ (e^4 \log a) \sim \delta \lambda G^{\mu\nu} k_\mu k_\nu $$
$$ (e^4 a^2) \sim \delta M^2 G^{\mu\nu} g_{\mu\nu} $$
$$ (e^4 \log a) \sim \delta g^a G_{\mu\nu} G^{\mu\nu} $$
$$ (e^4 \log a) \sim \delta g^b G^{\mu}_{\mu} G^{\nu}_{\nu} $$

Subdivergency in the ladder:
Calculated as the solution of the Bethe-Salpeter equation

Bethe-Salpeter -> $V_L^{\mu\nu} |_{k^*} = 0$

\[ \frac{\partial \Pi}{\partial k^2} |_{k^*} = 0 \]
\[ \Pi_L(k^*) = 0 \]
The 2PI pressure curve

Pressure is quartically divergent

$\rightarrow$ we calculate

$$p(T_1) - p(T_2)\over T_1^4 - T_2^4$$

$T_1$: find the counterterms, calculate the pressure

$T_2$: use the counterterms, calculate the pressure

(The regularized equations are solved)

$$P = T \over L_3 i \Gamma[\phi_0, D(\phi_0)]$$

$$S = \partial P \over \partial T$$

$$\varepsilon = -P + TS = T^2 \partial \over \partial T {P \over T}$$

(a scalar example)

Berges, SB, Reinosa, Serreau 2004
The pressure curve: QED
(parameter: gauge fixing)

see also Andersen & Strickland 2005

SB, Reinosa 2007
Restoration of gauge parameter independence

“Strong” gauge parameter independence:
(e.g. Perturbation theory)

\[ \text{pressure}(N,e,\xi) \text{ is } \xi \text{ independent for any } N \]

\( N \): loop order, \( e \) coupling
\( \xi \): gauge parameter

“Weak” gauge parameter independence:
(e.g. 2PI effective action)

\[ \text{pressure}(N,e,\xi) \text{ } \xi \text{ dependence at } O(e^{2N+2}) \]

\( N(e, \xi) \): order required for the required precision

\( N_{2PI} < N_{\text{pert}} \), and \( N_{2PI}(e, \xi=0) < N_{2PI}(e, \xi) \)

Arrizabalaga, Smit 2002
Conclusion

Long live 2PI!

Self-consistent, Cures secularity, Renormalisable (and we know how to renormalise)
Gives a prescription for symmetry-respecting propagators
Gauge symmetry is restored as we increase the order in $g$
(all gauges are equal, but some gauges are more equal)

We need:
more people, more jobs, more machines.