Aspects of thermodynamics of the $\mathcal{N} = 4$ theory on $S^3$

S. Prem Kumar

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(KITP, Santa Barbara)
• Gauge theories exhibit a rich variety of thermodynamic phases:
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E.g. QCD phase diagram

![Diagram](image)
- $SU(N)$ Pure Yang-Mills in 3+1 dimensions (+ adjoint matter)

- This theory has $Z_N$ center symmetry

- $T \neq 0$ thermodynamics: theory on $R^3 \times S^1$

- Order parameter for $Z_N$: $u_1 = 1/N \text{Tr} \exp i \int_0^\beta A_0 d\tau$

- Polyakov loop

- Low $T$: $\langle u_1 \rangle = 0 = \Rightarrow$ Confined Phase

- First Order Transition (Svetitsky, Yaffe)

- High $T$: $\langle u_1 \rangle \neq 0 = \Rightarrow$ Deconfined Phase $\rightarrow Z_N$ breaking

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Yang-Mills theories on finite volume can also have interesting thermodynamics as $N \rightarrow \infty$
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Motivation:

- $\text{AdS/CFT}$ correspondence.
- $\mathcal{N} = 4$ SUSY Yang-Mills at large $N \equiv \text{String theory on } \text{AdS}_5 \times S^5$
Yang-Mills theories on finite volume can also have interesting thermodynamics as $N \to \infty$.

**Motivation:**
- AdS/CFT correspondence.
- $\mathcal{N} = 4$ SUSY Yang-Mills at large $N \equiv$ String theory on $AdS_5 \times S^5$.

Field theory on $S^3 \times R_t \simeq$ conformal boundary of global $AdS_5$.

$T \neq 0$: $\mathcal{N} = 4$ SYM on $S^3 \times S^1 \simeq$ boundary of Euclidean $AdS_5$ space with thermal $S^1$. 

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• Radius of $S^3 = R$

\[ \text{Radius of } S^1 = \beta = \frac{1}{T} \]

\[ \implies \text{Two dimensionless tunable parameters:} \]

\[ \text{'t Hooft coupling } \lambda = g^2 N \text{ and Temperature } TR \]
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Two tractable regimes at $N = \infty$:

- $\lambda \to \infty$ Classical SUGRA
- $\lambda << 1$ Weakly coupled gauge theory on $S^3$
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  Radius of $S^1 = \beta = \frac{1}{T}$

$\Rightarrow$ Two dimensionless tunable parameters:

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Two tractable regimes at $\mathcal{N} = \infty$:

▶ $\lambda \rightarrow \infty$ Classical SUGRA

▶ $\lambda \ll 1$ Weakly coupled gauge theory on $S^3$

• SUGRA on $AdS_5$ yields $\lambda \rightarrow \infty$ field theory dynamics

• Can gauge theory at $\lambda \ll 1$ provide a window into AdS gravity?
Free theory \((\lambda = 0)\) on \(S^3 \times S^1\)

(Sundborg; Aharony, Marsano, Minwalla, Papadodimas, van Raamsdonk)

- Hamiltonian on \(S^3 = \Delta\): Dilatation operator on \(\mathbb{R}^4\).
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- Hamiltonian on $S^3 = \Delta$: Dilatation operator on $\mathbb{R}^4$.

- Physical states $\simeq$ All gauge-invariant words

  E.g. $\text{Tr} \left[ \phi_1 \phi_2 \ldots \phi_2 \phi_2 \ldots \right]$  

  Energy $\sim L$

- No. of states with energy $L \sim e^{\#L}$: Hagedorn density
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- $Z = \text{Tr} \ e^{-\beta \Delta}$ can be computed in a Wilsonian approach:

- $A_0$ has a zero mode on $S^3 \times S^1$

  $\alpha = \int_0^\beta A_0 d\tau \quad U \equiv e^{i\alpha}$
Integrating out all KK harmonics on $S^1 \times S^3$, obtain an effective action for the zero mode of $U = e^{i\alpha}$

$$Z = \int [dU] \exp\left[\sum_{m=1}^{\infty} a_m (TR) \text{Tr} U^m \text{Tr} U^\dagger m\right]$$

$\mathbb{Z}_N$-invariant effective action

$$u_n = \frac{1}{N} \text{Tr} U^n \quad n = 1, 2, \ldots$$
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$Z_N$-invariant effective action

$$u_n = \frac{1}{N} \text{Tr} U^n \ n = 1, 2, \ldots$$

• Eigenvalues $(\alpha_1, \alpha_2, \ldots \alpha_N)$ experience Vandermonde repulsion

$$\sim \log |\sin\left(\frac{\alpha_i - \alpha_j}{2}\right)| +$$

$T$-dependent attraction
• First order Hagedorn/Deconfinement transition at
  \[ T_H \approx 0.38 R^{-1} \]
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- \( \rho(\alpha) \)

\[ \begin{array}{c}
-\pi \\
T < T_H \\
\pi 
\end{array} \]

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• Change in free energy \( \mathcal{O}(N^2) \)
• First order Hagedorn/Deconfinement transition at $T_H \approx 0.38 R^{-1}$

- Change in free energy $\mathcal{O}(N^2)$

\[
\langle u_1 \rangle = 0 \quad \quad \langle u_1 \rangle \neq 0 \quad \mathbb{Z}_N \text{ breaking}
\]
• $\lambda = 0$ picture consistent with $\lambda = \infty$

At $\lambda = \infty$: first order Hawking-Page transition between Thermal AdS and the Big AdS-Schwarzschild Black Hole

\[
W = \langle \frac{1}{N} \text{Tr} e^{i \int_\beta (A_0 + \ldots)} \rangle = 0
\]

Th AdS

\[
W = e^{-S_{F1}} \neq 0.
\]

AdS BH
• The picture at $\lambda \ll 1$ unresolved. Depending on the sign of $b$ in

$$V = N^2(m^2(T)|u_1|^2 + b|u_1|^4); \quad b \sim \lambda^2$$
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**Possibility 2**

**Possibility 1**
Introducing Chemical Potentials

- Chemical potentials \((\mu_1, \mu_2, \mu_3)\) for \(U(1)^3 \subset SU(4)_R\) global symmetry.

- The \(\mathcal{N} = 4\) scalars \(\phi_i\) transform as a \(6\) of \(SU(4)_R\),
  Fermions \(\psi^A\) as a \(4\).
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\[
\mathcal{L}_E \rightarrow \mathcal{L}_E - \frac{1}{2} \mu^2_p \text{Tr} (\phi^2_p + \phi^2_{2p-1}) - i \mu_p \text{Tr} \phi_{2p} D_0 \phi_{2p-1} + \ldots
\]
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\mathcal{L}_E \rightarrow \mathcal{L}_E - \frac{1}{2} \mu_p^2 \Tr (\phi_p^2 + \phi_{2p-1}^2) - \frac{i}{2} \mu_p \Tr \phi_{2p} D_0 \phi_{2p-1} + \ldots
\]

- On \(S^3\), all scalars have a conformal mass \(\frac{1}{R^2}\)

\[
V_0 = \frac{N}{\lambda} \Tr \left( \frac{1}{2} (R^{-2} - \mu_p^2)(\phi_{2p}^2 + \phi_{2p-1}^2) - [\phi_a, \phi_b]^2 \right)
\]
Phase diagram at $\lambda = 0$

(Yamada, Yaffe)

Energy unbounded from below for $\mu > \mu_c \equiv \mathcal{R} - 1$

With $T \neq 0$, $\mu \leq \mu_c$ the grand canonical partition sum

$$Z = \text{Tr} e^{-\beta (\Delta - \mu p J_p)} = \int \left[ dU \right] \exp \left[ \sum_m a_m^{(\mu_p, T)} \text{Tr} U_m \text{Tr} U_m^\dagger \right]$$
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Small non-zero coupling

(Hollowood, SPK, Naqvi, to appear)

- For $\mu_p > \mu_c$, classical theory is still unstable along mutually commuting scalar directions.
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- New light, interacting scalar degrees of freedom appear for $\mu_p \simeq \mu_c$ and $T = 0$. 

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- New light, interacting scalar degrees of freedom appear for $\mu_p \approx \mu_c$ and $T = 0$.

- Classically, at $\mu_p = \mu_c$, flat directions parametrized by constant diagonal modes of $(\phi_{2p}, \phi_{2p-1})$. 
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- Along the classically flat directions

\[
\begin{pmatrix}
\phi_{a1} & . & . & . \\
. & \phi_{a2} & . & . \\
. & . & \phi_{a3} & . \\
. & . & . & .
\end{pmatrix}
\]

integrate out all heavy off-diagonal modes, $m^2 \sim |\phi_i - \phi_j|^2 + \ell^2$. 

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\[
S_S \left( 3^L \right) = \frac{1}{2} g_2^2 T \left( \nabla_i A^i + \bar{\nabla} A_0 - i [\phi, \bar{\phi}] \right)^2 + \bar{\psi} \left( - \bar{\nabla}^2 - \Delta (s) + [\phi,.] \right) \psi.
\]
Background field gauge on $S^3$

$$\mathcal{L}^{(gf)} = \frac{1}{2g^2} \text{Tr} \left[ \left( \nabla_i A^i + \tilde{D}_0 A^0 - i[\phi, \delta \phi] \right)^2 ight. 
\left. + \bar{c} \left(-\tilde{D}_0^2 - \Delta^{(s)} + [\phi, .]^2\right) c \right].$$
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+ \bar{c} (-\tilde{D}_0^2 - \Delta^{(s)} + [\phi, .]^2) c \right].$$

With $\mu_p \neq 0$, $A_0$ and scalar fluctuations mix; Fermions also mix.
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Fluctuation determinants yield Casimir sums at $T = 0$:

\[
V_{1\text{-loop}} \sim \sum_{\text{species}} \sum_{ij=1}^N \sum_{\ell} \deg(\ell) \varepsilon(\ell, |\phi_i - \phi_j|)
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With critical $\mu_p$ fermions can have integer moding on $S^3$:

$$\varepsilon_F = \sqrt{(\ell + \frac{1}{2})^2 + \phi_{ij}^2} \rightarrow \sqrt{(\ell + \frac{1}{2} \pm \frac{\mu_1}{2})^2 + \phi_{ij}^2 \pm \frac{\mu_2}{2} \pm \frac{\mu_3}{2}}.$$
Perform Casimir sums using energy cutoffs on $S^3$

Regularized Casimir sums at $T = 0$ and with critical $\mu_p$:
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Regularized Casimir sums at $T = 0$ and with critical $\mu_p$:

$$V_1^b = (2\pi^2 R^3)^{-1} \sum_{ij=1}^{N} \Lambda^4 R^3 - \frac{1}{2} R \Lambda^2 - R^3 \phi_{ij}^2 \Lambda^2 + \frac{1}{12 R} - \frac{1}{4} \phi_{ij}^2 R$$

$$+ \frac{1}{2} \phi_{ij}^4 R^3 \log \left( \frac{|\phi_{ij}| e^{1/4}}{2\Lambda} \right) + 8 \int_{R\phi_{ij}}^{\infty} \frac{x^2 \sqrt{x^2 R^{-2} - \phi_{ij}^2}}{e^{2\pi x} - 1}.$$
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Regularized Casimir sums at $T = 0$ and with critical $\mu_p$:

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$$V_1^f = (2\pi^2 R^3)^{-1} \sum_{ij=1}^{N} - \Lambda^4 R^3 + \frac{1}{2} R \Lambda^2 + R^3 \phi_{ij}^2 \Lambda^2 + \frac{5}{48 R}$$

$$+ \frac{1}{4} \phi_{ij}^2 R - \frac{1}{2} \phi_{ij}^4 R^3 \log \left( \frac{|\phi_{ij}|e^{1/4}}{2\Lambda} \right) - 8 \int_{R\phi_{ij}}^{\infty} \frac{x^2 \sqrt{x^2 R^{-2} - \phi_{ij}^2}}{e^{2\pi x} - 1}.$$
At critical chemical potential and $T = 0$, radiative corrections vanish, the classical flat directions are not lifted. For $\mu_1 = \mu_c; \mu_2 = \mu_3 = 0$, the new Hamiltonian $\Delta - J_1$ vanishes on all $1/2$ BPS states. These parametrize the ground states since $\{Q^\dagger, Q\} \sim \Delta - J_1$. At a generic point on this moduli space, there is a charged condensate. For two and three critical $\mu_p$, the ground states are the $1/4$ and $1/8$ BPS states.
\[ V_{1}^{b} + V_{1}^{f} = \frac{N^{2}}{Vol(S^{3})} \frac{3}{16R} \]

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- For two and three critical \( \mu_p \), the ground states are the \( \frac{1}{4} \) and \( \frac{1}{8} \) BPS states.
TR \ll 1 \text{ and } |\mu_1 - \mu_c| \lesssim O(\lambda)

- At \mu_1 = \mu_c, switch on a small non-zero T (TR \ll 1)

- Joint potential for \alpha_i and scalars:

\[ V_1 = \sum_{ij=1}^{N} \left( \frac{1}{\text{Vol}(S^3)} \left[ \frac{3}{16R} - 8Te^{-\frac{1}{TR}} \sqrt{1+R^2\phi_{ij}^2} \right] \cos \left( \frac{\alpha_i - \alpha_j}{T} \right) + O(e^{-2/TR}) \right) . \]
$TR \ll 1$ and $|\mu_1 - \mu_c| \lesssim O(\lambda)$

- At $\mu_1 = \mu_c$, switch on a small non-zero $T$ ($TR \ll 1$)
- Joint potential for $\alpha_i$ and scalars:
  \[ V_1 = \sum_{ij=1}^{N} \left( \frac{1}{\text{Vol}(S^3)} \left[ \frac{3}{16R} - 8Te^{-\frac{1}{TR}} \sqrt{1+R^2\phi_{ij}^2} \times \cos \left( \frac{\alpha_i-\alpha_j}{T} \right) + O(e^{-2/TR}) \right] \right). \]
- All $\alpha_i = 0$ – deconfined phase: $u_1 = 1$.
- 1-loop term vanishes at large $\phi_{ij}$, and has positive curvature near $\phi_i = 0$.
- For some values of $\mu \gtrsim \mu_c$, $V_1$ can overcome tree level instability near $\phi_i = 0$. 
For $TR \ll 1$ metastable state with $(\mu_1 - \mu_c) \leq 1$

- Thermal activation and tunnelling rates $\propto \exp\left(-\frac{1}{TR}\right)$

Aspects of thermodynamics of the $\mathcal{N} = 4$ theory on $S^3$
• For $TR \ll 1$ metastable state with $(\mu_1 - \mu_c) \leq \frac{1}{R} \lambda \exp(-\frac{1}{TR})$

• Thermal activation and tunnelling rates $\propto \exp(-Ne^{-\frac{1}{TR}})$
• Width of metastable band \( \sim \lambda e^{-1/TR} \)
At high temperatures $\sqrt{\lambda} \gg 1$, theory is deconfined ($\alpha_i = 0$).

 Scalars have a thermal mass $\lambda T^2$ near the origin $\phi_i = 0$.

 At large $|\phi_{ij}|$, quantum corrections vanish, effective potential has classical behaviour.

 Thus for $\mu_c < \mu < \sqrt{\lambda T^2} + \mu_c^2$, there is a metastable phase near the origin, with decay rate $\sim e^{-N/\lambda^{3/2}}$. 

(Yamada, Yaffe)
High T metastable phase

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- Thus for $\mu_c < \mu < \sqrt{\lambda T^2 + \mu_c^2}$, there is a metastable phase near the origin, with decay rate $\sim e^{-N/\lambda^2}$.
High $T$ metastable potential

\[ \frac{1}{\lambda} > (TR)^2 \gg 1 \]
Weak-strong comparison

\( \lambda \ll 1 \)

- Confined
- Unstable
- Metastable
- Deconfined

(Cvetic, Gubser; Behrndt, Cvetic, Sabra; Yamada)
Weak-strong comparison

(Cvetic, Gubser; Behrndt, Cvetic, Sabra; Yamada)
Further directions

- Unitary matrix model for on $S^3$, truncated to the '$b'$ term, as a model for extracting small black holes; blackhole-string phase transition. (Alvarez-Gaume, Gomez, Liu, Wadia; Basu, Wadia; Dutta, Gopakumar)

- An effective potential for the Polyakov loop from gravity. (Headrick)

- Eigenvalue distributions for the Polyakov-Maldacena both at weak and strong coupling. (Hartnoll, SPK)
Real time correlators at high temperature, $TR \to \infty$ - Poles vs. Cuts.

E.g. $\langle \text{Tr} F^2(t, \vec{x}) \text{Tr} F^2(0) \rangle^\text{ret}_{\omega, \vec{k}}$.

(Hartnoll, SPK)

More generally, branch cuts from graphs at $\lambda \ll 1$ should turn into poles corresponding to BH quasinormal frequencies at $\lambda \to \infty$. 
• Real time correlators as probes of black hole singularities. (Fidkowski, Hubeny, Kleban, Shenker)

\[ \langle O^+(t)O^-(−t) \rangle \sim \frac{1}{(t−t_c)^{2\Delta}}; \quad \Delta \gg 1. \]

• Exponential falloff of correlator at large imaginary frequency. (Festuccia, Liu)

• Remnants of such signals in weakly coupled field theory?