Transport Dynamics at Strong Coupling

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References

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Applications and Motivation

Quantum Phase Transition:
a phase transition between different quantum phases (phases of matter at $T = 0$). Quantum phase transitions can only be accessed by varying a physical parameter — such as magnetic field or pressure — at $T = 0$.

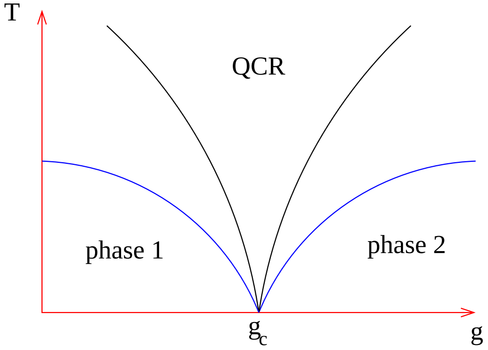


Figure: Phase diagram paradigm

Experimental relevance

Many important physical systems may have quantum critical points (QCPs). The QCP has an effective field theory description which continues to be valid at small "distances" away from the QCP. This quantum critical region may be in an experimentally accessible regime.
Examples:

- superfluid-insulator transition in thin films
- transitions between quantum Hall states
- high temperature, under-doped superconductors at $T > T_c$ and the Nernst effect

Thin Films

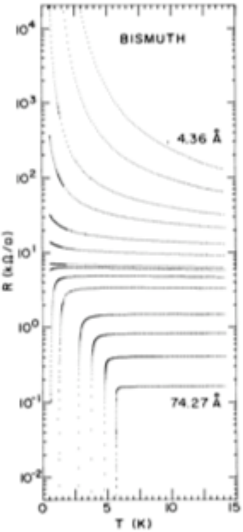


Figure: Haviland, Liu, Goldman, PRL, 62 (1989) 2180

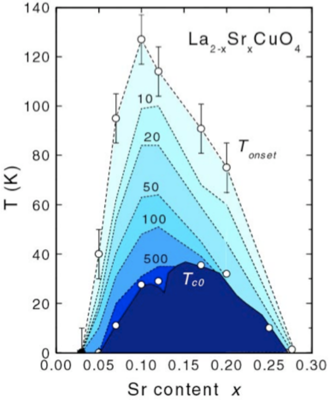
High T_c superconductors

- La_2CuO_4 is an antiferromagnetic insulator
- 2d physics: The Cu atoms arrange themselves into a square lattice on separated sheets.
- Hole doping: substitute some of the La with Sr, $La_{2-x}Sr_xCuO_4$

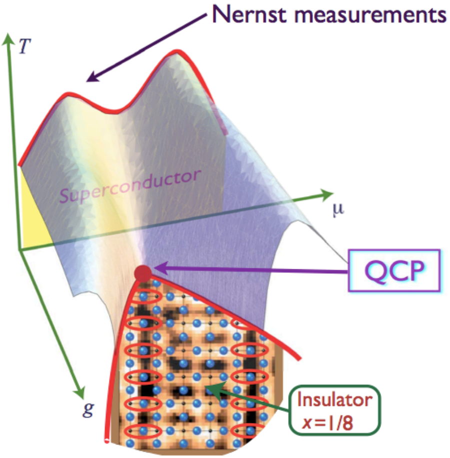
The Nernst effect

- Apply ∇T
- Apply $B \perp \nabla T$
- Measure $E \parallel B \times \nabla T$
- The Nernst coefficient is

$$\nu = \frac{E}{B|\nabla T|}$$



High T_c superconductors and quantum criticality



State of Theory

- There are many lattice models with quantum critical points — Boson-Hubbard model, quantum Ising and rotor models, etc.
- The effective field theory description of the fixed point is scale invariant.
- The field theory sometimes has a Lorentzian symmetry.

$$c \neq 3 \times 10^8 \text{ m/s}$$

- scale invariance + Lorentzian symmery $\implies$ conformal symmetry
- The description is often strongly interacting, e.g. a Wilson-Fisher fixed point

How do we analyze strongly interacting, Lorentzian conformal field theories?

The Sales Pitch

The AdS/CFT correspondence provides a tool to study a class of strongly interacting field theories with Lorentzian symmetry in d dimensions by mapping the field theories to classical gravity in $d+1$ dimensions.

- equation of state
- real time correlation functions
- transport properties — conductivities, diffusion constants, etc.

The ambitious program: There may be an example in this class of field theories which describes the quantum critical region of a real world material such as a high T_c superconductor.

The less ambitious program: By learning about this class of field theories, we may find universal features that could hold more generally for QCPs ($\eta/s = \hbar/4\pi k_B$).

A (hopefully) gentle introduction to AdS/CFT

Basic facts about string theory

- There are open strings (strings with end points) and closed strings (loops).
- Strings may split with a likelihood g_s.
- Strings have a tension $T_0 = 1/\ell_s^2$ where ℓ_s is the string scale.
- Open strings end on massive objects called D-branes. These D-branes have a $m \sim 1/g_s$.

Thinking of time as an extra dimension, strings become surfaces — worldsheets.

What is AdS/CFT?

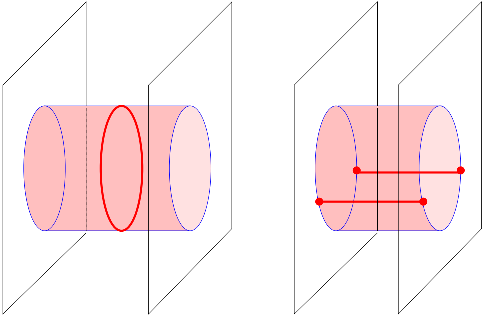


Figure: Open/closed duality

- D-branes are surfaces strings end on
- the lowest closed string mode is the graviton
- the lowest open string mode is a gauge boson

The original correspondence

maximally supersymmetric $SU(N)$ Super Yang Mills in 3+1 dimensions	$\sim$	type IIB closed string theory on $AdS_5 \times S^5$
$g_{YM}^2 N \equiv \lambda$	$=$	L^4/ℓ_s^4
g_{YM}^2	$=$	$4\pi g_s$

AdS_5: Five dimensional anti-de Sitter space. A hyperboloid with a time direction and a boundary.
S^5: Five dimensional sphere.

CFT: This Yang-Mills theory is conformal for all values of λ.

Classical strings: Take $N \to \infty$ with λ fixed means $g_s \to 0$. Strings don't split.
Supergravity: Take λ large. The radius of curvature is large compared to the string scale.

AdS/CFT for 2+1 dimensional field theories

While there exist many examples, the best known and oldest is the "M2-brane theory".

- Consider the maximally supersymmetric $SU(N)$ Yang-Mills theory in 2+1 dimensions.
- Now the coupling is relevant: $g_{YM}^2 = g_s/\ell_s$, $g_{\rm eff} = g_{YM}^2 N/\Lambda$.
- This gauge theory has an interacting superconformal fixed point at low energies.
- Itzhaki, Maldacena, Sonnenschein, and Yankielowicz, hep-th/9802042 conjectured the IR fixed point at large N is a SCFT described via AdS/CFT by 11 dimensional supergravity on $AdS_4 \times S^7$.
- There is no equivalently tuneable λ as there was for the AdS_5/CFT_4 correspondence.

A more concrete statement of the duality

Think of AdS as a half space

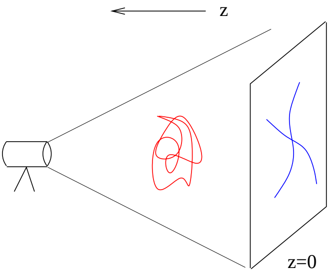


Figure: Bulk information is projected onto the boundary where the field theory lives.

- $\mathcal{O}(x)$ is a field theory operator.
- $\phi_0(x)$ is a source for $\mathcal{O}$ or a boundary value of a supergravity field $\phi(x,z)$.
- $W = \ln Z_{\rm gravity}$ is a generating functional for connected correlators in the field theory.

$$\left\langle \exp\left(\int d^d x \, \phi_0(x)\mathcal{O}(x)\right)\right\rangle_{\rm FT} = Z_{\rm gravity}\left[\phi(x,z)|_{z=0} = \phi_0(x)\right]$$

The Classical Gravitational Limit

- The limit $N \to \infty$ means we are in a classical limit.
- We can evaluate Z_{gravity} through a saddlepoint method.
- Solve Einstein's equations, plug in the solutions to the gravitational action, and call the result W.
- Evaluated on-shell, W has support only at $z = 0$.

Maldacena, Klebanov, Polyakov, Gubser, Witten

Important Field Theory Operators $\mathcal{O}$

- For $\mathcal{O} = T^{\mu\nu}$ the stress-energy tensor, $\phi(x,z) = \delta g_{\mu\nu}(x,z)$ fluctuations in the metric tensor.
- For $\mathcal{O} = J^{\mu}$ a conserved current, $\phi(x,z) = A_{\mu}(x,z)$ a vector potential of a gauge field.

By evaluating the classical gravity action for a solution to Einstein's equations, we can produce a generating functional for $\langle T^{\mu\nu}\rangle$, $\langle J^{\mu}\rangle$, $\langle T^{\mu\nu} J^{\lambda}\rangle$, etc!

N.B. We are computing correlators of a global current. To re-interpret the results for a gauge current, the gauge field has to be very weak. We ignore Coulomb interactions.

The Gravitational Action

The bosonic part of the eleven dimensional supergravity action is

$$\frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-\det(g_{\mu\nu})}R-\frac{1}{4\kappa_{11}^2}\int\left(F_4\wedge\star F_4+\frac{1}{3}A_3\wedge F_4\wedge F_4\right) .$$

Today, we can focus on a 4 dimensional "consistent truncation" of the 11 dimensional action:

$$\frac{1}{2\kappa^2}\int d^4x\sqrt{-\det(g_{\mu\nu})}\left[R-L^2F_{\mu\nu}F^{\mu\nu}+\frac{6}{L^2}\right]$$

Classical gravity with E&M and a negative cosmological constant.

The $F_{\mu\nu}$ is dual to the R-symmetry current in the M2-brane theory.

N.B. many AdS_4/CFT_3's have such a consistent truncation.

Dyonic Black Holes

One solution to this 4d action is a dyonic black hole in AdS_4.
Dyonic black holes have electric and magnetic charge.

- The Hawking temperature of the black hole is the temperature of the field theory.
- The magnetic field of the black hole is the magnetic field in the field theory.
- The electric field of the black hole becomes the charge density of the field theory.

One can freely tune the temperature and charges of the black hole.

Dyonic Black Holes II

The metric or line element:

$$\frac{1}{L^2} ds^2 = \frac{\alpha^2}{z^2}\left[-f(z)dt^2 + dx^2 + dy^2\right] + \frac{1}{z^2}\frac{dz^2}{f(z)}$$

The electric and magnetic fields:

$$\mathcal{B}_z = F_{xy} = h\alpha^2 \; ; \quad \mathcal{E}_z = F_{zt} = q\alpha$$

The warp factor:

$$f(z) = 1 + (h^2 + q^2)z^4 - (1 + h^2 + q^2)z^3$$

The temperature and entropy density ($g^2 = 2L^2/\kappa^2$):

$$T = \frac{(3 - h^2 - q^2)\alpha}{4\pi} \; ; \quad s = \frac{\pi\alpha^2}{g^2}$$

Why electric field becomes charge density

Consider $A_\mu(x,z)$ near the boundary $z=0$ in a gauge where $A_z = 0$:

$$A_\mu = a_\mu(x) + z\, b_\mu(x) + \mathcal{O}(z^2) \ .$$

On shell, the relevant piece of the supergravity action reduces to the boundary term

$$W = \frac{\alpha}{g^2} \int d^3x \left(A_t \partial_z A_t - A_x \partial_z A_x - A_y \partial_z A_y \right) \ .$$

The radial electric field at the boundary $\lim_{z\to 0} \partial_z A_t = \mathcal{E}_z = b_t$ and by our prescription the charge density is thus

$$\rho = \langle J^t \rangle = \frac{\delta W}{\delta a_t} = \alpha b_t / g^2 = \alpha \mathcal{E}_z / g^2 \ .$$

The $\mathcal{B}$ field, by contrast, is a source term $\mathcal{B}_z = \partial_x a_y - \partial_y a_x$.

Transport Coefficient Results from AdS/CFT

Two point functions

By considering small fluctuations around the background values of A_μ and $g_{\mu\nu}$, we can compute two point functions $\langle T^{\mu\nu} T^{\lambda\rho} \rangle$, $\langle T^{\mu\nu} J^{\lambda} \rangle$, and $\langle J^{\mu} J^{\nu} \rangle$. These two points functions determine the conductivities.

$$\begin{pmatrix} \vec{J} \\ \vec{Q} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sigma} & \hat{\boldsymbol{\alpha}} T \\ \hat{\boldsymbol{\alpha}} T & \bar{\boldsymbol{\kappa}} T \end{pmatrix} \begin{pmatrix} \vec{E} \\ -(\vec{\nabla} T)/T \end{pmatrix} .$$

Here $\vec{E}$ is electric field, T is temperature, $\vec{J}$ is charge current and the heat current $Q^{\nu} \equiv T^{0\nu} - \mu J^{\nu}$ where μ is the chemical potential.

Calculating the conductivity

The conductivity is

$$\sigma_{ij} = \frac{J^i}{E_j} = \lim_{z\to 0} \frac{\mathcal{B}_i}{g^2 \mathcal{E}_j} = \frac{B_i}{g^2 E_j}$$

We define

$$\sigma_\pm \equiv \sigma_{xy} + i\sigma_{xx}$$

and

$$E_\pm = E_x \pm iE_y \ ; \quad B_\pm = B_x \pm iB_y \ ; \quad J_\pm = J_x \pm iJ_y \ .$$

Thus Ohm's law becomes $J_\pm = \mp i\sigma_\pm E_\pm$, and

$$\sigma_\pm = \lim_{z\to 0} \frac{\mathcal{B}_\pm}{g^2 \mathcal{E}_\pm} = \frac{B_\pm}{g^2 E_\pm} \ .$$

Electric-magnetic duality

The action for an abelian gauge field

$$\frac{1}{2g^2}\int d^4x\sqrt{-\det(g_{\mu\nu})}F_{\mu\nu}F^{\mu\nu}$$

has electric-magnetic duality, $E \to B$, $B \to -E$, and $2\pi/g^2 \to g^2/2\pi$.

On the boundary, this duality switches ρ and B and, as should be clear from the previous slide, takes

$$2\pi\sigma_\pm \to -\frac{1}{2\pi\sigma_\pm} \ .$$

S duality.

An AdS/CFT result

A frequency independent $\sigma_\pm$.

- Consider the case $\rho = B = 0$. S-duality becomes a self-duality, ignoring the action on g.
- Because of electric-magnetic duality, the ratio $E_\pm / B_\pm = -B_\pm / E_\pm$
- Thus

$$\sigma_\pm = \frac{B_\pm}{g^2 E_\pm} = \pm \frac{i}{g^2} \ ,$$

  a result independent of ω.

$$\sigma_{xx} = \frac{1}{g^2}$$

Hydrodynamic Results

Small frequency (hydrodynamic) behavior at nonzero ρ and B:

$$\sigma_+ = i\sigma_Q \frac{\omega + i\omega_c^2/\gamma + \omega_c}{\omega + i\gamma - \omega_c}$$

$$\omega_c = \frac{B\rho}{\epsilon + P} \; ; \quad \gamma = \frac{\sigma_Q B^2}{\epsilon + P} \; ; \quad \sigma_Q = \frac{(sT)^2}{(\epsilon + P)^2}\frac{1}{g^2} \; .$$

- The structure is fixed by hydrodynamics, while the values of ϵ, P, s, σ_Q are fixed by the microscopic AdS/CFT theory.
- The cyclotron pole at $\omega = \omega_c - i\gamma$.
- Hall conductivity: $\lim_{\omega\to 0}\sigma_+ = \rho/B$ which implies $\sigma_{xy} = \rho/B$.

Numeric Results

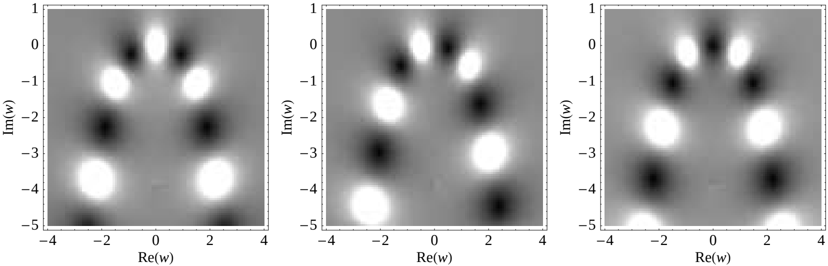


Figure: A density plot of $|\sigma_+|$ as a function of complex w. White areas are large in magnitude and correspond to poles while dark areas are zeroes of σ_+, from left to right, $h = 0$ and $q = 1$, $h = q = 1/\sqrt{2}$, and $h = 1$ and $q = 0$.

Remember $B = h\alpha^2$ and $\rho = -q\alpha^2/g^2$.

More symmetry constraints

These relations are Ward identities and are consistent with but independent of the AdS/CFT results.

$$\begin{aligned} \pm\hat{\alpha}_{\pm} T\omega &= (B \mp \mu\omega)\sigma_{\pm} - \rho \ , \\ \pm\bar{\kappa}_{\pm} T\omega &= (B \mp \mu\omega)\,\hat{\alpha}_{\pm} T - \epsilon - P + \mu\rho \ . \end{aligned}$$

They rely on the following assumptions:

- Gravitational interactions are unimportant.
- Electromagnetic interactions between components of the material are unimportant.
- The field theory is Lorentz invariant.
- The equilibrium state is time reversal invariant, rotationally symmetric, and reflectionally invariant.

Given σ, we can calculate $\hat{\alpha}$ and $\bar{\kappa}$.

The Nernst effect redux

The Nernst response is governed by

$$\vec{E} = -\boldsymbol{\theta}\vec{\nabla} T \qquad \text{where} \qquad \boldsymbol{\theta} = \boldsymbol{\sigma}^{-1}\hat{\boldsymbol{\alpha}}$$

The Nernst coefficient

$$\nu = \theta_{yx}/B \qquad (\text{Recall } \nu = E/B|\nabla T|)$$

In complex combinations

$$\theta_{\pm} = \mp i \frac{\alpha_{\pm}}{\sigma_{\pm}}$$

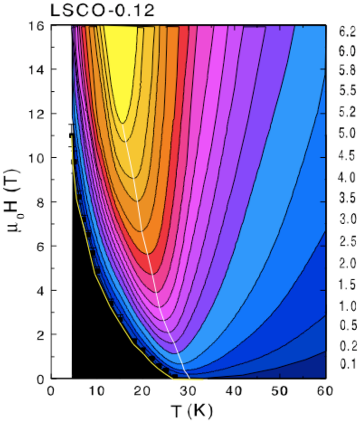
Impurities

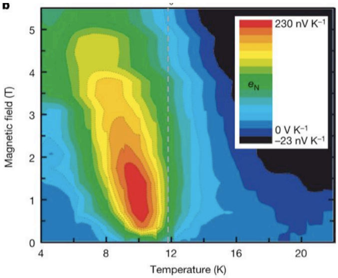
- To compare the Nernst effect with experiments, we have to add the effect of scattering from impurities, τ_{imp}

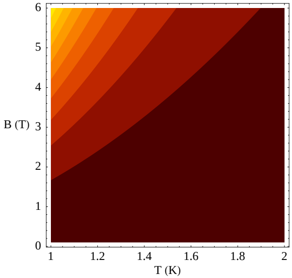
$$\nu = \frac{1}{T} \frac{1/\tau_{\mathrm{imp}}}{(\omega_c^2/\gamma + 1/\tau_{\mathrm{imp}})^2 + \omega_c^2}$$

- Impurities have been considered from a hydrodynamic context by Hartnoll and collaborators and in the AdS/CFT context by Hartnoll and me.
- When $\rho = 0$,

$$\nu = \frac{\tau_{\mathrm{imp}}}{T} \ .$$


LSCO-0.12
μ0H (T)
T (K)
0 10 20 30 40 50 60
0 2 4 6 8 10 12 14 16
6.2 6.0 5.8 5.5 5.2 5.0 4.5 4.0 3.5 3.0 2.5 2.0 1.5 1.0 0.5 0.2 0.1


b
Magnetic field (T)
Temperature (K)
4 8 12 16 20
0 1 2 3 4 5
230 nV K−1
eN
0 V K−1
−23 nV K−1


B (T)
T (K)
1 1.2 1.4 1.6 1.8 2
0 1 2 3 4 5 6

Remarks and Plans for the Future

- Tried to convince you that AdS/CFT is a useful tool for studying strongly interacting field theories — equations of state, correlation functions, transport properties.
- The hope is that these field theories may be relevant for understand real world condensed matter systems.
- Getting away from a translationally invariant system. How does one introduce impurities into AdS/CFT?
- Getting away from the quantum critical point. Can we find supergravity solutions that correspond to deforming the effective field theory by a relevant operator?

Extra Slides

Separating AdS/CFT from symmetry constraints

It is often difficult to figure out what AdS/CFT teaches us that the symmetries don't. In this case, one finds some clear lessons.

- Under electric-magnetic duality, $2\pi\sigma \to -1/2\pi\sigma$.
- The $B, \rho \to 0$ limit is ω independent. Follows from electric-magnetic duality.

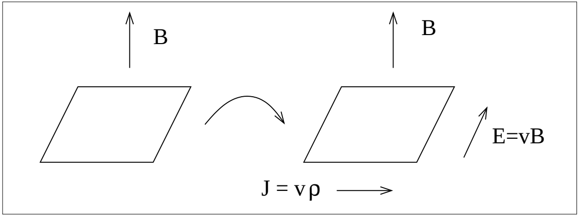
$$\lim_{B,\rho\to 0} \sigma_{xx} = \frac{1}{g^2}$$

- The full form of $\sigma_\pm$ as a function of ω with its horseshoe of zeros and poles stretching into the complex ω plane.

Simple argument for the Hall conductivity

The Hall conductivity at $\omega = 0$ follows from translation invariance alone.

- Consider a plate with charge density ρ in a transverse magnetic field.
- Perform a boost, $x \to x + vt$, v small.
- There is now a current $J^x = v\rho$ and an electric field $E_y = vB$. Thus, $\sigma_{xy} = J^x/E_y = \rho/B$.



A similar argument shows that the DC $\alpha_{xy} = (\epsilon + P - \mu\rho)/BT$.

There are many theoretical examples

The noncompact $\mathbb{CP}^1$ easy-plane model

$$\begin{aligned}\mathcal{S} \;=\; & \int d^2x\, dt \Big[\left|(\partial_\mu - iA_\mu)\, z_1\right|^2 + \left|(\partial_\mu - iA_\mu)\, z_2\right|^2 + s\left(|z_1|^2 + |z_2|^2\right) + \\ & u\left(|z_1|^2 + |z_2|^2\right)^2 + v|z_1|^2|z_2|^2 + \frac{1}{2e^2}\left(\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda\right)^2\Big]\end{aligned}$$

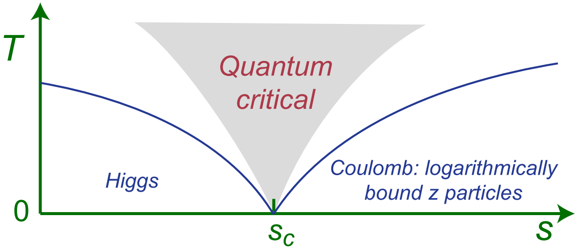


Figure: Phase diagram

Motrunich and Vishwanath, cond-mat/0311222

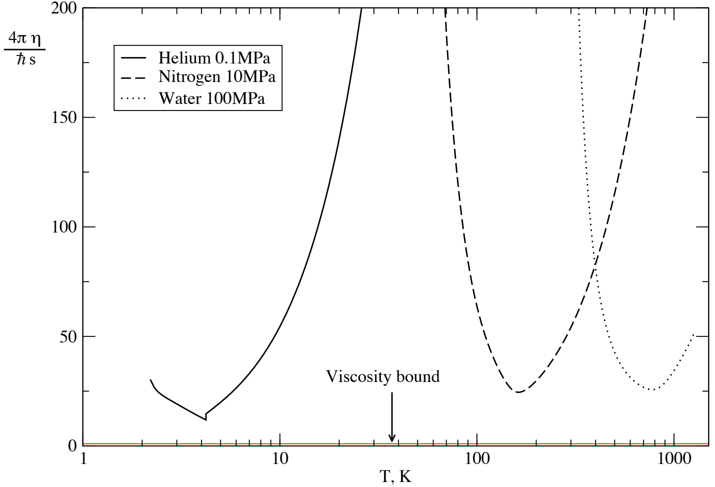


Figure: Viscosity to entropy density ratio

Current-current two-point functions at $B = \rho = 0$

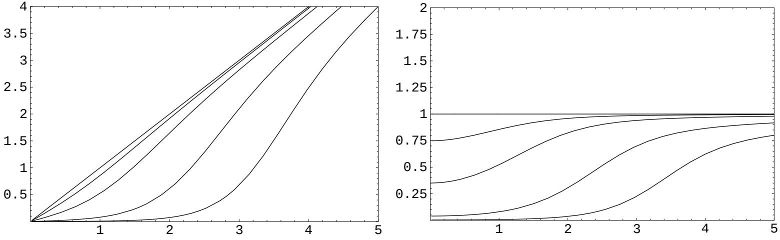


Figure: Imaginary part of the retarded function $C^{yy}(\omega, k)$, plotted in units of $(-\chi)$, as a function of dimensionless frequency $w \equiv 3\omega/(4\pi T)$, for several values of dimensionless momentum $q \equiv 3k/(4\pi T)$. Curves from left to right correspond to $q = 0, 0.5, 1.0, 2.0, 3.0$. Left: $\mathrm{Im}\, C^{yy}(w, q)$, Right: $\mathrm{Im}\, C^{yy}(w, q)/w$.

$$\chi = 4\pi T/3g^2$$

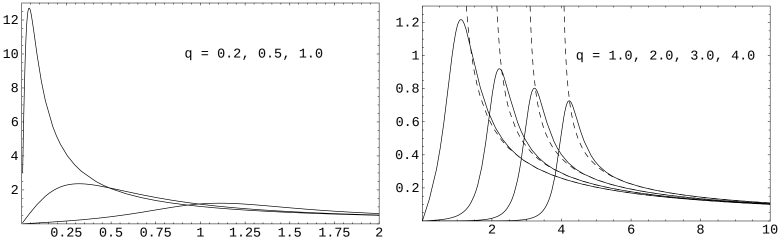


Figure: Imaginary part of the retarded function $C^{tt}(w,q)/q^2$, plotted in units of $(-\chi)$, as a function of dimensionless frequency $w \equiv 3\omega/(4\pi T)$, for several values of dimensionless momentum $q \equiv 3k/(4\pi T)$. Curves from left to right correspond to $q = 0.2, 0.5, 1.0$ (left panel), and $q = 1.0, 2.0, 3.0, 4.0$ (right panel). The dashed curves are plots of $1/\sqrt{w^2 - q^2}$.

$\chi = 4\pi T/3g^2$

small q: hydrodynamic peak at $w \sim q^2$

large q: collisionless peak at $w \sim q$

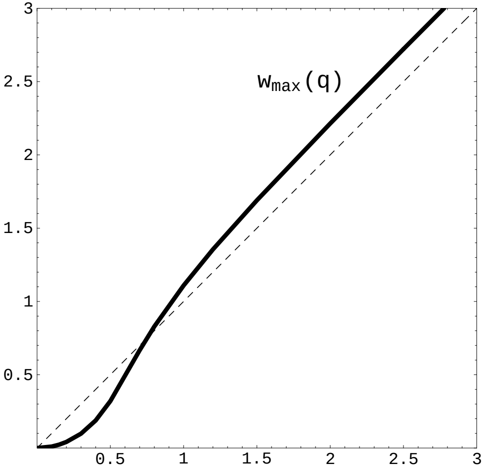


Figure: The position of the peak of the spectral function. The dashed line is $w = q$.

The cyclotron resonance

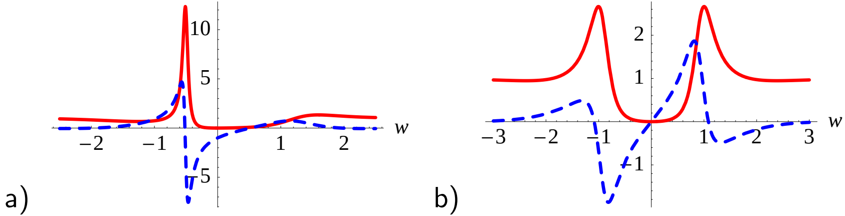


Figure: The dashed blue line is the $\mathrm{Im}(\sigma_+)$ while the solid red line is the $\mathrm{Re}(\sigma_+)$ as a function of w: a) $h = q = 1/\sqrt{2}$, b) $h = 1$ and $q = 0$.

Dyonic blackhole thermodynamics

$$T = \frac{\alpha(3 - h^2 - q^2)}{4\pi} .$$

$$B = h\alpha^2 , \quad m = -\frac{h\alpha}{g^2} , \quad \rho = -\frac{q\alpha^2}{g^2} , \quad \text{and} \quad \mu = -q\alpha \ .$$

$$s = \frac{\pi\alpha^2}{g^2} \ , \quad \epsilon = \frac{\alpha^3}{g^2}\frac{1}{2}(1 + h^2 + q^2) \ , \quad \text{and} \quad P = \epsilon/2 + mB \ .$$

$$\mathcal{P} = \langle T_{aa} \rangle = \epsilon/2 \ .$$

$$\frac{1}{g^2} = \frac{\sqrt{2}N^{3/2}}{6\pi} .$$