Lattice Simulations of Non-Equilibrium Dynamics

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Lattice

- non-perturbative as much as possible: computers

1. classical approximations

2. 2PI approximations & topological ‘defects’

Initial-value problem

scalar field $\phi(x)$

observable $O(x)$, e.g. $T^{\mu\nu}(x) = \partial^\mu\phi(x)\partial^\nu\phi(x) + \ldots$

e.o.m. for Heisenberg fields

$$\partial_t\hat{\phi} = \hat{\pi}, \quad \partial_t\hat{\pi} - \nabla^2\hat{\phi} + \mu^2\hat{\phi} + \lambda\hat{\phi}^3 = 0$$

state described by density operator $\hat{\rho}$

$$\langle\hat{O}(x)\rangle = \mathrm{Tr}\hat{O}(x)\hat{\rho}, \quad \langle\hat{O}(x)\hat{O}(y)\rangle = \mathrm{Tr}\hat{O}(x)\hat{O}(y)\hat{\rho}, \quad \ldots$$

operator $\hat{\phi}(x) \approx$ infinite-dimensional matrix

- need to truncate/approximate in relation to $\hat{\rho}$ and $\hat{O}$

- Langevin?

classical approximation

$\widehat{\phi}(x) \to \phi(x)$, $\widehat{\phi}(x) \to \pi(x)$

classical fields, solns of classical e.o.m. with initial values $\phi_{\mathsf{i}}(\mathbf{x})$, $\pi_{\mathsf{i}}(\mathbf{x})$

$$\widehat{\rho} \to \rho_c = \rho_c[\phi_{\mathsf{i}}, \pi_{\mathsf{i}}], \qquad \hat{O}(x) \to O[\phi_{\mathsf{i}}, \pi_{\mathsf{i}}; x]$$

classical average over initial values

$$\langle O(x) \rangle = \int D\phi_{\mathsf{i}}\, D\pi_{\mathsf{i}}\, \rho_c[\phi_{\mathsf{i}}, \pi_{\mathsf{i}}]\, O[\phi_{\mathsf{i}}, \pi_{\mathsf{i}}, x], \qquad \ldots$$

discretize space-time on a lattice, spacings a, $a_t \ll a$

1. draw initial values from ρ_c ensemble
2. solve field equations on computer
3. evaluate averages over ρ_c ensemble

when applicable?

intuition: large occupation numbers

define $n_{\mathbf{k}}$ and $\tilde{n}_{\mathbf{k}}$ from two-point correlators at $t = t_{\mathsf{i}}$

$$
\begin{aligned}
C_{\mathbf{k}}^{\phi\phi} &= \langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle - \langle \phi_{\mathbf{k}} \rangle \langle \phi_{-\mathbf{k}} \rangle \equiv \left(n_{\mathbf{k}} + \tfrac{1}{2} \right) \frac{1}{\omega_{\mathbf{k}}} \\
C_{\mathbf{k}}^{\phi\pi} &= \langle \phi_{\mathbf{k}} \pi_{-\mathbf{k}} \rangle - \langle \phi_{\mathbf{k}} \rangle \langle \pi_{-\mathbf{k}} \rangle \equiv \tilde{n}_{\mathbf{k}} + \tfrac{1}{2} i \\
C_{\mathbf{k}}^{\pi\pi} &= \langle \pi_{\mathbf{k}} \pi_{-\mathbf{k}} \rangle - \langle \pi_{\mathbf{k}} \rangle \langle \pi_{-\mathbf{k}} \rangle \equiv \left(n_{\mathbf{k}} + \tfrac{1}{2} \right) \omega_{\mathbf{k}}
\end{aligned}
$$

free field:

$$
n_{\mathbf{k}} = \langle a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \rangle, \quad \tilde{n}_{\mathbf{k}} = \mathrm{Im}\, \langle a_{\mathbf{k}} a_{-\mathbf{k}} \rangle
$$

in case $n_{\mathbf{k}} \gg 1$: expect classical approximation to be good for observables dominated by these $\mathbf{k}$-modes in case of weak coupling, can study classical approximation in perturbation theory*

*Aarts, JS, PLB 393 (1997) 395; NPB 511 (1998) 451; Aarts, Nauta, van Weert, PRD 61 (2000) 105002; Van der Meulen, JS, JCAP 023(2007)0711

initial conditions

gaussian: determine/choose

$\bar{\pi} = \langle\widehat{\phi}\rangle$, $\bar{\pi} = \langle\widehat{\pi}\rangle$, $C^{\phi\phi}$, $C^{\{\phi,\pi\}} = \frac{1}{2}(C^{\phi\pi} + C^{\pi\phi})$, $C^{\pi\pi}$

probability distribution for the fluctuations

$\phi' = \phi - \bar{\phi} \quad \pi' = \pi - \bar{\pi}$

$$
\begin{aligned}
P &\propto \exp\left[-\frac{1}{2}\begin{pmatrix}\phi' & \pi'\end{pmatrix} C^{-1}\begin{pmatrix}\phi' \\ \pi'\end{pmatrix}\right] \\
&= \exp\left[-\frac{1}{2}\sum_{\mathbf{k}}\left(\frac{|\xi^+_{\mathbf{k}}|^2}{n_k + 1/2 + \tilde{n}_k} + \frac{|\xi^-_{\mathbf{k}}|^2}{n_k + 1/2 - \tilde{n}_k}\right)\right]
\end{aligned}
$$

$$
\xi^{\pm}_{\mathbf{k}} = \frac{1}{\sqrt{2\omega_k}}\left(\omega_k\,\phi'_{\mathbf{k}} \pm \pi'_{\mathbf{k}}\right)
$$

(assumed homogeneous ensemble)

draw initial ϕ', π' from P, add mean fields to get initial

$$\phi_{\rm i} = \bar{\phi}_{\rm i} + \phi'_{\rm i}, \quad \pi_{\rm i} = \bar{\pi}_{\rm i} + \pi'_{\rm i}$$

two schemes:

1. initialize only modes with $n_{\mathbf{k}} > 1/2$

   no power in the UV, typically $|\mathbf{k}| < k_{\rm max} \ll 1/a$

   good for sensitive observables with many derivatives, e.g. $F\tilde{F}$, winding number densities

2. initialize all modes*

   power in the UV, need renormalization & matching

   however, non-local UV divergencies require finite a

*E.g. Khlebnikov, Tkachev, PRL 77 (1966) 219 beyond gaussian initial conditions

- $\rho = Z^{-1} e^{-H/T}$

need finite $1/a = \mathcal{O}(T)$, matching to quantum theory, 'power in the UV',

add 'HTL particles' in case of large separation of scales

examples: sphaleron rate*, correlators in SU(2)-Higgs model**

- recent studies of plasma instabilities (cf. this meeting)

*Ambjørn, Krasnitz, PLB 362 (1995) 97, NPB 506 (1997) 387; Tang, JS, NPB 482 (1996) 265; Arnold, Son, Yaffe, PRD 55 (1997) 6264; Arnold, PRD 55 (1997) 7781; Moore, Rummukainen, PRD 61 (2000) 105008; Bödeker, Moore, Rummukainen, PRD 61 (2000) 056003

**Tang, JS, NPB 510 (1998) 401; Arnold, Yaffe PRD 57 (1998) 1178

- quantum?

$\mathrm{Tr} e^{-\hat{H}/T} = \int D\phi \, e^{-S}$

sample (MC) ϕ_i, π_i from two imaginary-time slices

QCD two-point functions: large occupation numbers in the vacuum?

fermions

always quantum

classical approximation for Bose fields gives linear Dirac equation; use mode functions*

$$\begin{aligned}
\hat{\psi}(x) &= \sum_{\alpha} w_{\alpha}(x)\,\hat{\psi}_{\alpha} \\
0 &= [\gamma^{\mu}\partial_{\mu} - igA_{\mu}(x)]w_{\alpha}(x) \\
0 &= \partial_{\mu}F^{\mu\nu}(x) + g\sum_{\alpha,\beta}\langle\hat{\psi}_{\alpha}^{\dagger}\hat{\psi}_{\beta}\rangle\,\bar{w}_{\alpha}(x)i\gamma^{\nu}w_{\beta}(x)
\end{aligned}$$

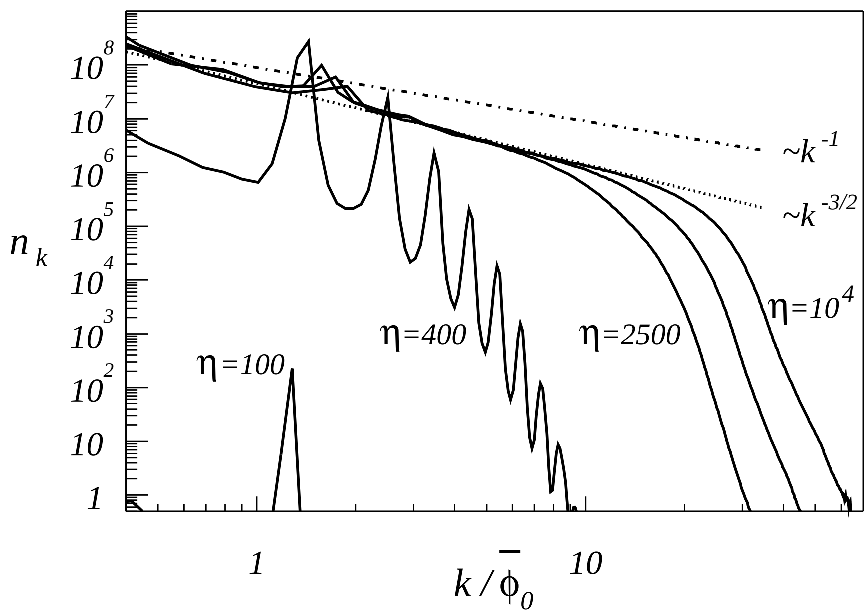
fermion doubling**

*Aarts, JS, NPB555 (1999) 355; PRD61 (2000) 025002
**avoidable? cf. Yukawa model Berges, Borsányi, Serreau, NPB 660 (2003) 51

Example: cascade to the UV

- has self similar features, turbulence*

*Son, PRD 54 (1996) 3745; Micha, Tkachev PRD 70 (2004) 043538, Boyanovsky, Destri, De Vega, PRD 69 (2004) 045003; Destri, De Vega, PRD 73 (2006) 025014; Díaz-Gil, García-Bellido, González-Arroyo, Pérez, POS Lat2005 (2005) 242



'cascade' to higher momenta: n_k vs $k/\overline{\Phi_0}$ ($\overline{\Phi_0}$ = amplitude of zero-mode oscillations) at various conformal times η (Micha & Tkachev)

Example: cold electroweak transition*

inflaton χ, Higgs doublet φ

after EW-scale inflation: universe is cold

χ-energy transferred to other d.o.f.

$$V(\chi,\cdots) = V(\chi) + (\lambda_{\chi\varphi}\chi^2 - \mu^2)\varphi^\dagger\varphi + \cdots$$

time-dependent effective φ-mass

$$\mu_{\rm eff}^2 = -\mu^2 + \lambda_{\chi\varphi}\chi^2$$

$\mu_{\rm eff}^2 < 0 \to$ tachyonic instability, large occupation numbers in χ and φ in limited momentum range: preheating

*scenario for baryogenesis, García-Bellido, Grigoriev, Kusenko, Shaposhnikov, PRD 60 (1999) 123504; García-Bellido, García-Pérez, González-Arroyo, PRD 67 (2003) 03501; PRD 69 (2004) 023504; Tranberg, JS, JHEP 12 (2002) 020, JHEP 0311 (2003) 016; JHEP 08 (2006) 012; Tranberg, Hindmarsh, JS, JHEP 01 (2007) 034 tachyonic transition at $t = t_c$, $\mu_{\text{eff}}(t_c) = 0$

gaussian approximation:

$$\ddot{\phi}_{\mathbf{k}} + (\mu_{\text{eff}}^2 + k^2)\phi_{\mathbf{k}} = 0$$

shortly after t_c:

$\mu_{\text{eff}}^2 = -M^3(t - t_c)$,

or consider a quench:

$$\begin{aligned}\mu_{\text{eff}}^2 &= +\mu^2,\ t < t_c,\\ &= -\mu^2,\ t > t_c\end{aligned}$$

for $k < k_{\text{max}}$: n_k and $\tilde{n}_k$ grow $\approx$ exponentially

$n_k + 1/2 - \tilde{n}_k \rightarrow 0$

$$\begin{aligned} k_{\text{max}} &= \sqrt{M^3(t - t_c)} \\ &= \mu, \quad \text{for the quench} \end{aligned}$$

initial distribution for classical approximation:

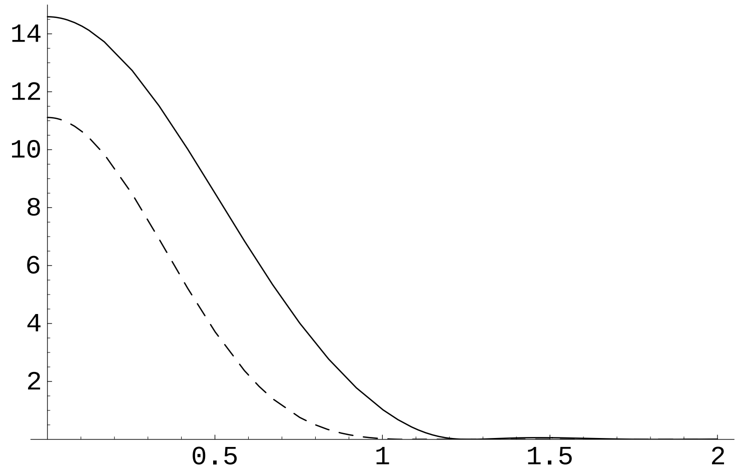
- at time $t_i > t_c$, when $n_k \gg 1$ and non-linear terms in e.o.m. still small

or

- at $t_i = t_c$, 'just the half'

1. $k < k_{\text{max}}$

2. all modes initialized



n_k versus k/μ in 1+1 D at time $t_{\mathsf{s}} = 2.05\,\mu^{-1}$ (full)
$10^{-2}\,n_k$ vs k/μ in 3+1 D at time $t_{\mathsf{s}} = 4.2\,\mu^{-1}$ (dashed)

initial quench ($t_c = 0$)

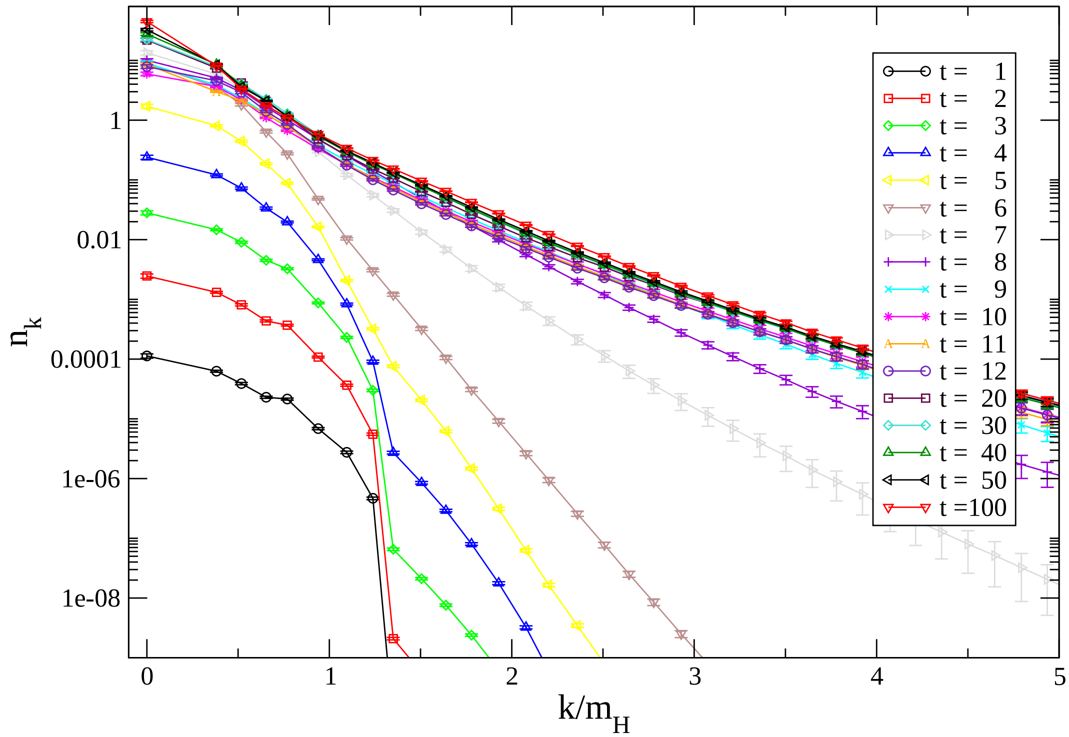
t_{s} = spinodal time at which back reaction becomes important particle numbers in the SU(2)-Higgs model*

$$-\mathcal{L} = \frac{1}{2g^2}\,\mathrm{tr}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^\dagger D^\mu\varphi - \mu^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2$$

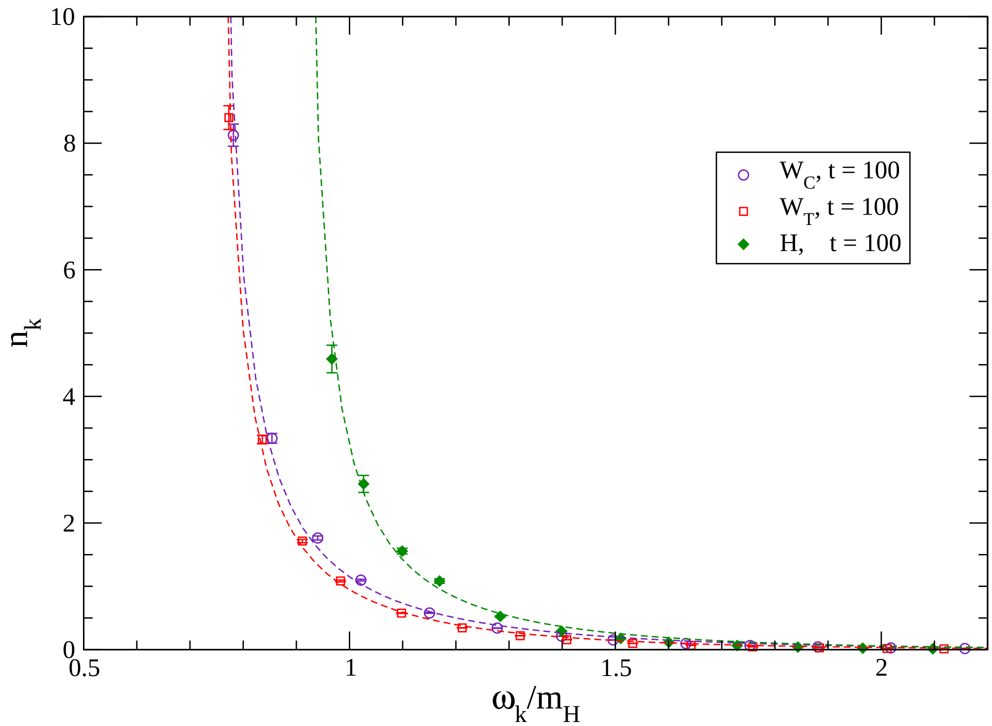
quench, 'just the half', $k_{\mathrm{max}} = \mu$

$\mathbf{A}_{\mathrm{init}} = 0$, $A_0 = 0$, Gauss' law

*Skullerud, JS, Tranberg, JHEP 0308 (2003) 045



Coulomb-gauge W-particle numbers as a function of momentum, time in units of m_H^{-1}



Higgs- and W-particle numbers at time $t = 100\, m_H^{-1}$; W_C: Coulomb gauge; W_T: unitary gauge, transverse; H: unitary gauge Higgs approx. gauge-independence of transverse n_k^W (Coulomb- and unitary-gauge) after $t = 50\, m_H^{-1}$ not much happens in $50 \lesssim t m_H < 100$

temperature $T \approx 0.4\, m_{\mathrm{H}}$

chemical potential $\mu_{\mathrm{ch}} \approx 0.7(0.9) m_{\mathrm{H}}$ for W(H)

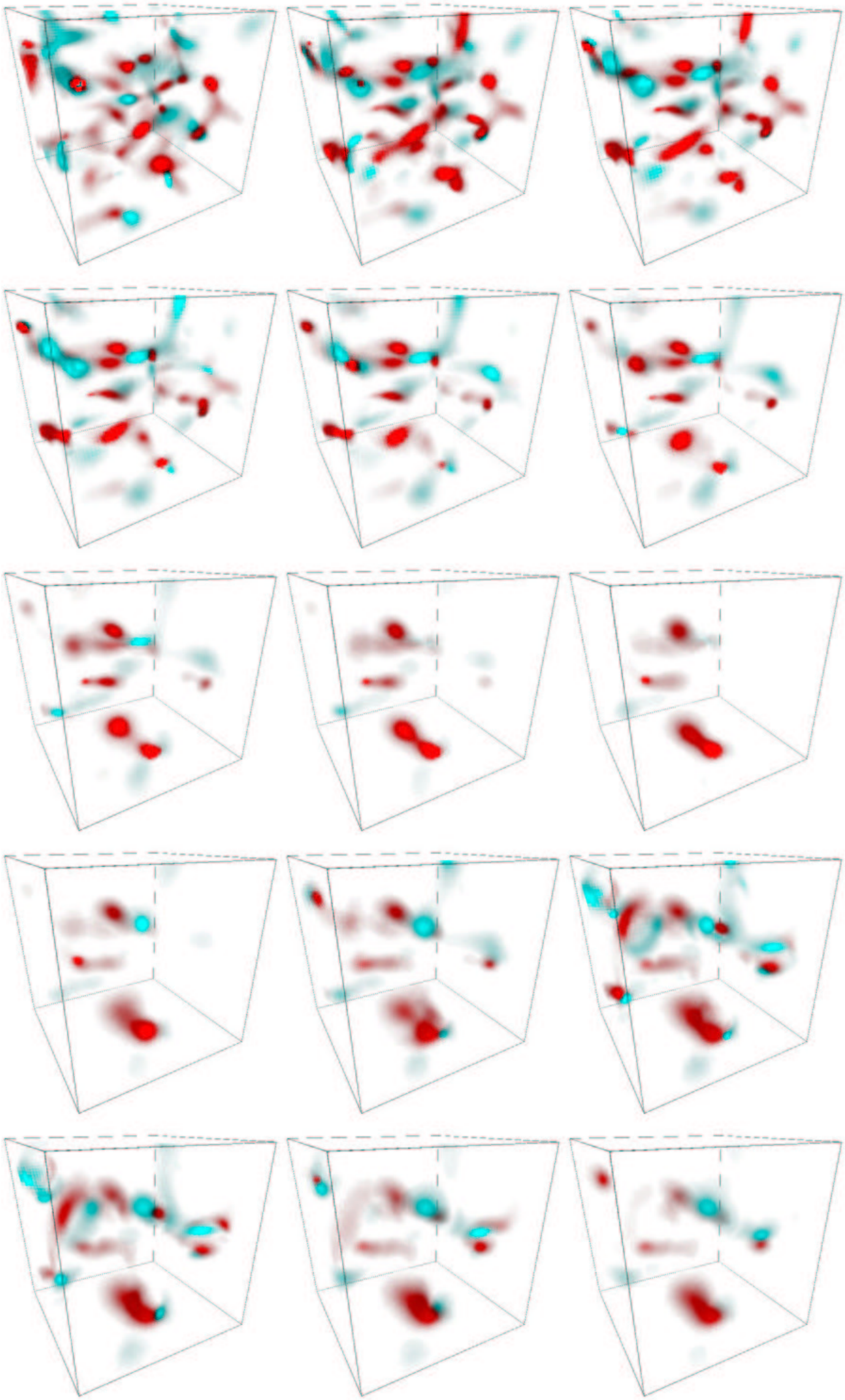
low temperature, but still many $n_k \gg 1$ because of chemical potential

longitudinal modes settle at slower rate winding-number density of the Higgs field

for $t\,m_{\rm H} = 1,\ \ldots,\ 15$

study of generation of Chern-Simons number in cold electroweak baryogenesis*

*Van der Meulen, Sexty, JS, Tranberg, JHEP02(2006)029



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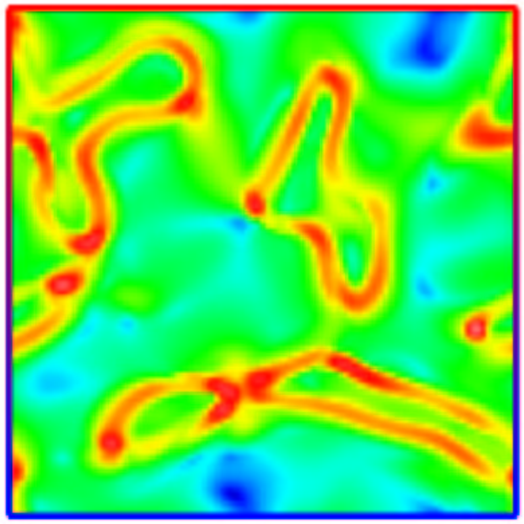
primordial magnetic field in U(1)$\times$SU(2) Higgs model*



2D contour plot of magnetic field flux tubes at $t\mu = 15$

*Díaz-Gil, García-Bellido, García Pérez, González-Arroyo, arXiv:0712.4263 classical approximation

good:

- gauge fields no problem
- topological defects naturally included

not so good:

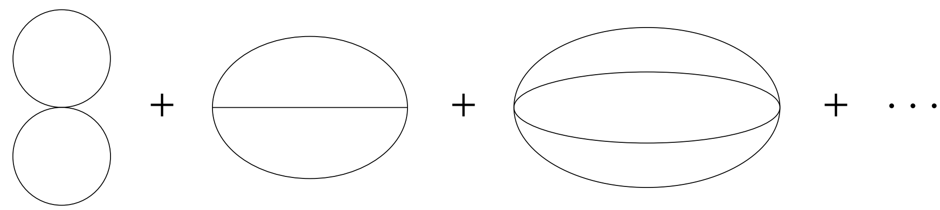
- limited time, ‘cascade’ to the UV leads to $n_{\mathbf{k}} < 1$
- lattice artefacts in case $1/a$ is physical scale (e.g. T)

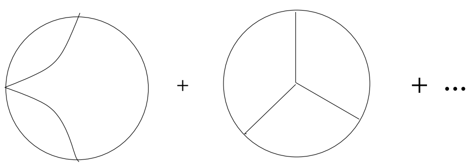
Quantum: 2PI approximation

Effective action

$$\Gamma[\phi, G] = S[\phi] - \frac{i}{2}\mathrm{Tr}\ln G + \frac{i}{2}\mathrm{Tr}\frac{\delta^2 S[\phi]}{\delta\phi\,\delta\phi}G + \Phi[\phi, G]$$

with

$i\,\Phi =$ 

$\ldots =$ 

presumably diagrammatic series does not converge, asymptotic series

each vertex $\rightarrow \int dx^0 \int d^3x \Rightarrow$ numerically expensive

use $1/N$ to sum diagrams when applicable*
($1/N$ exp. also asymptotic)

truncate $\Phi \Rightarrow \Phi$-derivable approximation

*Berges, NPA 699 (2002) 847

$\bar{\phi} = \langle\widehat{\phi}\rangle$ and $\bar{G} = \langle\widehat{\phi}\widehat{\phi}\rangle - \langle\widehat{\phi}\rangle\langle\widehat{\phi}\rangle$ determined by solving

$$\frac{\delta\Gamma[\bar{\phi},\bar{G}]}{\delta\bar{\phi}} = 0, \quad \frac{\delta\Gamma[\bar{\phi},\bar{G}]}{\delta\bar{G}} = 0$$

on the closed-time-path

$\Rightarrow$ e.o.m. for

$$\begin{aligned} F(x,y) &= \tfrac{1}{2}\langle\widehat{\phi}(x)\widehat{\phi}(y) + \widehat{\phi}(y)\widehat{\phi}(x)\rangle - \langle\widehat{\phi}(x)\rangle\langle\widehat{\phi}(y)\rangle \\ \rho(x,y) &= i\langle[\widehat{\phi}(x),\widehat{\phi}(y)]\rangle \end{aligned}$$

initial conditions as for classical, plus
$\langle[\widehat{\phi}_{\mathbf{k}},\widehat{\phi}_{-\mathbf{k}}]\rangle = 0$, $\langle[\widehat{\phi}_{\mathbf{k}},\widehat{\pi}_{-\mathbf{k}}]\rangle = i$

Note: classical $\phi \equiv \bar{\phi} + \phi' \neq \bar{\phi}$

'memory kernels'

ϕ^4 model:

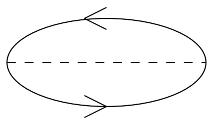
$$
\begin{aligned}
[\partial_0^2 - \nabla^2 + M^2(x)]F(x,y) &= \int_0^{x^0} dz^0 \int d^3z\, \Sigma_\rho(x,z)F(z,y) \\
&- \int_0^{y^0} dz^0 \int d^3z\, \Sigma_F(x,z)\rho(z,y) \\
[\partial_0^2 - \nabla^2 + M^2(x)]\rho(x,y) &= \int_{y^0}^{x^0} dz^0 \int d^3z\, \Sigma_\rho(x,z)\rho(z,y)
\end{aligned}
$$

$$
\begin{aligned}
M^2(x) &= m^2 + \tfrac{1}{2}\lambda\,\bar\phi^2(x) + \tfrac{1}{2}\lambda F(x,x) \\
0 &= \left[\partial_0^2 - \nabla^2 + m^2 + \tfrac{1}{6}\,\lambda F(x,x)\right]\bar\phi(x) + \tfrac{1}{6}\,\lambda\,\bar\phi(x)^3 \\
&\quad + \int_0^{x^0} dz^0 \int d^3z\, \tilde\Sigma_\rho(x,z)\bar\phi(z)
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_F(x,y) &= \tfrac{1}{2}\lambda^2\bar{\phi}(x)\bar{\phi}(y)\left[F(x,y)^2 - \tfrac{1}{4}\,\rho(x,y)^2\right] \\
&\quad +\lambda^2\left[\tfrac{1}{6}\,F(x,y)^3 - \tfrac{3}{24}\,F(x,y)\rho(x,y)^2\right] \\
&\quad +\ldots \\
\Sigma_\rho(x,y) &= \lambda^2\bar{\phi}(x)\bar{\phi}(y)F(x,y)\rho(x,y) + \tilde{\Sigma}_\rho(x,y) \\
\tilde{\Sigma}_\rho(x,y) &= \lambda^2\left[\tfrac{1}{2}\,F(x,y)^2\rho(x,y) - \tfrac{1}{24}\,\rho(x,y)^3\right] \\
&\quad +\ldots
\end{aligned}
$$

Example*: chiral quark-meson model

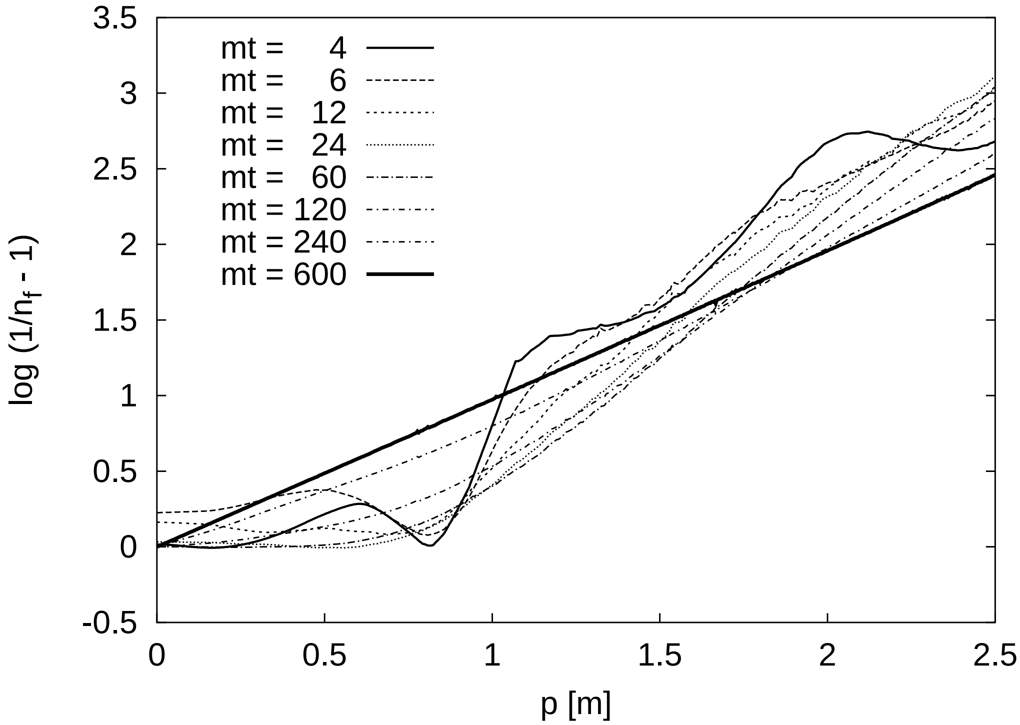
$$-\mathcal{L} = \bar{\psi}\gamma^\mu\partial_\mu\psi + \tfrac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\pi^a\partial^\mu\pi^a) + \tfrac{1}{2}\mu^2(\sigma^2 + \pi^a\pi^a) + g\bar{\psi}(\sigma + i\gamma_5\tau^a\pi^a)\psi$$

$i\,\Phi =$ 

e.o.m. discretized in terms of components, e.g.

$$\rho = \rho_S + \rho_P\, i\gamma_5 + \rho_V^\mu\gamma_\mu + \rho_A^\mu\gamma_\mu\gamma_5 + \rho_T^{\mu\nu}\sigma_{\mu\nu}/2$$

*Berges, Borsányi, Serreau, NPB 660 (2003) 51



thermalization: $\ln(-1 + 1/n_p) \to p/T$, Fermi-Dirac
$1/\gamma_{\mathrm{F}}^{\mathrm{therm}} \approx 95\, m^{-1}$, $g = 1$

good:

- large times OK
- lots of physics included by ‘self consistent’ resummation, e.g. quantum equilibration à la BE and FD

not so good:

- instabilities at large (effective) couplings $(\lambda\, n_{\mathbf{k}})$*

- RG β-function wrong at low orders

- gauge theory: gauge-fixing dependence; some 3-loop contributions to Φ numerically too expensive

- topological defects?

*however, see Garny, Müller, poster at this meeting

defects and other creatures

domain walls, strings, textures, monopoles, Q-balls, I-balls, oscillons, half knots, Chern-Simons numbers, . . .

homogeneous ensemble:

$$\langle\hat{\phi}(x)\rangle = \bar{\phi}(x^0), \quad F(x,y) = F(\mathbf{x}-\mathbf{y}; x^0, y^0), \quad \text{etc.}$$

finite volume: $\bar{\phi} = 0$

defects are expected to contribute to the 2-point correlators

can truncated 2PI action Φ incorporate 'defects'?

Example: $O(N)$ model

$$-\mathcal{L} = (1/2)\partial_\mu \phi_a \partial^\mu \phi_a - (\mu^2/2)\phi_a\phi_a + (\lambda/4)(\phi_a\phi_a)^2$$

$a = 1, \ldots, N$, $\lambda \propto 1/N$

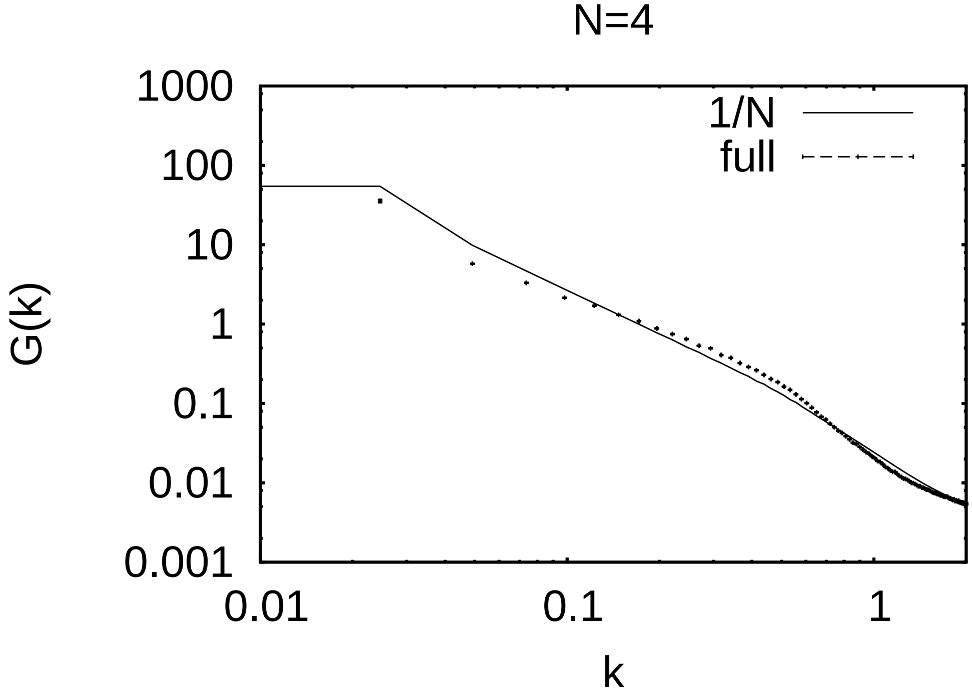
kinks and textures in 1+1 D*:

$N = 1$: kinks $\phi(x) = v \tanh(\mu x/\sqrt{2}), \quad L\mu \gg 1$

$N = 2$: textures $\phi_1(x) + i\phi_2(x) = v\, e^{i2\pi N_w x/L}$

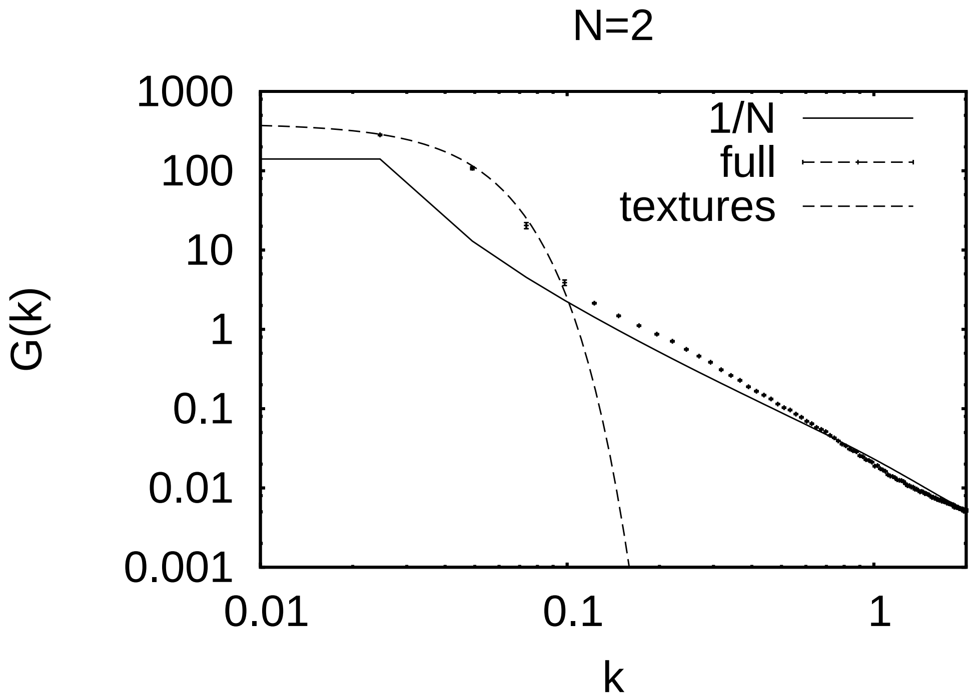
damping added* to relax initial randomized state to low energy

*Rajantie, Tranberg, JHEP11 (2006)020



full: equal-time F correlator at late time in 2PI-NLO large N, $N = 4$

dotted: classical



similar, $N = 2$;

dashed: fit to expected form due to textures cold electroweak quench in 3+1 D*

$N = 4$

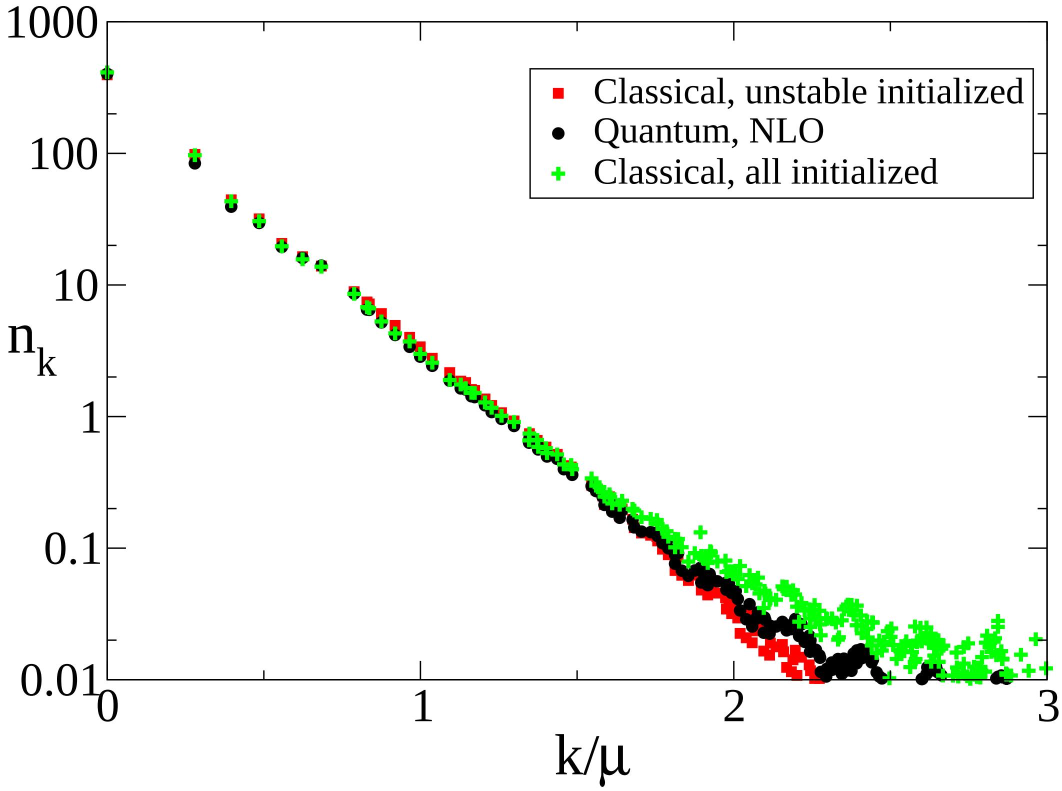
next three slides:

comparison of n_k in NLO 1/N with two 'just the half' classical approximations:

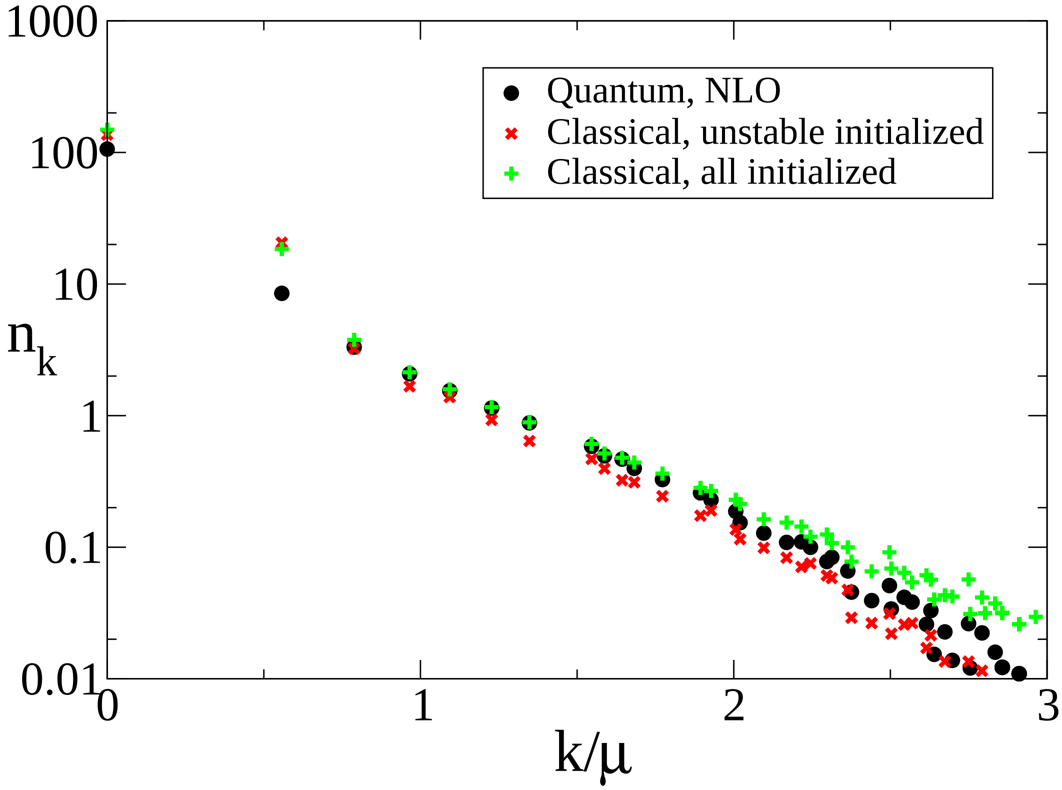
1. only unstable modes initialized ($k_{\text{max}} = \mu$)

2. all modes initialized

*Arrizabalaga, JS, Tranberg, JHEP 0410 (2004) 017

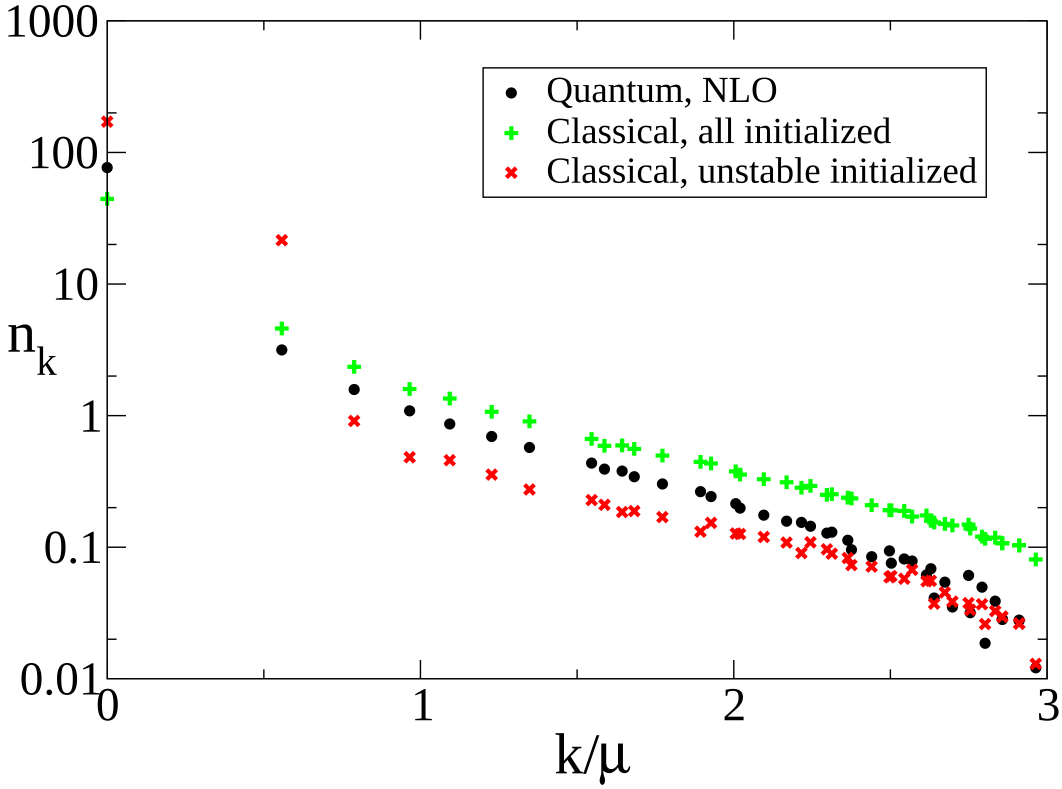


particle numbers at time $t\mu = 14$; $\lambda = 1/24$, $L\mu = 22.4$



particle numbers at time $t\mu = 100$

$\lambda = 1/4$ ($m_{\mathrm{H}} = 174$ GeV), $L\mu = 11.2$



similar, time $t\mu = 700$

differences at $k/\mu \approx 0.5$ for $t \gtrsim 100\,\mu^{-1}$ due to textures?

sample quantum mean field

$$\widehat{\rho} = \int D\phi \, D\pi \, \rho_q(\phi, \pi) \, |\phi, \pi\rangle\langle\phi, \pi|$$

representation in terms of coherent states $|\phi, \pi\rangle$

$\widehat{\rho}$ homogeneous, $\langle\phi, \pi|\widehat{\phi}(x)|\phi, \pi\rangle$ *in*-homogeneous

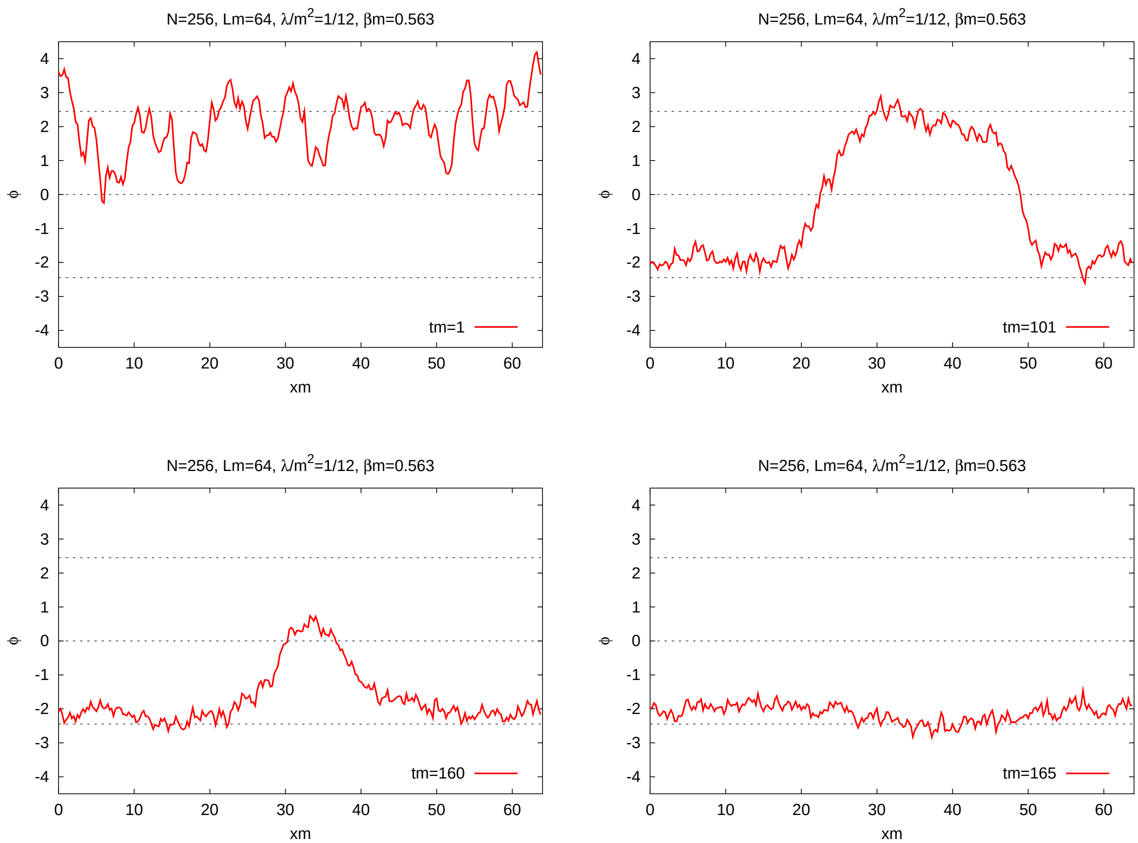
'Hartree ensemble approximation'*

next slide:

Thermal kink-antikink cleation and annihilation

snapshots of a realization of an initial BE ensemble**

*Salle, JS, Vink, PRD 64 (2001) 025016; PRD 76 (2003) 116006
**Salle, PRD 69 (2004) 025005


N=256, Lm=64, λ/m²=1/12, βm=0.563
φ
xm
tm=1
N=256, Lm=64, λ/m²=1/12, βm=0.563
φ
xm
tm=101
N=256, Lm=64, λ/m²=1/12, βm=0.563
φ
xm
tm=160
N=256, Lm=64, λ/m²=1/12, βm=0.563
φ
xm
tm=165

outlook

- classical approximation powerful, develop further
- more work comparing classical and Φ-derivable
- Φ-derivable for gauge fields

$N = 4$ texture in 3+1 D

$(\vec{\phi}, \phi_4) = v\,(\hat{x}\sin\chi(r), \cos\chi(r))$

$\mathbf{x} = r\hat{x},\ \chi(r) = \pi(1 - e^{-r/d})$