Quantum Backreaction and Particle Production in QED and QCD in external fields
Talk Given by Fred Cooper at the KITP Conference on Nonequilibrium Phenomena, February 2008

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I. OUTLINE OF TALK

- Early History and issues in Nonequilibrium QFT
- Schwinger Pair Production in QED- Transverse Distributions-Analytic
- Schwinger Pair Production in QCD- Transverse Distributions-Analytic
- Some Analytic Results for Time dependent Fields
- Backreaction and particle production in Leading order large-N-QED
- Backreaction in QCD
- Thermalization vs. Expansion time scales

II. EARLY HISTORY AND ISSSUES

Crucial questions in the 1980's

- How to add quantum corrections to classical treatment of inflation [1] of Guth, Linde and Starobinsky
- Understanding quantum backreaction and pair production in de-Sitter space as a possible solution to the cosmological constant (Mottola) [2]
- How to correctly describe the initial state in Large-N or Gaussian approximation and how to renormalize mean field (large-N) time evolution problems. [6] [7]
- Can we understand Particle Production at RHIC using a Flux Tube picture [8]. and Schwinger Mechanism? Verify Semiclassical Transport approach.
- How to correctly (without secular terms [4]) go beyond leading order Gaussian approximation and renormalize using the CTP formalism. [3]
- How to preserve symmetries in approximation schemes.

It was in order to better understand back reaction as a solution of the cosmological constant problem that Cooper and Mottola studied as a "toy" model, backreaction in the Electric Field case. However interest in particle production following Relativistic Heavy Ion collisions using a Flux Tube model, converted this toy problem into one with experimental consequences [8]. In light of the renewed interest in particle production from semi-classical gluonic fields we have recently undertaken a study of the quantum back reaction problem in SU(3) QCD in 3+1 dimensions in the hope of seeing what one can learn about jet production at RHIC and LHC and also what one can learn about the initial gluon condensate state.

III. PAIR PRODUCTION FROM A STRONG ELECTRIC FIELD

In order to pop a pair of fermions (or bosons) out of the vacuum one must supply an energy eEx in a Compton wave length $x \approx \hbar/mc$. Since this needs to produces (at) rest a pair with rest energy $2mc^2$ then it is clear that the critical value of the electric field for this to happen is of order

$$eE\hbar/mc = 2mc^2 \tag{1}$$

or

$$E_{critical} \approx 2m^2c^3/e\hbar. \tag{2}$$

Simple tunneling picture: One imagines that one has an electron bound in a potential well of order $|V_0| \approx 2mc^2$ one then applies a constant electric field with potential energy eEx to the (say square well) potential of depth $2mc^2$. The ionization probability is proportional to the WKB barrier penetration factor:

$$\exp\left[-\frac{2}{\hbar}\int_0^{V_0/eE} dx \left(2m(V_0 - |eE|x)^{1/2}\right)\right] = e^{-(8m^2c^3/3\hbar eE)}. \tag{3}$$

In what follows we will set $\hbar = 1; c = 1$. When $E > E_{critical}$ there is no exponential suppression of pair production.

A. Constant Electric and Chromoelectric Field Results

In a classic paper [9] in 1951 Schwinger derived the following one-loop non-perturbative formula

$$\frac{dW}{d^4x} = \frac{e^2E^2}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2}\, e^{-\frac{n\pi m^2}{|eE|}} \tag{4}$$

for the probability of e^+e^- pair production per unit time per unit volume using proper time method. For charged scalars instead one has:

$$\frac{dW}{d^4x} = \frac{e^2E^2}{8\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\, e^{-\frac{n\pi m^2}{|eE|}}. \tag{5}$$

The result of Schwinger was extended to QCD using proper time methods by Claudon, Yildiz and Cox [12] for the case of massless (but not massive) quarks. However the p_T distribution $\frac{dW}{d^4xd^2p_T}$, could not be obtained by the proper time method. A WKB approximation was done by Casher et. al. [10]. The exact Path Integral calculation done recently by Gouranga et. al. [11]. For QED the WKB analysis is exact:

$$\frac{dW}{d^4xd^2p_T} = -\frac{|eE|}{4\pi^3}\mathrm{Log}[1 - e^{-\pi\frac{p_T^2+m^2}{|eE|}}]. \tag{6}$$

Scalar electrodynamics:

$$\frac{dW}{d^4xd^2p_T} = \frac{|eE|}{8\pi^3}\text{Log}[1+e^{-\pi\frac{p_T^2+m^2}{|eE|}}]. \tag{7}$$

The QCD result depends on two independent Casimir invariants: $C_1 = [E^aE^a]$ and $C_2 = [d_{abc}E^aE^bE^c]^2$ where E^a is the constant chromo-electric field with color index $a, b, c = 1, 2, ..8$[11].

$$\begin{aligned}\frac{dN_{q,\bar{q}}}{dtd^3xd^2p_T} &= -\frac{1}{4\pi^3}\sum_{j=1}^{3}|g\lambda_j|\,\ln[1-e^{-\frac{\pi(p_T^2+m^2)}{|g\lambda_j|}}]\,, \\ &= \frac{1}{4\pi^3}\sum_{j=1}^{3}|g\lambda_j|\sum_{n=1}^{\infty}\frac{e^{-\frac{\pi(p_T^2+m^2)}{|g\lambda_j|}}}{n}\end{aligned} \tag{8}$$

where m is the mass of the quark. This result is gauge invariant because it depends on the following gauge invariant eigenvalues

$$\begin{aligned}\lambda_1 &= \sqrt{\frac{C_1}{3}}\cos\theta\,, \\ \lambda_2 &= \sqrt{\frac{C_1}{3}}\cos\left(2\pi/3-\theta\right), \\ \lambda_3 &= \sqrt{\frac{C_1}{3}}\cos\left(2\pi/3+\theta\right),\end{aligned} \tag{9}$$

where θ is given by

$$\cos^2 3\theta = 3C_2/C_1^3\,. \tag{10}$$

The integration over p_T in eq. (31) yields

$$\frac{dW}{d^4x} = \frac{1}{4\pi^3} \sum_{j=1}^{3} g^2\lambda_j^2 \sum_{n=1}^{\infty} \frac{e^{-\frac{n\pi m^2)}{|g\lambda_j|}}}{n^2}. \tag{11}$$

This result depends on both Casimirs, except for massless fermions when the series can be summed to give:

$$\frac{dW}{d^4x} = \frac{1}{4\pi^3} \sum_{j=1}^{3} g^2\lambda_j^2 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{g^2}{4\pi^3} \sum_{1}^{8} E^a E^a \zeta(2) \tag{12}$$

which reproduces Schwinger's proper time result for massless fermions, extended to QCD, for the total production rate dN/d^4x [12].The exact result in eq. (31) can be contrasted with the following formula obtained by the WKB tunneling method [28]

$$\frac{dN_{q,\bar{q}}}{dtd^3xd^2p_T} = \frac{-|gE|}{4\pi^3} \ln[1 - e^{-\frac{\pi(p_T^2+m^2)}{|gE|}}], \tag{13}$$

For soft gluon production Nayak and van Nieuwenhuizen [11] found in the Feynman T'Hooft gauge [13]

$$\frac{dN_{gg}}{dtd^3xd^2p_T} = \frac{1}{4\pi^3} \sum_{j=1}^{3} |g\lambda_j| \ln[1 + e^{-\frac{\pi p_T^2}{|g\lambda_j|}}]. \tag{14}$$

This was shown to be indepedent of the gauge fixing choice by Cooper and Nayak [15].

B. Background Field Method and Schwinger Pair production in SU(3) Gauge Theory

In the background field method of QCD the gauge field is the sum of a classical background field and the quantum gluon field:

$$A^a_\mu \;\rightarrow\; A^a_\mu \;+\; Q^a_\mu \tag{15}$$

where in the right hand side A^a_μ is the classical background field and Q^a_μ is the quantum gluon field. The gauge field Lagrangian density is given by

$$\mathcal{L}_{\rm gauge} \;=\; -\frac{1}{4}F^a_{\mu\nu}[A+Q]F^{\mu\nu a}[A+Q]. \tag{16}$$

The background gauge fixing is given by by [13]

$$D_\mu[A]Q^{\mu a} \;=\; 0, \tag{17}$$

where the covariant derivative is defined by

$$D^{ab}_\mu[A] \;=\; \delta^{ab}\partial_\mu \;+\; gf^{abc}A^c_\mu. \tag{18}$$

The gauge fixing Lagrangian density is

$$\mathcal{L}_{\rm gf} \;=\; -\frac{1}{2\alpha}[D_\mu[A]Q^{\mu a}]^2 \tag{19}$$

where α is any arbitrary gauge parameter, and the corresponding ghost contribution is given by

$$\mathcal{L}_{ghost} \ = \ \overline{\chi}^a D_\mu^{ab}[A] D^{\mu,bc}[A+Q]\chi^c \ = \ \overline{\chi}^a \ K^{ab}[A,Q] \ \chi^b. \qquad (20)$$

Now adding eqs. (16) and (19) and (20)we get the Langrangian density for gluons interacting with a classical background field

$$\mathcal{L}_{\rm gluon} \ = \ -\frac{1}{4}F_{\mu\nu}^a[A+Q]F^{\mu\nu a}[A+Q] \ - \ \frac{1}{2\alpha}[D_\mu[A]Q^{\mu a}]^2 + \overline{\chi}^a \ K^{ab}[A,Q] \ \chi^b. \qquad (21)$$

To discuss gluon pair production at the one-loop level on considers just the part of this Lagrangian which is quadratic in quantum fields. This quadratic Lagrangian is invariant under a restricted class of gauge transformations. The quadratic Lagrangian for a pair of gluon interacting with background field A_μ^a is given by

$$\mathcal{L}_{\rm gg} \ = \ \frac{1}{2}Q^{\mu a}M_{\mu\nu}^{ab}[A]Q^{\nu b} \qquad (22)$$

where

$$M_{\mu\nu}^{ab}[A] \ = \ \eta_{\mu\nu}[D_\rho(A)D^\rho(A)]^{ab} \ - \ 2gf^{abc}F_{\mu\nu}^c \ + \ (\frac{1}{\alpha}-1)[D_\mu(A)D_\nu(A)]^{ab}$$

For our purpose we write

$$M_{\mu\nu}^{ab}[A] \ = \ M_{\mu\nu,\alpha=1}^{ab}[A] \ + \ \alpha'[D_\mu(A)D_\nu(A)]^{ab} \qquad (23)$$

where $\alpha' = (\frac{1}{\alpha} - 1)$. The matrix elements for α=1 is given by

$$M^{ab}_{\mu\nu,\alpha=1}[A] = \eta_{\mu\nu}[D_\rho(A)D^\rho(A)]^{ab} - 2gf^{abc}F^c_{\mu\nu} \tag{24}$$

which was studied in [11]. In this approximation the ghost Lagrangian density is given by

$$\mathcal{L}_{ghost} = \overline{\chi}^a D^{ab}_\mu[A]D^{\mu,bc}[A]\chi^c = \overline{\chi}^a \ K^{ab}[A] \ \chi^b \tag{25}$$

The vacuum-to-vacuum transition amplitude in pure gauge theory in the presence of a background field A^a_μ is given by:

$$_+ < 0|0 >^A_- = \int [dQ][d\chi][d\bar{\chi}] \ e^{i(S+S_{gf}+S_{\rm ghost})}. \tag{26}$$

For the gluon pair part this can be written by

$$_+ < 0|0 >^A_- = \frac{Z[A]}{Z[0]} = \frac{\int [dQ] \ e^{i\int d^4x \ Q^{\mu a}M^{ab}_{\mu\nu}[A]Q^{\nu b}}}{\int [dQ] \ e^{i\int d^4x \ Q^{\mu a}M^{ab}_{\mu\nu}[0]Q^{\nu b}}} = e^{iS^{(1)}_{eff}} \tag{27}$$

where $S^{(1)}_{eff}$, the one-loop effective action, is given by

$$S^{(1)}_{eff} = -i{\rm Ln}\frac{({\rm Det[M^{ab}_{\mu\nu}[A]]})^{-1/2}}{({\rm Det[M^{ab}_{\mu\nu}[A]]})^{-1/2}} = \frac{i}{2}Tr[{\rm Ln}M^{ab}_{\mu\nu}[A] - {\rm Ln}M^{ab}_{\mu\nu}[0]]. \tag{28}$$

The trace Tr contains an integration over d^4x and a sum over color and Lorentz indices. To the above action, we need to add the ghost action. The ghost action is gauge independent and eliminates the unphysical gluon degrees of freedom and is given by

$$S^{(1)}_{ghost} \;=\; -\,i\;Tr\int_0^\infty \frac{ds}{s}[e^{is\;[K[0]+i\epsilon]} \;-\,e^{is\;[K[A]+i\epsilon]}] \tag{29}$$

where $K^{ab}[A]$ is given by (25). Since the total action is the sum of the gluon and ghost actions, the gauge parameter dependent part proportional to $(\frac{1}{\alpha}-1)$ can be evaluated as an addition to the $\alpha = 1$.

The non-perturbative gluon pair production per unit volume per unit time is related to the imaginary part of this effective action via

$$\frac{dN}{dtd^3x} \;\equiv\; \text{Im}\mathcal{L}_{eff} \;=\; \frac{\text{Im}S^{(1)}_{eff}}{d^4x}. \tag{30}$$

This expression was evaluated for $\alpha = 1$ in [11] where for gluon pair production it was found

$$\frac{dN_{g,g}}{dtd^3xd^2p_T} \;=\; \frac{1}{4\pi^3}\;\sum_{j=1}^{3}\;|g\lambda_j|\;\text{Ln}[1\;+\;e^{-\frac{\pi p_T^2}{|g\lambda_j|}}]. \tag{31}$$

After this calculation was done in $\alpha = 1$ gauge, Cooper and Gouranga showed by explicit evaluation of the extra term proportion to $\alpha - 1$ that the result for the particle production rate was independent of the Gauge Fixing parameter α [15].

Recently there has also been some progress to extending this result to time dependent fields. Using a formal operator shift theorem [14], Nayak and Cooper were able to show:

$$\frac{dW}{d^4x d^2p_T} = \frac{|eE(t)|}{8\pi^3}\mathrm{Log}[1 + e^{-\pi\frac{p_T^2+m^2}{|eE(t)|}}]. \tag{32}$$

For Fermion pair production they obtained instead

$$\frac{dW}{d^4x d^2p_T} = -\frac{|eE(t)|}{4\pi^3}\mathrm{Log}[1 - e^{-\pi\frac{p_T^2+m^2}{|eE(t)|}}]. \tag{33}$$

These results came from evaluating the one loop Action. For the Boson case they found

$$S_B^{(1)} = \frac{i}{16\pi^3}\int_0^\infty \frac{ds}{s}\int d^4x \int d^2p_T e^{is(p_T^2+m^2+i\epsilon)}[\frac{1}{s} - \frac{eE(t)}{\sinh(seE(t))}]. \tag{34}$$

wheras in the fermion case they obtained

$$S^{(1)} = \frac{i}{8\pi^3}\int_0^\infty \frac{ds}{s}\int d^4x \int d^2p_T e^{is(p_T^2+m^2+i\epsilon)}[eE(t)\,\coth(seE(t)) \;-\; \frac{1}{s}].] \tag{35}$$

To do this calculation one has that the action can be written as

$$S^{(1)} = \frac{-i}{(2\pi)^2}\int_0^\infty \frac{ds}{s}\int d^2x_T \int d^2p_T e^{is(p_T^2+m^2+i\epsilon)}$$
$$[\int_{-\infty}^{+\infty} dt < t|\int_{-\infty}^{+\infty} dz < z|e^{-is[(-i\frac{d}{dt}+eE(t)z)^2-\hat{p}_z^2]}|z > |t >$$
$$-\int dt \int dz \frac{1}{4\pi s}] \qquad (36)$$

This expression contains the noncommuting quantities $E(t)$ and $\frac{d}{dt}$

To evaluate these terms we derived a shift theorem [14]

$$\int_{-\infty}^{+\infty} dx < x|e^{-[(a(y)x+h\frac{d}{dy})^2+b(\frac{d}{dx})+c(y)]}|x > f(y)$$
$$= \int_{-\infty}^{+\infty} dx < x - \frac{h}{a(y)}\frac{d}{dy}|e^{-[a^2(y)x^2+b(\frac{d}{dx})+c(y)]}|x - \frac{h}{a(y)}\frac{d}{dy} > f(y). \qquad (37)$$

This Shift Theorem also implies the important corollary

$$\int_{-\infty}^{+\infty} dxe^{-(f(y)x+\frac{d}{dy})^2}g(y) = \int_{-\infty}^{+\infty} dxe^{-f^2(y)x^2}g(y) = \sqrt{\pi}\frac{g(y)}{f(y)}$$

These results have the remarkable feature that they are equivalent to Schwinger's original expressions for the effective action with the substitution $E \to E(t)$. Fried and Woodard [21], found a similar result using Fradkin's formulation of the Path Integral, for the case of an Electric field which depended on the light cone time coordinate

$x^+ = (x^0 + x^3)$. For the Fermion Action they obtained:

$$\Gamma_1[A] = -iL[A] \ , \tag{38}$$

$$= \frac{1}{8\pi^2} \int d^4x \int_0^\infty \frac{ds}{s^3} e^{-ism^2} \left\{ eE(x^+) s \coth \big(eE(x^+) s \big) - 1 \right\} \ . \tag{39}$$

IV. PARTICLE PRODUCTION AND BACK REACTION IN BOOST INVARIANT QED

We assume in what follows that the kinematics of ultrarelativistic high energy collisions results in a boost invariant dynamics in the longitudinal (z) direction (here z corresponds to the axis of the initial collision) so that all expectation values (such as energy densities) are functions of the proper time $\tau = \sqrt{t^2 - z^2}$. [18]. [19] [20]. The back-reaction problem was first discussed [8] in a semi-classical approximation using a Vlasov Equation with a Schwinger source term in the adiabatic approximation. In 1+1 dimensions this leads to the equation for the phase space distribution function:

$$\frac{\partial f}{\partial \tau} + eF_{\eta\tau}(\tau)\frac{\partial f}{\partial p_\eta} = \pm[1 \pm 2f(\mathbf{p},\tau)]e\tau|E(\tau)| \times \ln\left[1 \pm \exp\left(-\frac{\pi(m^2 + \mathbf{p}_\perp^2)}{e|E(\tau)|}\right)\right]\delta(p_\eta). \tag{40}$$

And the back reaction equation (Maxwell Equation)

$$-\tau\frac{dE}{d\tau} = j_\eta = j_\eta^{cond} + j_\eta^{pol}\,, \tag{41}$$

where j^{cond} is the conduction current and j_μ^{pol} is the polarization current due to pair creation [27]

$$
\begin{aligned}
j_\eta^{cond} &= 2e \int \frac{dp_\eta}{2\pi\tau p_\tau} p_\eta f(p_\eta, \tau) \\
j_\eta^{pol} &= \frac{2}{F^{\tau\eta}} \int \frac{dp_\eta}{2\pi\tau p_\tau} p^\tau \frac{Df}{D\tau} \\
&= \pm[1 \pm 2f(p_\eta = 0, \tau)] \frac{me\tau}{\pi} \mathrm{sign}[E(\tau)] \ln\left[1 \pm \exp\left(-\frac{\pi m^2}{|eE(\tau)|}\right)\right].
\end{aligned}
\tag{42}
$$

Since in solving these equations one did not know the validity of either the semi-classical approximation or the adiabatic equation, it was important to actually solve the quantum back reaction problem to understand whether Nuclear Theorists using transport theory to model the quark gluon plasma were making reasonable assumptions. In the field theory calculation,the assumption that the electric field can be treated "classically" gets translated into this approximation being the first term in a large-N approximation of QED, where N refers to having N electron or quark flavors We were also interested in knowing whether a hydrodynamic picture emerges from the field theory calculation. [31].

We introduce the light cone variables τ and η, which will be identified later with fluid proper time and rapidity . These coordinates are defined in terms of the ordinary lab-frame Minkowski time t and coordinate along the beam direction z by

$$z = \tau \sinh \eta \quad , \quad t = \tau \cosh \eta \, . \tag{43}$$

The Minkowski line element in these coordinates has the form

$$ds^2 = -d\tau^2 + dx^2 + dy^2 + \tau^2 d\eta^2 \, . \tag{44}$$

The QED action in curvilinear coordinates is:

$$\begin{aligned} S \; &= \int d^{d+1}x \, (\det V)[-\frac{i}{2}\bar{\Psi}\tilde{\gamma}^{\mu}\nabla_{\mu}\Psi + \frac{i}{2}(\nabla^{\dagger}_{\mu}\bar{\Psi})\tilde{\gamma}^{\mu}\Psi \\ &-im\bar{\Psi}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}], \end{aligned} \tag{45}$$

where

$$\nabla_{\mu}\Psi \equiv (\partial_{\mu} + \Gamma_{\mu} - ieA_{\mu})\Psi \tag{46}$$

Varying the action leads to the Heisenberg field equation:

$$\left[\gamma^0 \left(\partial_\tau + \frac{1}{2\tau}\right) + \gamma_\perp \cdot \partial_\perp + \frac{\gamma^3}{\tau}(\partial_\eta - ieA_\eta) + m\right]\Psi = 0 \, , \tag{47}$$

and the Maxwell equation: $E = E_z(\tau) = -\dot{A}_\eta(\tau)$

$$\frac{1}{\tau}\frac{dE(\tau)}{d\tau} = \frac{e}{2}\left\langle\left[\bar{\Psi}, \tilde{\gamma}^{\eta}\Psi\right]\right\rangle = \frac{e}{2\tau}\left\langle\left[\Psi^{\dagger}, \gamma^0\gamma^3\Psi\right]\right\rangle . \tag{48}$$

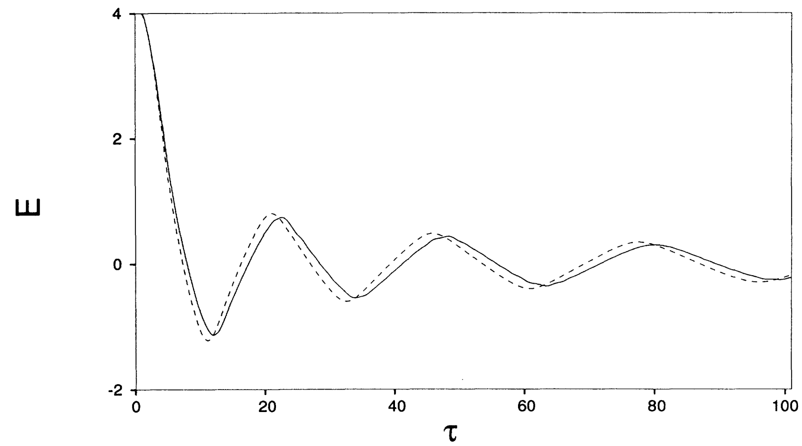

FIG. 1: Proper-time evolution of the electric field $E(\tau)$ for an initial $E = 4$.

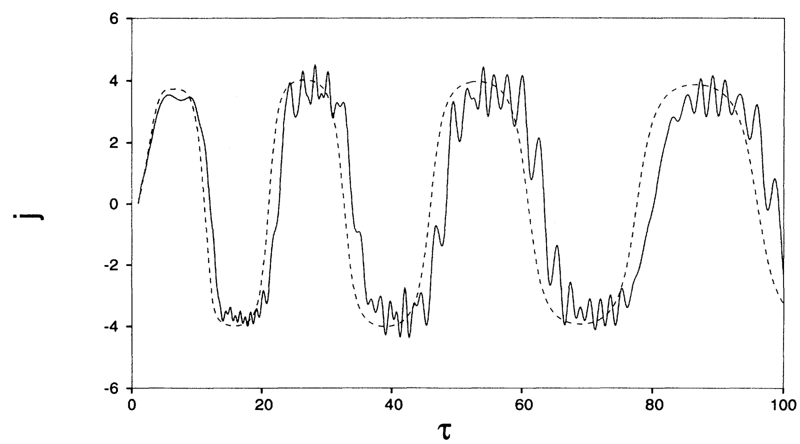

FIG. 2: Proper-time evolution of the fermionic current $j_\eta(\tau)$.

To solve the problem numerically we expand the fermion field in terms of Fourier modes at fixed proper time: τ, and used a grid in momentum space with a maximum momentum Λ.

$$\Psi(x) = \int [d\mathbf{k}] \sum_s [b_s(\mathbf{k})\psi^+_{\mathbf{k}s}(\tau)e^{ik\eta}e^{i\mathbf{p}\cdot\mathbf{x}} \\ +d^\dagger_s(-\mathbf{k})\psi^-_{-\mathbf{k}s}(\tau)e^{-ik\eta}e^{-i\mathbf{p}\cdot\mathbf{x}}]. \tag{49}$$

The $\psi^\pm_{\mathbf{k}s}$ then obey

$$\left[\gamma^0\left(\frac{d}{d\tau}+\frac{1}{2\tau}\right)+i\gamma_\perp\cdot\mathbf{k}_\perp+i\gamma^3\pi_\eta+m\right]\psi^\pm_{\mathbf{k}s}(\tau)=0, \tag{50}$$

Squaring the Dirac equation:

$$\psi^\pm_{\mathbf{k}s}=\left[-\gamma^0\left(\frac{d}{d\tau}+\frac{1}{2\tau}\right)-i\gamma_\perp\cdot\mathbf{k}_\perp-i\gamma^3\pi_\eta+m\right]\chi_s\frac{f^\pm_{\mathbf{k}s}}{\sqrt{\tau}}\,. \tag{51}$$

$$\gamma^0\gamma^3\chi_s=\lambda_s\chi_s \tag{52}$$

with $\lambda_s = 1$ for $s = 1, 2$ and $\lambda_s = -1$ for $s = 3, 4$, we then get the mode equation:

$$\left(\frac{d^2}{d\tau^2}+\omega^2_{\mathbf{k}}-i\lambda_s\dot{\pi}_\eta\right)f^\pm_{\mathbf{k}s}(\tau)=0, \tag{53}$$

$$\omega^2_{\mathbf{k}}=\pi^2_\eta+\mathbf{k}^2_\perp+m^2;\quad \pi_\eta=\frac{k_\eta-eA}{\tau}. \tag{54}$$

The back-reaction equation in terms of the modes is

$$\frac{1}{\tau}\frac{dE(\tau)}{d\tau}=-\frac{2e}{\tau^2}\sum_{s=1}^{4}\int[d\mathbf{k}](\mathbf{k}^2_\perp+m^2)\lambda_s|f^+_{\mathbf{k}s}|^2, \tag{55}$$

Renormalization is done by realizing eE is renormalization invariant, so multiplying both sides of the Maxwell equation by e and recognizing that

$$e^2 = Z^{-1}(\Lambda, m) e_R^2(m^2) \tag{56}$$

where

$$Z(\Lambda, m) = 1 - \frac{e_R^2(m^2)}{6\pi^2} \ln\left(\frac{\Lambda}{m}\right) \tag{57}$$

The finite result which is independent of Λ for large Λ [31] can be written as

$$\frac{e_R dE_R(\tau)}{d\tau} = -Z^{-1}(\Lambda, m) \frac{2e_R^2}{\tau} \sum_{s=1}^{4} \int [d\mathbf{k}](\mathbf{k}_\perp^2 + m^2)\lambda_s |f_{\mathbf{k}s}^+|^2, \tag{58}$$

This straightforward method of renormalization is to be compared with our original approach which was based on an adiabatic expansion of a WKB parameterization [7] [25]. Namely we can write:

$$f_{\mathbf{k}s}^+(\tau) = N_{\mathbf{k}s} \frac{1}{\sqrt{2\Omega_{\mathbf{k}s}}} \exp\left\{ \int_0^\tau \left(-i\Omega_{\mathbf{k}s}(\tau') - \lambda_s \frac{\dot{\pi}_\eta(\tau')}{2\Omega_{\mathbf{k}s}(\tau')} \right) d\tau' \right\} \tag{59}$$

where $\Omega_{\mathbf{k}s}$ obeys the real equation

$$\frac{1}{2}\frac{\ddot{\Omega}_{\mathbf{k}s}}{\Omega_{\mathbf{k}s}} - \frac{3}{4}\frac{\dot{\Omega}_{\mathbf{k}s}^2}{\Omega_{\mathbf{k}s}^2} + \frac{\lambda_s}{2}\frac{\ddot{\pi}_\eta}{\Omega_{\mathbf{k}s}} - \frac{1}{4}\frac{\dot{\pi}_\eta^2}{\Omega_{\mathbf{k}s}^2} - \lambda_s \frac{\dot{\pi}_\eta \dot{\Omega}_{\mathbf{k}s}}{\Omega_{\mathbf{k}s}^2} = \omega_{\mathbf{k}}^2(\tau) - \Omega_{\mathbf{k}s}^2\,. \tag{60}$$

Ω has an adiabatic expansion:

$$\Omega_s^2 = \omega^2 - \frac{1}{2\omega^2}\left[\pi\ddot{\pi} + \dot{\pi}^2\left(1 - \frac{\pi^2}{\omega^2}\right)\right] + \frac{3}{4}\frac{\pi^2\dot{\pi}^2}{\omega^4} + \frac{\dot{\pi}^2}{4\omega^2} + \frac{\lambda_s\dot{\pi}^2\pi}{\omega^3} - \frac{\lambda_s\ddot{\pi}}{2\omega} + \cdots$$

Using this expansion we find

$$\sum_{s=1}^{4}(k_\perp^2 + m^2)(-2\lambda_s))\frac{|f_{\mathbf{k}s}^+|^2}{\tau} = \frac{2\pi_\eta}{\tau\omega_{\mathbf{k}}} - \left(\frac{\ddot{\pi}_\eta}{2\omega_{\mathbf{k}}^5} - \frac{5\dot{\pi}_\eta^2\pi_\eta}{4\omega_{\mathbf{k}}^7}\right)\frac{(\omega_{\mathbf{k}}^2 - \pi_\eta^2)}{\tau} - R_{\mathbf{k}}(\tau)\,,$$

where $R_{\mathbf{k}}(\tau)$ falls faster than ω^{-3}. This then yields

$$\begin{aligned}\frac{dE}{d\tau} &= \frac{e^2}{2\tau^2}\int[d\mathbf{k}]\,\frac{k_\perp^2 + m^2}{\omega_{\mathbf{k}}^5}\left\{\left(\ddot{A} - 2\frac{\dot{A}}{\tau}\right) + \frac{5\dot{A}\pi_\eta^2}{\tau\omega_{\mathbf{k}}^2}\right\} - e\int[d\mathbf{k}]R_{\mathbf{k}}(\tau) \\ &= -\frac{e^2}{6\pi^2}\ln\left(\frac{\Lambda}{m}\right)\frac{dE}{d\tau} - e\int[d\mathbf{k}]R_{\mathbf{k}}(\tau).\end{aligned} \tag{61}$$

where Λ is the cutoff in the transverse momentum integral which has been reserved for last.

Defining $\delta e^2 = (1/6\pi^2)\ln(\Lambda/m)$ as usual we obtain

$$e\frac{dE}{d\tau}(1 + e^2\delta e^2) = -e^2\int[d\mathbf{k}]R_{\mathbf{k}}(\tau)\,, \tag{62}$$

after multiplying both sides of the equation by e. Using $e_R E_R = eE$ we obtain

$$\frac{dE_R}{d\tau} = -e_R\int[d\mathbf{k}]R_{\mathbf{k}}(\tau), \tag{63}$$

where $R_{\mathbf{k}}(\tau)$ is defined by Eq. (61), and the integral is now completely convergent. This method of renormalizing was originally used for quantum fields in curved space, but we see that it is very cumbersome and unnecessary.

Our original simulations were in $1 + 1$ dimensions, and typical proper time evolution of E and j are shown in figs. 1 and 2. The dotted line corresponds to the solution of the transport equations discussed above.

A. Spectrum of Particles and Effective Hydrodynamics

Although particle number is not conserved, at each τ on can diagonalize the Hamiltonian and define an effective particle number which is the adiabatic particle number which interpolates from the initial particle number to the final one if . Namely our boundary condition will be :

$$\langle b_0^\dagger(k,\tau_0) b_0(k,\tau_0)\rangle = \langle b^\dagger(k) b(k)\rangle = 0. \tag{64}$$

with a similar condition on d_0. Introducing the adiabatic bases for the fields via:

$$\begin{aligned}\Psi(x) &= \int [d\mathbf{k}] \sum_s [b_s^0(\mathbf{k};\tau) u_{\mathbf{k}s}(\tau) e^{-i\int \omega_{\mathbf{k}} d\tau} \\ &\quad + d_s^{(0)\dagger}(-\mathbf{k};\tau) v_{-\mathbf{k}s}(\tau) e^{i\int \omega_{\mathbf{k}} d\tau}] e^{i\mathbf{k}\cdot\mathbf{x}}.\end{aligned} \tag{65}$$

The operators $b_s(\mathbf{k})$ and $b_s^{(0)}(\mathbf{k};\tau)$ are related by a Bogolyubov transformation:

$$\begin{aligned}b_r^{(0)}(\mathbf{k};\tau) &= \sum \alpha_{\mathbf{k}r}^s(\tau) b_s(\mathbf{k}) + \beta_{\mathbf{k}r}^s(\tau) d_s^\dagger(-\mathbf{k}) \\ d_r^{(0)}(-\mathbf{k};\tau) &= \sum \beta_{\mathbf{k}r}^{*s}(\tau) b_s(\mathbf{k}) + \alpha_{\mathbf{k}r}^{*s}(\tau) d_s^\dagger(-\mathbf{k})\end{aligned} \tag{66}$$

The interpolating phase space number density is given by:

$$n(\mathbf{k};\tau) = \sum_{r=1,2} \langle 0_{in}| b_r^{(0)\dagger}(\mathbf{k};\tau) b_r^{(0)}(\mathbf{k};\tau)|0_{in}\rangle = \sum_{s,r} |\beta_{\mathbf{k}r}^s(\tau)|^2 \tag{67}$$

The phase space distribution of particles (or antipartcles) in light cone variables is

$$n_{\mathbf{k}}(\tau) = f(k_\eta, k_\perp, \tau) = \frac{d^6N}{\pi^2 dx_\perp^2 dk_\perp^2 d\eta dk_\eta}. \tag{68}$$

We introduce the particle rapidity y and $m_\perp = \sqrt{k_\perp^2 + m^2}$ defined by the particle 4-momentum in the center of mass coordinate system

$$k_\mu = (m_\perp \cosh y, k_\perp, m_\perp \sinh y) \tag{69}$$

The boost that takes one from the center of mass coordinates to the comoving frame where the energy momentum tensor is diagonal is given by $\tanh\eta = v = z/t$, so that one can define the "fluid" 4-velocity in the center of mass frame as

$$u^\mu = (\cosh\eta, 0, 0, \sinh\eta) \tag{70}$$

We then find that the variable

$$\omega_k = \sqrt{m_\perp^2 + \frac{k_\eta^2}{\tau^2}} \equiv k^\mu u_\mu \tag{71}$$

has the meaning of the energy of the particle in the comoving frame. The variables

$$\tau = (t^2 - z^2)^{1/2} \qquad \eta = \frac{1}{2}\ln\left(\frac{t+z}{t-z}\right)$$

have as their canonical momenta

$$k_\tau = Et/\tau - k_z z/\tau = k_\mu u^\mu \quad k_\eta = -Ez + tk_z. \tag{72}$$

The interpolating phase-space density f of particles depends on k_η, $\mathbf{k}_\perp$, τ, and is η-independent. In order to obtain the physical particle rapidity and transverse momentum distribution, we change variables from (η, k_η) to (z, y) at a fixed τ where y is the particle rapidity. We have

$$E\frac{d^3N}{d^3k} = \frac{d^3N}{\pi dy\, dk_\perp^2} = \int \pi dz\ dx_\perp^2\ J\ f(k_\eta, k_\perp, \tau) \tag{73}$$

where the Jacobian J is evaluated at a fixed proper time τ

$$J = \frac{m_\perp \cosh(\eta - y)}{\cosh \eta} = \frac{\partial k_\eta}{\partial z}|_\tau. \tag{74}$$

We also have

$$k_\tau = m_\perp \cosh(\eta - y); \quad k_\eta = -\tau m_\perp \sinh(\eta - y)\,. \tag{75}$$

Calling the integration over the transverse dimension the effective transverse size of the colliding ions $A_\perp$ we then obtain that:

$$\frac{d^3N}{\pi dy\, dk_\perp^2} = A_\perp \int dk_\eta f(k_\eta, k_\perp, \tau) \equiv \frac{d^3N}{\pi d\eta\, dk_\perp^2} \tag{76}$$

The distribution of particles in particle rapidity is the same as the distribution of particles in fluid rapidity, verifying that in the boost-invariant regime that Landau's intuition based on a hydronamic picture was correct.

We now want to make contact with the hydrodynamic approach to calculating particle spectra, namely the Cooper- Frye formula [32]. First we note that the interpolating number density depends on k_η and $k_\perp$ only through the combination:

$$\omega_k = \sqrt{m_\perp^2 + \frac{k_\eta^2}{\tau^2}} \equiv k^\mu u_\mu \tag{77}$$

Thus $f(k_\eta, k_\perp) = f(k_\mu u^\mu)$ and so it depends on exactly the same variable as the comoving thermal distribution! We also have that a constant τ surface (which is the freeze out surface of Landau) is parametrized as:

$$d\sigma^\mu = A_\perp(dz, 0, 0, dt) = A_\perp d\eta(\cosh\eta, 0, 0, \sinh\eta) \tag{78}$$

We therefore find

$$k^\mu d\sigma_\mu = A_\perp m_\perp \tau \cosh(\eta - y) = A_\perp |dk_\eta| \tag{79}$$

Thus we can rewrite our expression for the field theory particle spectra as

$$\frac{d^3N}{\pi dy\, dk_\perp^2} = A_\perp \int dk_\eta f(k_\eta, k_\perp, \tau) = \int f(k^\mu u_\mu, \tau) k^\mu d\sigma_\mu \tag{80}$$

where in the second integration we keep y and τ fixed. Thus with the replacement of the thermal single particle distribution by the interpolating number operator, we get via the coordinate transformation to the center of mass frame the Cooper-Frye formula.

B. Hydrodynamic Variables

The Energy Momentum tensor is diagonal in the$(\tau, \eta, x_\perp)$ coordinate system which is a comoving one.

$$T^{\mu\nu} = \text{diagonal}\ \{\varepsilon(\tau), p_\parallel(\tau), p_\perp(\tau), p_\perp(\tau)\} \tag{81}$$

Only the longitudinal pressure enters into the "entropy" equation.

Only the longitudinal pressure enters into the "entropy" equation

$$\begin{aligned} \varepsilon + p_\parallel &= Ts \qquad (82) \\ \frac{d(\varepsilon\tau)}{d\tau} + p_\parallel &= Ej_\eta \\ \frac{d(s\tau)}{d\tau} &= \frac{Ej_{eta}}{T} \end{aligned}$$

In the out regime we find as in the Landau Model

$$s\tau = constant$$

basis. The energy density as a function of proper time is shown in fig.3.

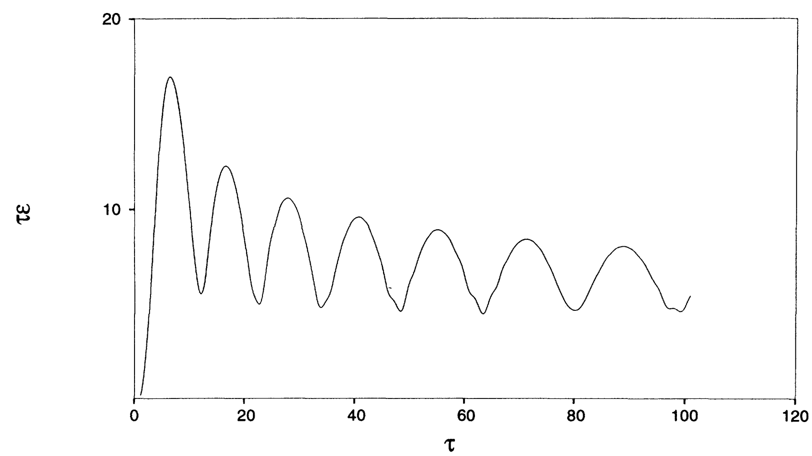

FIG. 3: Proper-time evolution of $\tau\,\epsilon(\tau)$.

For our one-dimensional boost invariant flow we find that the energy in a bin of fluid rapidity is just:

$$\frac{dE}{d\eta} = \int T^{0\mu} d\sigma_\mu = A_\perp \tau \cosh\eta \varepsilon(\tau) \tag{83}$$

which is just the (1 + 1) dimensional hydrodynamical result. Here however ε is obtained by solving the field theory equation rather than using an ultrarelativistic equation of state. Our result does not depend on any assumptions of thermalization. We can ask if we can directly calculate the particle rapidity distribution from the ansatz:

$$\frac{dN}{d\eta} = \frac{1}{m\cosh\eta}\frac{dE}{d\eta} = \frac{A_\perp}{m}\varepsilon(\tau)\tau. \tag{84}$$

We see from fig. 4. that this ansatz works well even in our case

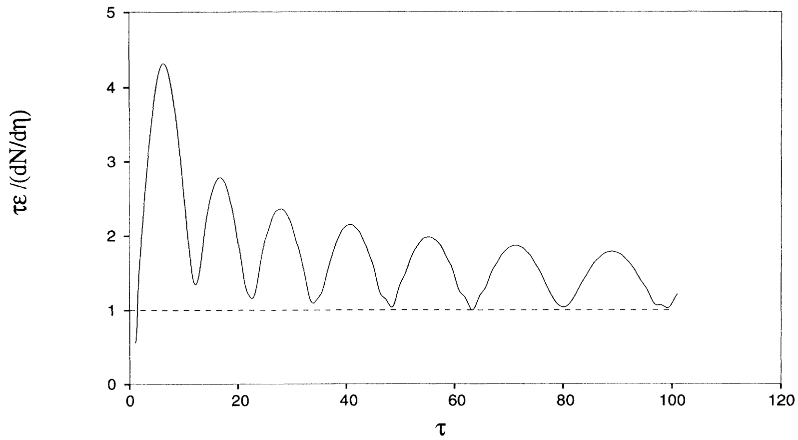

FIG. 4: Proper-time evolution of $\tau\epsilon/(dN/d\eta))$.

where we have ignored interactions between the fermions, so that we are not in thermal equilibrium.

Explicitly we have in the fermion case.

$$\varepsilon(\tau) =< T_{\tau\tau} >= \tau\Sigma_s \int [dk] R_{\tau\tau}(k) + E_R^2/2$$

where

$$R_{\tau\tau}(k) = 2(p_\perp^2 + m^2)(g_0^+|f^+|^2 - g_0^-|f^-|^2) - \omega$$
$$-(p_\perp^2 + m^2)(\pi + e\dot{A})^2/(8\omega^5\tau^2)$$
$$p_\parallel(\tau)\tau^2 =< T_{\eta\eta} >= \tau\Sigma_s \int [dk]\lambda_s \pi R_{\eta\eta}(k) - \frac{1}{2}E_R^2\tau^2 \quad (85)$$

where

$$
\begin{aligned}
R_{\eta\eta}(k) \ &= 2|f^+|^2 - (2\omega)^{-1}(\omega + \lambda_s \pi)^{-1} - \lambda_s e\dot{A}/8\omega^5\tau^2 \\
&\quad -\lambda_s e\dot{E}/8\omega^5 - \lambda_s \pi/4\omega^5\tau^2 + 5\pi\lambda_s(\pi + e\dot{A})^2/(16\omega^7\tau^2)
\end{aligned}
\tag{86}
$$

and

$$
\begin{aligned}
&p_\perp(\tau) =< T_{yy} >=< T_{xx} > \\
&= (4\tau)^{-1}\sum_s \int [dk]\{p_\perp^2(p_\perp^2 + m^2)^{-1}R_{\tau\tau} - 2\lambda\pi p_\perp^2 R_{\eta\eta}\} \\
&+E_R^2/2.
\end{aligned}
\tag{87}
$$

Thus we are able to numerically determine the effective time dependent equation of state $p_i = p_i(\varepsilon)$ as a function of τ. A typical result is shown in fig. 5.

V. QCD BACK REACTION PROBLEM WITH CYLINDRICAL SYMMETRY

Pair production from constant External fields in QCD suggest that event by event the transverse distribution of jets might depend on the values of the casimirs and not just the initial energy density. So the question is does this continue after expansion and how does thermalization compete with expansion. First we consider the expansion

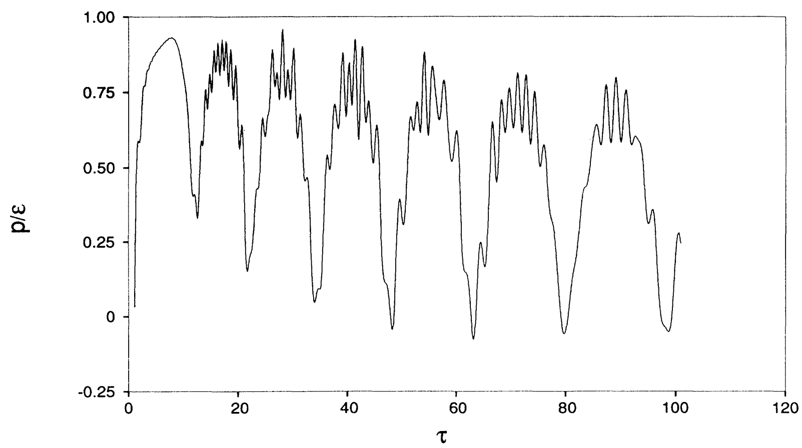

FIG. 5: Proper-time evolution of p/ϵ.

problem with azimuthal symmetry and boost invariance. For the quarks described by $\psi(x)$ and satisfying Dirac's equation:

$$\left[\gamma^\mu \left(\partial_\mu - g\, A_\mu(x) \right) + m \right] \psi(x) = 0\,, \tag{88}$$

interacting with a classical Yang-Mills field $A_\mu(x) = A^a_\mu(x)\, T^a$, where T^a are the generators of the $SU(3)$ algebra, and satisfying a back-reaction equation given by:

$$D^{ab}_\mu\, F^{b,\mu\nu}(x) = g\, \langle\, [\, \hat{\bar{\psi}}(x), \tilde{\gamma}^\nu(x)\, T^a\, \hat{\psi}(x)\,]_{\_}\, \rangle/2\,, \tag{89}$$

with $D^{ab}_\mu = \delta^{ab}\partial_\mu + g\, f^{abc}\, A^c_\mu(x)$.

$$t = \tau \cosh\eta\,, \qquad z = \tau \sinh\eta\,, \qquad x = r\cos\theta\,, \qquad y = r\sin\theta\,. \tag{90}$$

For a boost invariant expansion, the classical gauge fields are restricted to be in the η-direction and depend only on τ. We also consider only the $a = 3$ and $a = 8$ gauge fields which carry all colors. Then, using the Gell-Mann representation for the λ^a matrices,

$$\begin{aligned}
\tilde{\gamma}^\mu A_\mu =& \frac{1}{2}\,\tilde{\gamma}^\eta(x)\,\big[\,A^3_\eta(\tau)\,\lambda^3 + A^8_\eta(\tau)\,\lambda^8\,\big] \\
=& \frac{1}{2}\,\tilde{\gamma}^\eta(x)\,\begin{pmatrix} A^3_\eta(\tau) + A^8_\eta(\tau)/\sqrt{3} & 0 & 0 \\ 0 & -A^3_\eta(\tau) + A^8_\eta(\tau)/\sqrt{3} & 0 \\ 0 & 0 & -2\,A^8_\eta(\tau)/\sqrt{3} \end{pmatrix} ,
\end{aligned} \tag{91}$$

We choose an axial gauge, so that only the electric field terms:

$$E^a_\eta(\tau) = -\frac{\partial A^a_\eta(\tau)}{\partial \tau}\,, \tag{92}$$

for $a = 3$ and $a = 8$ contribute. Then, in our coordinate system, Eq. (89) becomes:

$$\frac{1}{\tau}\frac{\partial}{\partial \tau}\Big(\,\tau\,E^a_\eta(\tau)\,\Big) = -g\,\langle\,[\,\hat{\bar{\psi}}(x), \tilde{\gamma}^\eta(x)\,T^a\,\hat{\psi}(x)\,]_{\_}\,\rangle/2\,. \tag{93}$$

Eqs. (88) and (93) are the equations we want to solve.

The two Casimir invariants for $SU(3)$ are given by:

$$C_1 = E^a E^a\,, \quad \text{and} \quad C_2 = \big[\,d^{abc}\,E^a\,E^b\,E^c\,\big]^2\,, \tag{94}$$

Choosing E as arbitary linear combination of E^3 and E^8 allows one to cover the range of possible Casimir invariants.

In Cylindrical Coordinates the canonical quark fields obey (suppressing all SU(3) indices):

$$[\,\hat{\phi}_\alpha(\tau,\rho,\theta,\eta),\hat{\phi}^\dagger_\beta(\tau,\rho',\theta',\eta')\,]_+ = \delta_{\alpha,\beta}\,\frac{\delta(\rho-\rho')}{\sqrt{\rho\rho'}}\,\delta(\theta-\theta')\,\delta(\eta-\eta')\,, \tag{95}$$

We write the Dirac field operator in terms of solutions of the Dirac equation in cylindrical coordinate times appropriate creation and annihilation operators.

$$\begin{aligned}\hat{\phi}(\tau,\rho,\theta,\eta) &= \int_{-\infty}^{\infty}\frac{\mathrm{d}k_\eta}{2\pi}\int_0^\infty\frac{k_\perp\,\mathrm{d}k_\perp}{2\pi}\sum_{h=\pm1}\sum_{m=-\infty}^{+\infty}\\ &\times\left\{\,\hat{b}^{(h)}_{k_\eta,k_\perp,m}\,\phi^{(h,+)}_{k_\eta,k_\perp,m}(\tau,\rho,\theta,\eta)+\hat{d}^{(h)\,\dagger}_{k_\eta,k_\perp,m}\,\phi^{(-h,-)}_{-k_\eta,k_\perp,-m}(\tau,\rho,\theta,\eta)\,\right\}.\end{aligned} \tag{96}$$

where:

$$\phi^{(h)}_{k_\perp,m}(\tau,\rho,\theta,\eta) = \begin{pmatrix}\phi^{(h)}_{(+);k_\perp}(\tau,\eta)\,\chi^{(h)}_{k_\perp,m}(\rho,\theta)\\ \phi^{(h)}_{(-);k_\perp}(\tau,\eta)\,\chi^{(-h)}_{k_\perp,m}(\rho,\theta)\end{pmatrix}\,, \tag{97}$$

with $\lambda = hk_\perp$, and where $h = \pm 1$.

$$\chi^{(h)}_{k_\perp,m}(\rho,\theta) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{im\theta} J_m(k_\perp \rho) \\ h\, e^{i(m+1)\theta} J_{m+1}(k_\perp \rho) \end{pmatrix} , \tag{98}$$

with eigenvalues $\lambda = hk_\perp$ and the helicity $h = \pm 1$ Orthogonality is given by the relation:

$$\int_0^{+\infty} \rho\, \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta\ \chi^{(h)\dagger}_{k_\perp,m}(\rho,\theta)\, \chi^{(h')}_{k'_\perp,m'}(\rho,\theta)$$
$$= \pi\, \delta_{m,m'} \int_0^{+\infty} \rho\, \mathrm{d}\rho \left\{ J_m(k_\perp\rho)\, J_m(k'_\perp\rho) + hh'\, J_{m+1}(k_\perp\rho)\, J_{m+1}(k'_\perp\rho) \right\}$$
$$= \delta_{h,h'}\, \delta_{m,m'}\, (2\pi)\, \frac{\delta(k_\perp - k'_\perp)}{\sqrt{k_\perp k'_\perp}} \,. \tag{99}$$

The Dirac equation acting on the quantum field leads to the following matrix equation for the functions $\phi(\tau,\eta)$:

$$\begin{pmatrix} i\partial_\tau + 1 & (\, i\partial_\eta + g\, A(\tau)\,)/\tau - ihk_\perp \\ (\, i\partial_\eta + g\, A(\tau)\,)/\tau + ihk_\perp & i\partial_\tau - 1 \end{pmatrix} \begin{pmatrix} \phi^{(h)}_{(+);k_\perp}(\tau,\eta) \\ \phi^{(h)}_{(-);k_\perp}(\tau,\eta) \end{pmatrix} = 0 \tag{100}$$

which is independent of m. Introducing the Fourier transform:

$$\phi^{(h)}_{(\pm);k_\perp}(\tau,\eta) = e^{ik_\eta \eta}\, \phi^{(h)}_{(\pm);k_\eta,k_\perp}(\tau) \,, \tag{101}$$

gives an equation involving τ alone:

$$\begin{pmatrix} i\partial_\tau + 1 & -\pi_{k_\eta}(\tau) - ihk_\perp \\ -\pi_{k_\eta}(\tau) + ihk_\perp & i\partial_\tau - 1 \end{pmatrix} \begin{pmatrix} \phi^{(h)}_{(+);k_\eta,k_\perp}(\tau) \\ \phi^{(h)}_{(-);k_\eta,k_\perp}(\tau) \end{pmatrix} = 0\,, \quad (102)$$

where we have defined $\pi_{k_\eta}(\tau)$ by:

$$\pi_{k_\eta}(\tau) = (\, k_\eta - g\, A(\tau)\,)/\tau\,. \quad (103)$$

Eq. (102) is the equation we want to solve numerically as a function of τ for some given initial spinor at $\tau = \tau_0$. Since we have scaled all variable with the fermion mass m we choose $\tau_0 = 1$. Initially in Axial Gauge the electromagnetic field A can be chosen to be zero. This allows us to use a complete set of solutions to the "free" Dirac equation corresponding to the vacuum state as initial conditions. These solutions are also chosen to be adiabatic in that they will be assumed to hold near $\tau = 1$ also. That is *near* $\tau = 1$, Eq. (102) becomes:

$$\begin{pmatrix} i\partial_\tau + 1 & -k_\eta - ihk_\perp \\ -k_\eta + ihk_\perp & i\partial_\tau - 1 \end{pmatrix} \begin{pmatrix} \phi^{(h)}_{0\,(+);k_\eta,k_\perp}(\tau) \\ \phi^{(h)}_{0\,(-);k_\eta,k_\perp}(\tau) \end{pmatrix} = 0\,, \quad (104)$$

which have positive and negative frequency solutions of the form:

$$\phi^{(h,+)}_{0\,;k_\eta,k_\perp}(\tau) = \sqrt{\frac{\omega_{0;k_\eta,k_\perp} - 1}{2\omega_{0;k_\eta,k_\perp}}} \begin{pmatrix} 1 \\ +\dfrac{k_\eta - ihk_\perp}{\omega_{0;k_\eta,k_\perp} - 1} \end{pmatrix} \exp[\,-i\omega_{0;k_\eta,k_\perp}(\tau - 1)\,] \,, \tag{105a}$$

$$\phi^{(h,-)}_{0\,;k_\eta,k_\perp}(\tau) = \sqrt{\frac{\omega_{0;k_\eta,k_\perp} - 1}{2\omega_{0;k_\eta,k_\perp}}} \begin{pmatrix} -\dfrac{k_\eta - ihk_\perp}{\omega_{0;k_\eta,k_\perp} - 1} \\ 1 \end{pmatrix} \exp[\,+i\omega_{0;k_\eta,k_\perp}(\tau - 1)\,] \,, \tag{105b}$$

where $\omega_{0;k_\eta,k_\perp} = \sqrt{k_\eta^2 + k_\perp^2 + 1}$. These solutions are orthogonal:

$$\sum_{\alpha=\pm} \phi^{(h,\lambda)\,*}_{0\,(\alpha);k_\eta,k_\perp}(\tau)\, \phi^{(h,\lambda')}_{0\,(\alpha);k_\eta,k_\perp}(\tau) = \delta_{\lambda,\lambda'} \,, \tag{106}$$

and complete:

$$\sum_{\lambda=\pm 1} \phi^{(h,\lambda)}_{0\,(\alpha);k_\eta,k_\perp}(\tau)\, \phi^{(h,\lambda)\,*}_{0\,(\beta);k_\eta,k_\perp}(\tau) = \delta_{\alpha,\beta} \,. \tag{107}$$

So at $\tau = 1$, we choose our solutions of Eq. (102) so that:

$$\phi^{(h,\lambda)}_{(\alpha);k_\eta,k_\perp}(1) = \phi^{(h,\lambda)}_{0\,(\alpha);k_\eta,k_\perp}(1) \,, \tag{108}$$

for $\alpha = \pm$ and where $\lambda = \pm 1$ labels the initial positive and negative frequency solutions of Eq. (104). The τ-dependent solutions will then be numerically stepped out from the values at $\tau = 1$.

Maxwell's equation becomes:

$$\partial_\tau E(\tau) = -\frac{g}{\tau} \int_{-\infty}^{\infty} \frac{\mathrm{d}k_\eta}{2\pi} \int_0^{\infty} \frac{k_\perp \, \mathrm{d}k_\perp}{2\pi} \sum_{h=\pm 1} j^{(h)}_{k_\eta,k_\perp}(\tau) \;, \tag{109}$$

where $j^{(h)}_{k_\eta,k_\perp}(\tau)$ is given by the positive energy solutions of the Dirac equation only:

$$\begin{aligned} j^{(h)}_{k_\eta,k_\perp}(\tau) &= \phi^{(h,+)\,*}_{(+);k_\eta,k_\perp}(\tau)\, \phi^{(h,+)}_{(-);k_\eta,k_\perp}(\tau) + \phi^{(h,+)\,*}_{(-);k_\eta,k_\perp}(\tau)\, \phi^{(h,+)}_{(+);k_\eta,k_\perp}(\tau) \;, \\ &= \phi^{(h,+)\,\dagger}_{k_\eta,k_\perp}(\tau)\, \sigma_x\, \phi^{(h,+)}_{k_\eta,k_\perp}(\tau) \;. \end{aligned} \tag{110}$$

Here, $\phi^{(h,+)}_{k_\eta,k_\perp}(\tau)$ is the two-component positive energy spinor:

$$\phi^{(h,+)}_{k_\eta,k_\perp}(\tau) = \begin{pmatrix} \phi^{(h,+)}_{(+);k_\eta,k_\perp}(\tau) \\ \phi^{(h,+)}_{(-);k_\eta,k_\perp}(\tau) \end{pmatrix} \;, \tag{111}$$

and σ_x the Pauli matrix. Dirac's Eq. (102) and Maxwell's Eq. (109), are the update equations we want to solve simultaneously.

VI. INTERPOLATING NUMBER OPERATOR

We choose to define our interpolating wave functions in terms of the exact solutions of the Dirac equation in the *absence* of external fields. These zeroth order spinors are given by:

$$\phi^{(h,\lambda)}_{0;k_\eta,k_\perp,m}(\tau,\rho,\theta,\eta) = e^{ik_\eta\eta}\begin{pmatrix}\phi^{(h,\lambda)}_{0\,(+);k_\eta,k_\perp}(\tau)\,\chi^{(h)}_{k_\perp,m}(\rho,\theta)\\ \phi^{(h,\lambda)}_{0\,(-);k_\eta,k_\perp}(\tau)\,\chi^{(-h)}_{k_\perp,m}(\rho,\theta)\end{pmatrix}\ , \quad (112)$$

where $\phi^{(h,\lambda)}_{0;k_\eta,k_\perp}(\tau)$ given by Eqs. (105). These spinors are also orthogonal and complete. Expansion of the field operator in the *zeroth order* spinors then requires that the creation and annihilation operators $\hat{A}^{(h,\lambda)}_{0;k_\eta,k_\perp,m}(\tau)$ become time-*dependent*. That is:

$$\begin{aligned}\hat{\phi}(\tau,\rho,\theta,\eta) &= \int_{-\infty}^{\infty}\frac{\mathrm{d}k_\eta}{2\pi}\int_0^{\infty}\frac{k_\perp\,\mathrm{d}k_\perp}{2\pi}\sum_{h=\pm1}\sum_{\lambda=\pm1}\\ &\times\sum_{m=-\infty}^{+\infty}\hat{A}^{(h,\lambda)}_{0;k_\eta,k_\perp,m}(\tau)\,\phi^{(h,\lambda)}_{0;k_\eta,k_\perp,m}(\tau,\rho,\theta,\eta)\ , \quad (113)\end{aligned}$$

Because of the orthogonality of the initial spinors, we see that the $\hat{A}^{(h,\lambda)}_{0;k_\eta,k_\perp,m}(\tau)$ operators obey the same commutation relations as the time-*independent* ones at equal time:

$$[\,\hat{A}^{(h,\lambda)}_{0;k_\eta,k_\perp,m}(\tau), \hat{A}^{(h',\lambda')\,\dagger}_{0;k'_\eta,k'_\perp,m'}(\tau)\,]_+ = \delta_{\lambda,\lambda'}\,\delta_{h,h'}\,\delta_{m,m'}\,(2\pi)^2\,\delta(k_\eta-k'_\eta)\,\frac{\delta(k_\perp-k'_\perp)}{\sqrt{k_\perp k'_\perp}}, \quad (114)$$

and are a reasonable interpolating number operators at time τ. The interpolating particle and anti-particle operators at time τ are

$$\hat{A}^{(h,+)}_{0;k_\eta,k_\perp,m}(\tau) = \hat{b}^{(h)}_{0;k_\eta,k_\perp,m}(\tau), \qquad \text{and} \qquad \hat{A}^{(h,-)}_{0;k_\eta,k_\perp,m}(\tau) = \hat{d}^{(-h)\,\dagger}_{0;-k_\eta,k_\perp,-m}(\tau). \tag{115}$$

As before we can determine the adiabatic number operator from the Bogoliubov transformation. The overlap between the adiabatic wave functions and the exact ones is : $C^{(h;\lambda,\lambda')}_{k_\eta,k_\perp}(\tau)$ is given by:

$$C^{(h;\lambda,\lambda')}_{k_\eta,k_\perp}(\tau) = \phi^{(h,\lambda)\,\dagger}_{0;k_\eta,k_\perp}(\tau)\,\phi^{(h,\lambda')}_{k_\eta,k_\perp}(\tau)\,, \tag{116}$$

and is independent of m. So the creation and annihilation operators are related by the expression:

$$\hat{A}^{(h,\lambda)}_{0;k_\eta,k_\perp,m}(\tau) = \sum_{\lambda=\pm} C^{(h;\lambda,\lambda')}_{k_\eta,k_\perp}(\tau)\,\hat{A}^{(h,\lambda')}_{k_\eta,k_\perp,m}\,, \tag{117}$$

which is a Bogoliubov transformation of the operators.

Calling $n^{(h)}_{k_\perp,\phi,k_\eta}(\tau)$ be the phase space number density we find

$$n^{(h)}_{k_\eta,k_\perp,\phi}(\tau) = |\,C^{(h;+,-)}_{k_\eta,k_\perp}(\tau)\,|^2 = 1 - |\,C^{(h;+,+)}_{k_\eta,k_\perp}(\tau)\,|^2\,, \tag{118}$$

and is *independent* of ϕ. Explicitly, $|\,C^{(h;+,+)}_{k_\eta,k_\perp}(\tau)\,|^2$ is

$$|\,C^{(h;+,+)}_{k_\eta,k_\perp}(\tau)\,|^2 = \frac{\omega_{k_\eta,k_\perp}-1}{2\omega_{k_\eta,k_\perp}}\left|\,\phi^{(h,+)}_{(+);k_\eta,k_\perp}(\tau) + \frac{k_\eta + ih\,k_\perp}{\omega_{k_\eta,k_\perp}-1}\,\phi^{(h,+)}_{(-);k_\eta,k_\perp}(\tau)\,\right|^2, \tag{119}$$

which has unit value at $\tau = 1$, as required. That is, no particles are produced at $\tau = 1$. Right now we are in the process of doing these calculations. The renormalization method we are using for the back reaction equation will mimic the direct method of using the cutoff value for the the multiplicative charge renormalization $Z(\Lambda, m)$, in analogy the QED result found in Eq. [58]

VII. DOES THE PLASMA THERMALIZE?

In order to discuss whether the plasma thermalizes, one needs to have a robust enough approximation which leads to thermalization for non expanding plasmas. It has been shown that the 2-PI 1/N approximation does have that property. One then needs to discover whether the expansion rate will preclude or slow-down the thermalization of the quarks and gluons produced. To include interactions among the quarks and gluons one would solve the coupled Schwinger Dyson equations using the CTP formalism and a 2-PI Action expanded in 1/N. [31]. Here one would need to keep the background field formalism also to handle the background Chromoelectric Field. Below we sketch some features of the calculation that we are about to begin in order to answer the important question of whether the interactions will drastically change the transverse distribution of jets from that predicted in the case of noninteracting fermions and gluons. This formalism has already been used in QCD to determine transport coefficients by Aarts and Resco [33] and we follow their notation here. The action for N_f identical fermion fields ψ_a $(a = 1, \ldots, N_f)$ then reads

$$S = \int_x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}_a \left(i \not{D} - m \right) \psi_a \right] + S_{\mathrm{gf}} + S_{\mathrm{gh}}, \qquad (120)$$

with

$$\not{D} = \gamma^\mu D_\mu, \qquad D_\mu = \partial_\mu + \frac{ie}{\sqrt{N_f}} A_\mu, \qquad (121)$$

and we use the notation

$$\int_x = \int_{\mathcal{C}} dx^0 \int d^3x, \qquad (122)$$

where $\mathcal{C}$ refers to the CTP contour in the complex-time plane. We follow the closed time path formalism of Schwinger where all the Green's function can be thought of as ordered according to the closed time path or equivalently as 2×2 matrix Green's functions. The 2PI effective action is an effective action for the contour-ordered two-point

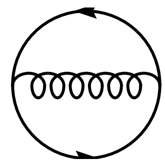

FIG. 6: NLO contribution to the 2PI effective action in the $1/N_f$ expansion.

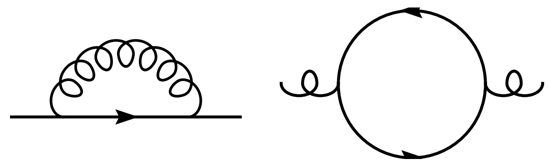

FIG. 7: Self energies at at NLO in large N_f QCD.

functions

$$D_{\mu\nu}(x,y) = \langle|T_{\mathcal{C}}\left(A_\mu(x)A_\nu(y)\right)|\rangle \qquad S_{ab}(x,y) = \langle|T_{\mathcal{C}}\left(\psi_a(x)\bar{\psi}_b(y)\right)|\rangle, \tag{123}$$

and can be written schematically as

$$\Gamma[S,D] = \frac{i}{2}\mathrm{tr}\ln D^{-1} + \frac{i}{2}\mathrm{tr}D_0^{-1}(D-D_0) \tag{124}$$

$$- i\mathrm{tr}\ln S^{-1} - i\mathrm{tr}S_0^{-1}(S-S_0) + \Gamma_2[S,D] + \text{ghosts}, \tag{125}$$

where D_0^{-1} and S_0^{-1} are the free inverse propagators.

For the gauge theory the NLO Schwinger Dyson equations that result from varying the 2-PI action are:

$$S^{-1} = S_0^{-1} - \Sigma, \qquad D^{-1} = D_0^{-1} - \Pi, \tag{126}$$

with S the fermion and D the gauge field propagator. The self energies, depending on full propagators, are shown in Fig 7. The Back Reaction equation is given by:

$$\nabla_\mu F^{\mu\nu} = \langle j^\nu \rangle = -ig^2 Tr \gamma^\nu S \qquad (127)$$

To determine the interpolating number densities of quarks and antiquarks one can follow the procedure of Berges, Borsanyi and Serreau [34] and define these from the current. Namely the associated 4–current for each given flavor is $\sim \bar{\psi}\gamma^\mu\psi$. Fourier transforming with respect to spatial momenta, the expectation value of the latter can be written as $J^\mu_f(t,p) = \mathrm{tr}[\gamma^\mu S^<(t,t,p)]$, In terms of the equal-time two point function, its temporal and spatial components are

$$J^0_f(t,p) = 2\left[1 - 2\,F^0_V(t,t;p)\right],$$
$$\vec{J}_f(t,p) = -4\,F_V(t,t;p)\,.$$

To obtain an effective particle number, Berges et. al. identify these expressions with the corresponding ones in a quasi-particle description with free-field expressions. These are given by

$$J^{0\,(\mathrm{QP})}_f(t,p) = 2\left[1 + Q_f(t,p)\right],$$
$$\vec{J}^{(\mathrm{QP})}_f(t,p) = -2\left[1 - 2N_f(t,p)\right],$$

where $Q_f(t,p) = n_f - \bar{n}_f$ is the difference between particle and anti-particle effective number densities and $N_f(t,p) = (n_f + \bar{n}_f)/2$ is their half-sum. The physical content of these expressions is simple: the temporal component J^0 directly represents the net-charge density per mode $Q_f(t,p)$, whereas the spatial part $\vec{J}$ is the net current density per mode and is therefore sensitive to the sum of particle and anti-particle number densities. Identifying the above expressions, theydefine

$$\frac{1}{2} Q_f(t,p) = -F_V^0(t,t;p)\,, \tag{128}$$

$$\frac{1}{2} - N_f(t,p) = F_V(t,t;p)\,. \tag{129}$$

Using these definitions and solving the backreatction problem to NLO in 2-PI 1/N we would also be able to discover if there is time for the produced quarks and antiquarks to thermalize before hadronization time scale and to see if the constant field result for the transverse distribution will be modified by the interactions.

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[1] A.H. Guth, Phys, Rev. D23, 347 (1981); A.D. Linde, Phys. Lett. 108 B, 389 (1982); A.A. Starobinsky, Phys. Lett. 117B, 175 (1982)

[2] Emil Mottola, Phys. Rev. D31, 754 (1985).

[3] J. Schwinger, *J. Math. Phys.* **2**, 407 (1961);
K.T. Mahanthappa, *Phys. Rev.* **126**, 329 (1962);
P. M. Bakshi and K. T. Mahanthappa, *J. Math. Phys.* **4**, 1 (1963); **4**, 12 (1963);
L. V. Keldysh, *Zh. Eksp. Teo. Fiz.* **47**, 1515 (1964) [*Sov. Phys. JETP* **20**, 1018 (1965)];
G. Zhou, Z. Su, B. Hao and L. Yu, *Phys. Rep.* **118**, 1 (1985);E. Calzetta and B. L. Hu, *Phys. Rev.* **D35**, 495 (1987).

[4] F. Cooper, J. Dawson and B. Mihaila, Phys.Rev.D63:096003,2001

[5] A. H. Guth and So-Young Pi, Phys. Rev. D32, 1899 (1985)

[6] F. Cooper and E. Mottola, Phys. Rev. D 36, 3114 (1987) , So-Young Pi and M. Samiullah, Phys. Rev. D 36, 3128 (1987).

[7] S.A. Fulling, L. Parker, and B.L. Hu, Phys. Rev. D 10, 3905 (1974).

[8] A. Bialas and W. Czyz, Phys. Rev. D30 (1984) 2371; A. Bialas, W. Czyz, A. Dyrek and W. Florkowski, Nucl. Phys. B296 (1988) 611; K. Kajantie and T. Matsui, Phys. Lett. 164B (1985) 373; G. Gatoff, A. K. Kerman and T. Matsui, Phys. Rev. D36 (1987) 114.

[9] J. Schwinger, Phys. Rev. 82 (1951) 664.

[10] A. Casher, H. Neuberger, and S. Nussinov, Phys. Rev. D 20, 179 (1979).

[11] G. Nayak and P. van Nieuwenhuizen Phys. Rev. D71 (2005) 125001G. C. Nayak, Phys. Rev. D72 (2005) 0510052.

[12] M. Claudson, A. Yildiz and P. H. Cox, Phys. Rev. D22 (1980) 2022.

[13] G. 't Hooft, Nucl. Phys. B62 (1973) 444.

[14] F. Cooper and G. C. Nayak, hep-th/0609192.

[15] F. Cooper and G. Nayak Phys.Rev.D73:065005,2006. e-Print: hep-ph/0511053.

[16] F. Cooper and G. Nayak e-Print: hep-th/0612292; hep-th/0611125

[17] Y. Kluger, J. M. Eisenberg, B. Svetitsky, F. Cooper and E. Mottola, Phys. Rev. Lett. 67 (1991) 2427; Y. Kluger, J. M. Eisenberg, B. Svetitsky, F. Cooper and E. Mottola, Phys. Rev. D **45** (1992)4659

[18] F. Cooper, G. Frye and E. Schonberg Phys. Rev. **D11**, 192 (1975).

[19] J. D. Bjorken, Phys. Rev. D**27**, 140 (1983).

[20] L. Bettencourt, F. Cooper and K. Pao, Phys.Rev.Lett.89:112301,2002.

[21] H. M. Fried and R. P. Woodard, Phys. Lett. B 524 (2002) 233.

[22] C. Itzykson and J-B. Zuber, Quantum Field Theory, page-194, Dover Publication, Inc. Mineola, New York.

[23] A. A. Grib, V. M. Mostepanenko, and V. M. Frolov. Teoret. i Matem. Fizika, **13, No. 3** 377 (1972).

[24] Y. Kluger, E. Mottola, and J. Eisenberg, Phys. Rev. D. **58**, 125015 (1998)

[25] F. Cooper and E. Mottola, Phys. Rev. D **40**, 456 (1989); . F. Cooper, J.M. Eisenberg, Y. Kluger, E. Mottola, and B. Svetitsky, Phys. Rev. **D 48** (1993) 190. hep-ph/9212206; F.Cooper, J. Dawson, Y. Kluger and H. Shepard, Nuclear Physics A566 (1994) 395c.

[26] N. B. Narozhnyi and A. I. Nikishov, Yad. Fiz **11** 1072 (1970) Sov. J. Nucl. Phys. 11 (1970) 596.

[27] G. Gatoff, A. K. Kerman, and T. Matsui, Phys. Rev. D **36**, 114 (1987).

[28] A. Casher, H. Neuberger and S. Nussinov, Phys. Rev. D20 (1979) 179.

[29] N. K. Glendenning and T. Matsui, Phys. Rev. D28 (1983) 2890.

[30] C. Itzykson and J-B. Zuber, Quantum Field Theory, Mc-Graw Hill (1980) page 193; L. Alvarez-Gaume and E. Witten, Nucl. Phys. B234 (1984) 234; P. van Nieuwenhuizen, *Anomalies in Quantum Field Theory: Cancellation of Anomalies in d=10 Supergravity*, Lecture Notes in

Mathematical and Theoretical Physics, Vol 3, Leuven University Press (1988), page 46.

[31] F. Cooper,S. Habib, Y.Kluger,E. Mottola, J. Paz, and Paul Anderon, Phys.Rev.D50:2848-2869,1994. e-Print: hep-ph/9405352;B. Mihaila, F. Cooper and J. Dawson arXiv:hep-ph-0006254, Phys.Rev. **D63** 096003,2001;J. Berges, Nucl.Phys.A699:847-886,2002, G. Aarts, D. Ahrensmeier, R. Baier, J. Berges, J. Serreau, Phys.Rev.D66:045008,2002; Fred Cooper, John F. Dawson, Bogdan Mihaila e-Print Archive: hep-ph/0209051 Phys.Rev.D67:056003,2003

[32] F. Cooper and G. Frye, Phys. Rev. **D10**, 186 (1974); F. Cooper, G. Frye and E. Schonberg Phys. Rev. **D11**, 192 (1975).

[33] Gert Aarts and Martinez Resco, JHEP 0503:074,2005. e-Print: hep-ph/0503161

[34] J. Berges, S. Borsanyi and J. Serreau, Nucl.Phys.B660:51-80,2003. e-Print: hep-ph/0212404