Equations of State for Astrophysical Applications

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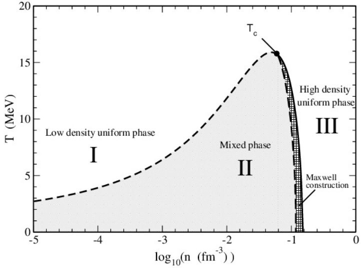
	Core-collapse supernovae	**Proto-neutron stars**	**Mergers of compact binary stars**
Baryon Density(n_0)	$10^{-8} - 10$	$10^{-8} - 10$	$10^{-8} - 10$
Temperature(MeV)	$0 - 30$	$0 - 50$	$0 - 100$
Entropy(k_B)	$0.5 - 10$	$0 - 10$	$0 - 100$
Proton Fraction	$0.35 - 0.45$	$0.01 - 0.3$	$0.01 - 0.6$

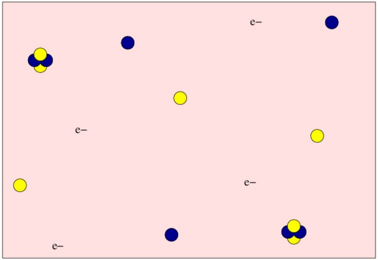
$n_0 \simeq 0.16$ fm^{-3} (equilibrium density of symmetric nuclear matter)

- Objective: Construction of EOS based on the best nuclear physics input for use in hydrodynamic simulations of core-collapse supernova explosions and binary mergers as well as the thermal evolution of proto-neutron stars.

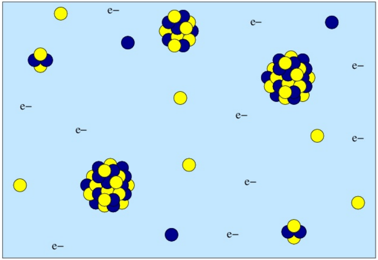
Nuclear Matter Phases



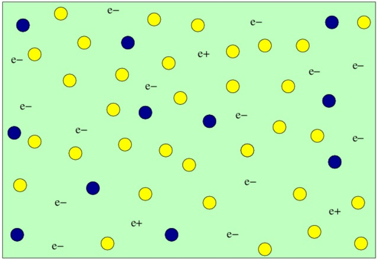
Phase I– Low Density Regime



Phase II– Mixed Phase



Phase III– $n > n_s$ Dissociated



Matter near the Equilibrium Point ($n = n_0, \alpha = 0$) of SNM

- About $\alpha = 0$,
  $E(n,\alpha) \simeq E_0(n) + S_2(n)\alpha^2 + \mathcal{O}(\alpha^4)$
- About $n = n_0$,
  $E_0(n) \simeq \mathcal{E}_0 + \frac{1}{2}K_0\left(\frac{n-n_0}{3n_0}\right)^2 + \ldots$
  $S_2(n) \simeq S_v + L\left(\frac{n-n_0}{3n_0}\right) + \ldots$
- $P(n,\alpha) = n^2\frac{\partial E}{\partial n} \simeq n^2\left[\frac{K_0}{3n_0}\left(\frac{n-n_0}{3n_0}\right) + \frac{L}{3n_0}\alpha^2\right] + \ldots$
- $\hat{\mu} = \mu_n - \mu_p = 2\frac{\partial E}{\partial \alpha} \simeq 4\alpha S_2(n)$

- Saturation density, $n_0 = 0.16 \pm 0.01\ \mathrm{fm}^{-3}$
  High-energy electron scattering: $r_0 \propto \pi/qR$, $n_0 = \left(\frac{4}{3}\pi r_0^3\right)^{-1}$
- Energy per particle, $\mathcal{E}_0 = -16 \pm 1$ MeV
  Fits to masses of atomic nuclei :
  $B(N,Z) = \mathcal{E}_o A - b_{surf} A^{2/3} - S_v \frac{(N-Z)^2}{A} - b_{Coul} Z^2 A^{-1/3}$
- Symmetry energy, $S_v = 30 - 35$ MeV
  (fits to masses of atomic nuclei)
- Slope of S_2, $L = 40 - 70$MeV
  (variety of experiments)
- Compression modulus, $K_0 = 240 \pm 30$ MeV
  Giant monopole resonances : $E_{GMR} = \left(\frac{K_A}{m<r^2>}\right)^{1/2}$
  $K_A = K_0 + K_{surf} A^{-1/3} + K_\tau \frac{(N-Z)^2}{A^2} + K_{Coul} \frac{Z^2}{A^{4/3}}$
- Effective mass, $M^*/M = 0.8 \pm 1$
  Neutron evaporation spectra : $N(E_n) \propto a$, $a_{FermiGas} = \frac{\pi^2 m^*}{2k_F^2}$.

Approaches to the Nucleon-Nucleon Interaction

- High-precision interactions fitted to NN scattering data
  - meson-exchange models
    e.g. Nijmegen, Paris, Juelich-Bonn
  - sums of local operators
    e.g. Urbana, Argonne

- Interactions from chiral EFT

- RG-evolved potentials

Extension of the above to bulk matter by a variety of techniques: SCGF, BHF, variational, etc.

- Phenomenological approaches: Skyrme, Gogny, RMFT
  Typically treated at the HF level, fitted to extracted quantities.

- $\hat{V}_{NN} = \sum_{i<j} \hat{V}_{ij} + \sum_{i<j<k} \hat{V}_{ijk}$, zero-range
- Evaluating $\hat{H} = \hat{T} + \hat{V}_{NN}$ in the HF approximation gives

$$\begin{aligned} \mathcal{H}_{Skyrme} &= \frac{\hbar^2}{2m_n}\tau_n + \frac{\hbar^2}{2m_p}\tau_p \\ &+ n(\tau_n + \tau_p)\left[\frac{t_1}{4}\left(1 + \frac{x_1}{2}\right) + \frac{t_2}{4}\left(1 + \frac{x_2}{2}\right)\right] \\ &+ (\tau_n n_n + \tau_p n_p)\left[\frac{t_2}{4}\left(\frac{1}{2} + x_2\right) - \frac{t_1}{4}\left(\frac{1}{2} + x_1\right)\right] \\ &+ \frac{t_o}{2}\left(1 + \frac{x_o}{2}\right) n^2 - \frac{t_o}{2}\left(\frac{1}{2} + x_o\right)(n_n^2 + n_p^2) \\ &\left[\frac{t_3}{12}\left(1 + \frac{x_3}{2}\right) n^2 - \frac{t_3}{12}\left(\frac{1}{2} + x_3\right)(n_n^2 + n_p^2)\right] n^\epsilon \end{aligned}$$

- Due to Akmal & Pandharipande, Phys. Rev. C. 56, 2261 (1997) (AP):

$$v_{NN} = v_{18,ij} + V_{IX,ijk} + \delta v(\mathbf{P}_{ij})$$

where

$$v_{18,ij} = \sum_{p=1,18} v^p(r_{ij}) O^p_{ij} + v_{em} \quad \text{(Argonne)}$$

$$V_{IX,ijk} = V^{2\pi}_{ijk} + V^R_{ijk} \quad \text{(Urbana)}$$

$$\delta v(\mathbf{P}) = -\frac{P^2}{8m^2} u + \frac{1}{8m^2} [\mathbf{P.r}\ \mathbf{P.}\nabla, u] + \frac{1}{8m^2} [(\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_j) \times \mathbf{P.}\nabla, u]$$

(relativistic boost correction)

APR: Hamiltonian Density

- Parametric fit by Akmal, Pandharipande, and Ravenhall (Phys. Rev. C. 58, 1804 (1998)).
- Hamiltonian density:

$$\begin{aligned} \mathcal{H}_{APR} &= \left[\frac{\hbar^2}{2m} + (p_3 + (1-x)p_5)ne^{-p_4 n}\right]\tau_n \\ &+ \left[\frac{\hbar^2}{2m} + (p_3 + xp_5)ne^{-p_4 n}\right]\tau_p \\ &+ g_1(n)(1-(1-2x)^2) + g_2(n)(1-2x)^2 \end{aligned}$$

$\tau_{n(p)}$ - neutron(proton) kinetic energy density
p_i - fit parameters

- Skyrme-like $\Rightarrow$ Landau effective mass:

$$m_i^* = \left(\frac{\partial \varepsilon_{k_i}}{\partial k_i}\bigg|_{k_{Fi}}\right)^{-1} k_{Fi} = \left[\frac{1}{m} + \frac{2}{\hbar^2}(p_3 + Y_i p_5)ne^{-p_4 n}\right]^{-1}$$

x - proton fraction, $Y_p = x,\ Y_n = 1 - x$

- Single-particle energy spectrum:

$$\varepsilon_i = k_i^2 \frac{\partial \mathcal{H}}{\partial \tau_i} + \frac{\partial \mathcal{H}}{\partial n_i} \equiv \varepsilon_{k_i} + V_i$$

- $n_i = \frac{1}{2\pi^2} \left(\frac{2m_i^* T}{\hbar^2} \right)^{3/2} F_{1/2i}$

$$\tau_i = \frac{1}{2\pi^2} \left(\frac{2m_i^* T}{\hbar^2} \right)^{5/2} F_{3/2i}$$

$$F_{\alpha i} = \int_0^\infty \frac{x_i^\alpha}{e^{-\psi_i} e^{x_i} + 1} dx_i$$

$$x_i = \frac{\varepsilon_{ki}}{T}, \quad \psi_i = \frac{\mu_i - V_i}{T} = \frac{\nu_i}{T}$$

- Rest of state variables :
  - Energy density $\varepsilon = \frac{\hbar^2}{2m_n^*} \tau_n + \frac{\hbar^2}{2m_p^*} \tau_p + U(n)$
  - Chemical potentials $\mu_i = T\psi_i + V_i$
  - Entropy density $s_i = \frac{1}{T} \left[\frac{5}{3} \frac{\hbar^2}{2m_i^*} \tau + n_i (V_i - \mu_i) \right]$
  - Pressure $P = T(s_n + s_p) + \mu_n n_n + \mu_p n_p - \varepsilon$
  - Free energy density $\mathcal{F} = \varepsilon - Ts$

- Nucleons, Ψ, coupled to σ, ω, and $\vec{\rho}$ mesons:

$$\begin{aligned}\mathcal{L} &= \bar{\Psi}\left[\gamma_\mu\left(i\partial^\mu - g_\omega\omega^\mu - \frac{g_\rho}{2}\vec{\rho}^\mu.\vec{\tau}\right) - (M - g_\sigma\sigma)\right]\Psi \\ &+\frac{1}{2}\left[\partial_\mu\sigma\partial^\mu\sigma - m_\sigma^2\sigma^2 - \frac{\kappa}{3}(g_\sigma\sigma)^3 - \frac{\lambda}{12}(g_\sigma\sigma)^4\right] \\ &+\frac{1}{2}\left[-\frac{1}{2}f_{\mu\nu}f^{\mu\nu} + m_\omega^2\omega^\mu\omega_\mu\right] \\ &+\frac{1}{2}\left[-\frac{1}{2}\vec{B}_{\mu\nu}\vec{B}^{\mu\nu} + m_\rho^2\vec{\rho}^\mu\vec{\rho}_\mu\right]\end{aligned}$$

- Extensions: Hyperons, scalar-isovector δ-meson, density-dependent couplings, derivative couplings, nucleon form factors, etc.

- Negligible meson-field fluctuations, uniform and static system $\Rightarrow$

$$\begin{aligned}
\sigma_0 &= \frac{g_\sigma}{m_\sigma^2} < \bar{\Psi}\Psi > - \frac{1}{m_\sigma^2}\left(\frac{\kappa}{2} g_\sigma^3 \sigma_0^2 + \frac{\lambda}{6} g_\sigma^4 \sigma_0^3\right) \\
&= \frac{g_\sigma}{m_\sigma^2} n_s - \frac{1}{m_\sigma^2}\left(\frac{\kappa}{2} g_\sigma^3 \sigma_0^2 + \frac{\lambda}{6} g_\sigma^4 \sigma_0^3\right) \\
\omega_0 &= \frac{g_\omega}{m_\omega^2} < \bar{\Psi}\gamma^0\Psi >= \frac{g_\omega}{m_\omega^2} n \\
\rho_0 &= \frac{g_\rho}{2m_\rho^2} < \bar{\Psi}\gamma^0\tau_3\Psi >= -\frac{g_\rho}{2m_\rho^2} n(1-2x)
\end{aligned}$$

- Spectrum: $\epsilon_{i\pm} = \pm(p_i^2 + M^{*2})^{1/2} + \frac{g_\omega^2}{m_\omega^2} n + \frac{g_\rho^2}{4m_\rho^2}(n_i - n_j)$
- Diagonal elements of the stress-energy tensor $T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)}\partial_\nu q_i - g_{\mu\nu}\mathcal{L}$, give:
  - Energy density, $\varepsilon =< T_{00} >$
  - Pressure, $P = \frac{1}{3} < T_{ii} >$
- Dirac mass, M^*, derived from the requirement $\frac{\delta\Omega}{\delta\sigma} = 0$

- In NR models, repulsive contributions to the energy per particle $E(u = n/n_s)$ that vary faster than linear in u give rise to acausal behavior at large u.
- The causality condition

  $\left(\frac{c_s}{c}\right)^2 \equiv \beta = \frac{\partial P}{\partial \epsilon}\Big|_S \leq 1$

  can be written as a 1st order DE

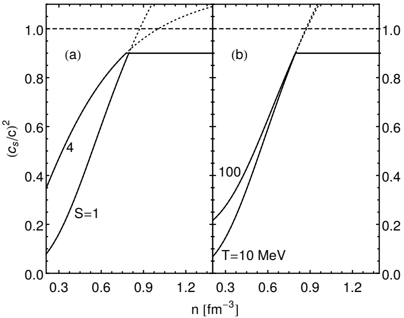
  $\frac{\partial \xi}{\partial n}\Big|_{S,N} - \frac{\beta}{n}\xi = \frac{\beta m}{n}$

  where $\xi = \frac{\partial \varepsilon}{\partial n}\Big|_{S,N} = \mu + TS$.
- For a fixed β_f, integration leads to

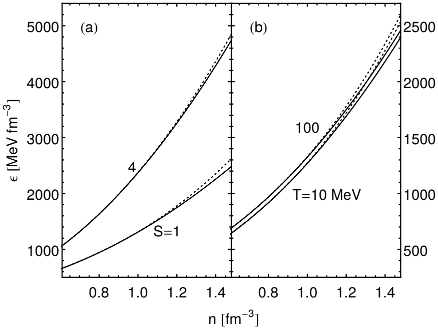
  $\varepsilon = -mn + \frac{c_1 n^{\beta_f+1}}{\beta_f+1} + c_2$

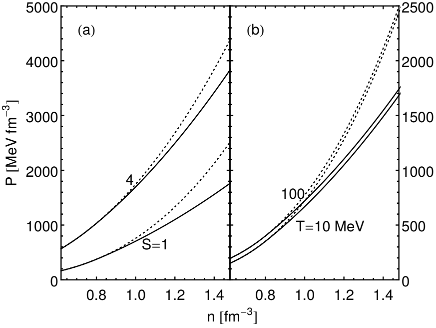
  $P = c_1 \frac{\beta_f}{\beta_f+1} n^{\beta_f+1} - c_2$

  where the constants c_1 and c_2 depend only on the causality-fixing density n_f.



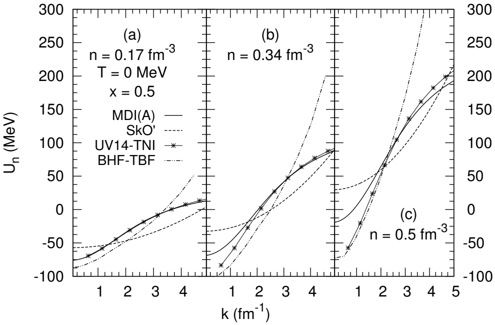
Causal State Variables



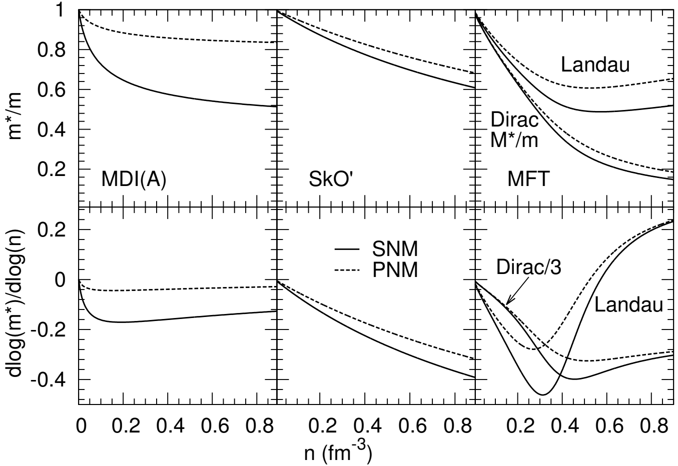


Choice of β_f determines deviation from original results.

- Single particle potential, $U(\rho, p)$ in both Skyrme and MFT models grows monotonically with momentum;
  inconsistent with optical model fits to nucleon-nucleus reaction data.
- Microscopic calculations (RBHF, variational calculations, etc.) show distinctly different behaviors in their momentum dependence, consistent with optical model fits.
- The above features were found necessary to account for heavy-ion data on transverse momentum and energy flow (in conjuction with $K \sim 240$ MeV).

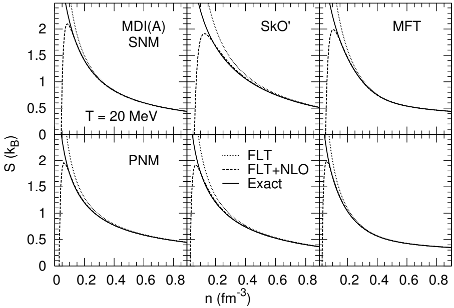


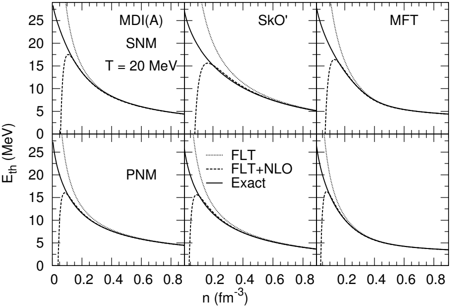
- SkO': P. G. Reinhard *et al.*, PRC **60**, 014316 (1999)
- BHF-TBF: W. Zuo *et al.* PRC **74**, 014317 (2006)
- UV14-TNI: R. B. Wiringa PRC **38**, 2967 (1988)



- For n large, $\frac{dm^*}{dn} \simeq 0$
- $m^* = \frac{m}{1+\beta(x)n}$
- $m^* = E_F^* = (p_F^2 + M^{*2})^{1/2}$
- Minimum at n s.t. $\frac{p_F}{M^*} + \frac{dM^*}{dp_F} = 0$

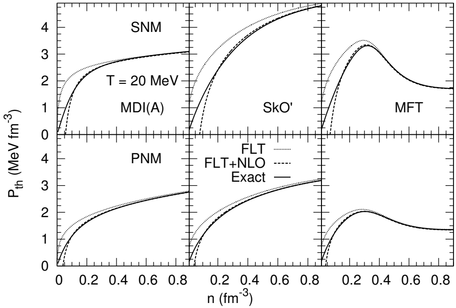
Results: S and E_{th}

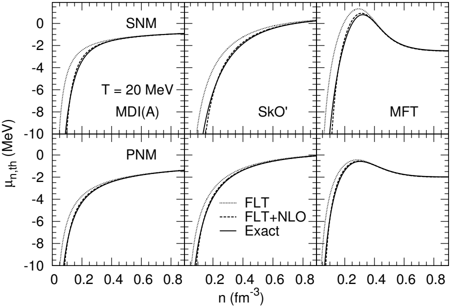




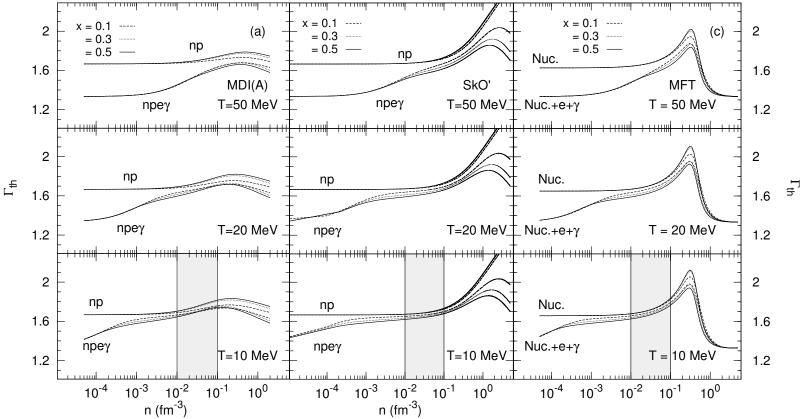
- The three models produce quantitatively similar results.
- Agreement with exact results extended to $n \simeq 0.1$ fm^{-3}.
- Better agreement for PNM than for SNM.

Results: P_{th} and μ_{th}





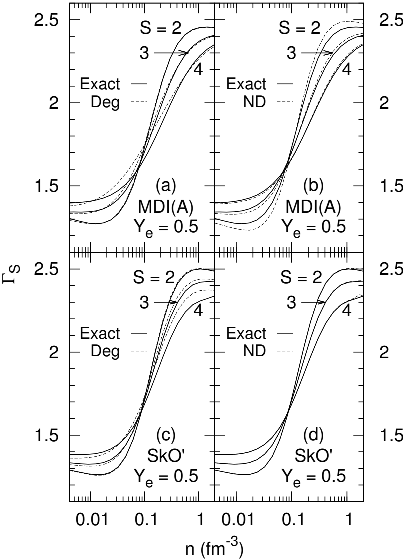
- Model dependence is evident- due to $\frac{dm^*}{dn}$.
- Agreement with exact results extended to $n \simeq 0.1$ fm^{-3}.
- Better agreement for PNM than for SNM.



- $\Gamma_{th} = 1 + \frac{P_{th}}{\varepsilon_{th}}$
- Weak x and T dependence but n dependence cannot be ignored.
- APR/Skyrme: $P_{th}(n,T) = P_{th}^{id}(n,T;m^*)Q$; $\varepsilon_{th}(n,T) = \varepsilon_{th}^{id}(n,T;m^*)$

$$\frac{P_{th}^{id}}{\varepsilon_{th}^{id}} = \frac{2}{3} \quad \Rightarrow \Gamma_{th} \propto \frac{m^*}{m} \text{ (Single-component system)}$$

- Significant contributions from electrons.



- Relevant during the early inspiralling phase of a merger when the two objects interact only gravitationally.
- $\Gamma_S(n,S) = \left.\frac{\partial \ln P}{\partial \ln n}\right|_S = \frac{n}{P} \left.\frac{\partial P}{\partial n}\right|_S$
- $\Gamma_S(n,T) = \frac{C_P}{C_V}\frac{n}{P} \left.\frac{\partial P}{\partial n}\right|_T$
- Relation to sound speed, c_s: $\left(\frac{c_s}{c}\right)^2 = \Gamma_S \, \frac{P}{h+mn}$
- Only nucleons ⇒ mechanical instability
- With leptons, nuclear matter is stable
- n_X such that $\frac{1}{P_0}\frac{dP_0}{dn} = \frac{1}{P_{th}} \left.\frac{\partial P_{th}}{\partial n}\right|_S$ (indep. of S in Deg. and ND Limits)

- Nuclear statistical equilibrium
  - Statistical ensemble of nucleons and nuclei in thermodynamic equilibrium
  - Chemical potentials of nuclei, $\mu_a = N_a\mu_n + Z_a\mu_p$
  - Maxwell-Boltzmann statistics
  - Abundances determined by the Saha equation
  - Requires nuclear binding energies as input
- Lattimer-Swesty
  - Single nucleus approximation (SNA).
  - Hard-sphere interactions
  - Equilibrium obtained by minimizing the free energy
- Virial expansion
  - Nondegenerate limit expansion of the grand potential in small fugacity: $z = \exp[(\mu - m)/T] \ll 1$
  - Coefficients depend on scattering phase shifts corresponding to the interaction
  - Supplements NSE
- Nucleons-in-cell: Molecular dynamics, Thomas-Fermi, Hartree-Fock ...

Lattimer-Swesty approach (Nucl. Phys. A **535**, 331 (1991)):
$F_{total} = F_N + F_\alpha + F_{bulk} + F_e + F_\gamma$

- Single representative species of heavy nucleus; described by the compressible liquid-drop model: $F_N = F_{bulk,in} + F_s + F_C + F_{tr}$.
  - F_S - Semi-infinite matter in TF approximation; curvature and neutron skin ignored.
  - $F_C = \frac{4}{5}\pi e^2 x_{in}^2 n_{in} r_N \mathcal{D}(u)$;
    $\mathcal{D}(u) = 1 - \frac{3}{2}u^{1/3} + \frac{1}{2}u$ (point + lattice + finite size)
  - $F_{tr} = \frac{u(1-u)n_{in}}{A_0} h(T) T \left\{ \ln \left[\frac{u(1-u)n_{in}}{n_Q A_0^{5/2}} \right] - 1 \right\}$
    (massive non-interacting point particles w/ modifications)
- α-particles represent light nuclei; treated as non-interacting Boltzmann gas having hard sphere interactions with nuclei:
  $F_\alpha = (1-u) n_\alpha f_\alpha$
- Nucleons have hard-sphere interactions with α's and nuclei:
  $F_{bulk} = (1-u)(1 - v_\alpha n_\alpha) n_{out} f_{bulk}$
- Pasta : Use smooth generalized functions which modify F_C and F_S appropriately.

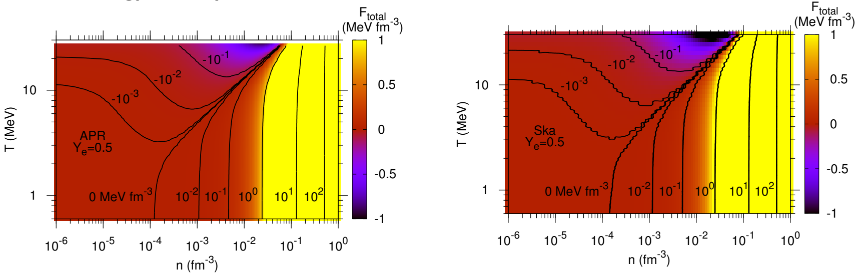
To find tbe equilibrium state of the system, F_{total} must be minimized with respect to nuclear radius r, proton fraction x_{in}, nucleon density n_{in}, volume occupied by nuclei u, and density of α-particles n_α:

- $\frac{\partial F}{\partial r} = 0$ (nuclear size optimization)
- $\frac{1}{un_{in}} \frac{\partial F}{\partial x_{in}} = 0$ (chemical equilibrium)
- $\frac{\partial F}{\partial n_i} - \frac{x_i}{n_i} \frac{\partial F}{\partial x_i}$
- $\frac{n_{in}}{u} \frac{\partial F}{\partial n_{in}} - \frac{\partial F}{\partial u} = 0$ (pressure equilibrium)
- $\frac{\partial F}{\partial n_\alpha} = 0$ (chemical equil. of α's with nucleons)

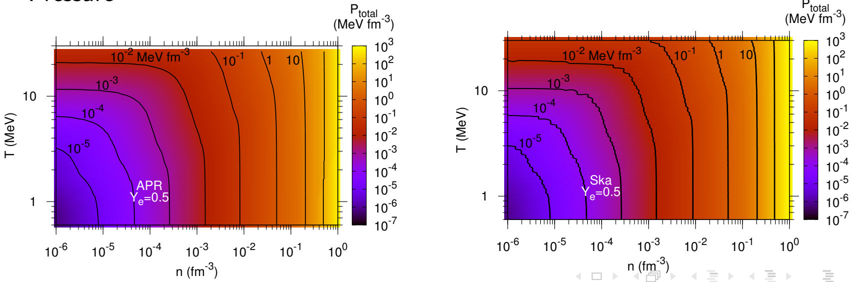
Constraints:

- $n = un_{in} + (1-u)[4n_\alpha + (n_{n,out} + n_{p,out})(1 - n_\alpha v_\alpha)]$
(baryon # conservation)
- $nY_e = ux_{in}n_{in} + (1-u)[2n_\alpha + n_{p,out}(1 - n_\alpha v_\alpha)]$
(charge conservation)

Free energy density



Pressure



Conclusions

- Matter in astrophysics covers a wide range in (n, x, T)-space; much of which cannot be accessed by terrestrial experiments.
- Upcoming astrophysical applications and laboratory experiments involving heavy ions/rare isotopes will probe higher densities and temperatures allowing a tighter grip on the EOS.
- The structure of the interaction, and thus m^*, is crucial in the determination of thermal effects. These, in turn, are important in neutrino and gravitational radiation emission.
- Importance of leptons in the stability of nuclear matter.
- APR EOS at finite T with consistent treatment of the subnuclear regime and of the transition to a pion-condensed phase; to serve as a benchmark for other nonrelativistic calculations.
- Many other approaches to the construction of the nuclear EOS exist; these include (but are not limited to) NR potential models based on the Skyrme and the Gogny forces, relativistic MFT, microscopic approaches using Green's functions methods, the renormalization group, chiral perturbation theory etc.

Future

- Apply LS to MFTs and models w/ finite-range interactions; comparison with the Shen EOS, Phys. Rev. C 83, 035802 (2011).
- Better treatment of nuclear properties (e.g. HFB description of heavy nuclei, etc.).
- SN, PNS, BM simulations using the full APR.
- Relax SNA, add more light nuclear species in the dissociated region.

Open questions:

- Neutron star M_{max}
- Emergence of non-nucleonic DoF.
- Phase transitions