Avalanches and Functional Renormalization Group:

Same universality class for the critical behavior in and out of equilibrium in a quenched random field

Gilles Tarjus (LPTMC, CNRS-Univ. PARIS 6)

Matthieu Tissier (LPTMC), Ivan Balog (Physics Institute, Zagreb)

KITP 14

Outline

- The role of avalanches in the critical behavior of the random field Ising model (RFIM).
- Problem 1: Escaping dimensional reduction!
- Problem 2: Are the out-of-equilibrium (hysteresis) critical point and the equilibrium one in same universality class?
- The method: Superfield theory and nonperturbative functional renormalization group (NP-FRG).

The random field Ising model (RFIM)

- Magnetism: lattice hard-spin version

$$\mathcal{H}[\{S_i\};\{h_i\}] = -J\sum_{<i,j>} S_i S_j + \sum_i h_i S_i$$

with Ising variables $S_i = \pm 1$ => magnetization $m_i = <S_i>$

a quenched random field (e.g., Gaussian): $\overline{h_i} = 0,\ \overline{h_i h_j} = \Delta_B\,\delta_{ij}$

\+ coupling to an applied magnetic field H

- Under some coarse-graining: Field theory

$$S[\varphi;h] = S_B[\varphi] - \int_x h(x)\varphi(x);\ S_B = \int d^dx \left\{\frac{1}{2}(\partial\varphi(x))^2 + \frac{\tau}{2}\varphi(x)^2 + \frac{u}{4!}\varphi(x)^4\right\}$$

with a real scalar field $\boldsymbol{\varphi}$(x) => classical field Φ(x) = < $\boldsymbol{\varphi}$(x) >

a quenched random ``source'' (e.g., Gaussian): $\overline{h(x)} = 0,\ \overline{h(x)h(y)} = \Delta_B\,\delta^{(d)}(x-y)$

\+ coupling to an applied source J

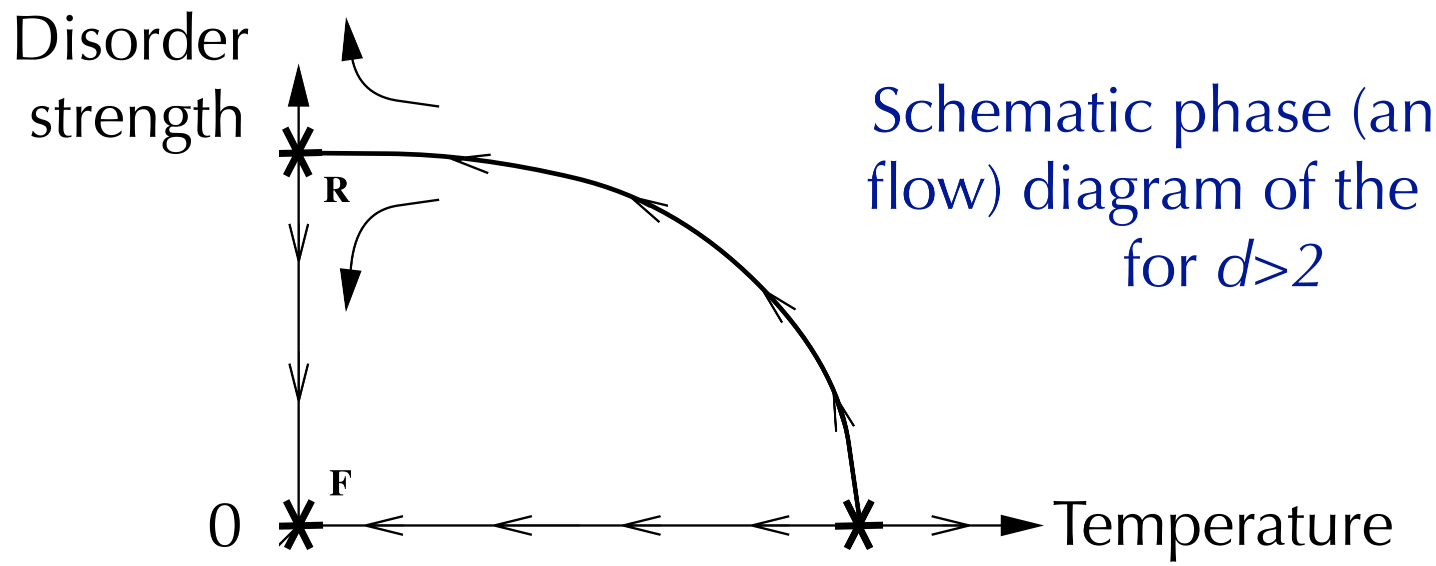
Why are avalanches relevant at all for equilibrium criticality ?

- RFIM equilibrium partition function in a given random-field sample *h* :

$$Z_h[J] = \int \mathcal{D}\varphi \, e^{-\frac{1}{T}\left(S[\varphi;h] - \int_x J(x)\varphi(x)\right)}$$

- At equilibrium: PM to FM transition (critical point) as a function of T at constant Δ_B or as a function of Δ_B at constant T
- Yet, avalanches are zero-T phenomena!
- But the long-distance (critical) physics at equilibrium is dominated by sample-to-sample (disorder) fluctuations, not by thermal fluctuations...

The equilibrium critical point is controlled by a zero-temperature fixed point



Schematic phase (and RG flow) diagram of the RFIM for *d>2*

- **Additional exponent for the temperature flow**: $\theta > 0$
- Two distinct pair correlation functions:

$$\overline{<\phi(x)><\phi(x')>} \sim \frac{1}{|x-x'|^{d-4+\bar{\eta}}}\ , \ \text{with}\ \ \theta = 2+\eta-\bar{\eta},$$

$$\overline{<\phi(x)\phi(x')> - <\phi(x)><\phi(x')>} \sim \frac{T}{|x-x'|^{d-2+\eta}}$$

- For *T>0*: very slow "activated" critical dynamics.

Dimensional reduction

- Conventional perturbation theory to **all** orders and the Parisi-Sourlas supersymmetric approach [PRL '79] both predict that the critical behavior of the RFIM is the same as that of the pure Ising model in two dimensions less:

d -> d-2 "dimensional reduction" property

- OK at the upper critical dimension ($d_{uc,RFIM}$=6 -> $d_{uc,pure}$=4) but rigorously proven wrong in d=2 and 3! [J.Z. Imbrie '84, J. Bricmont and A. Kupianen '87]

=> Problem 1

Generic difficulties for theories of disordered systems

- Due to quenched disorder (*h*), one loses translational invariance:

  $W_h[J] = \ln Z_h[J]$ is a random functional of the source =>

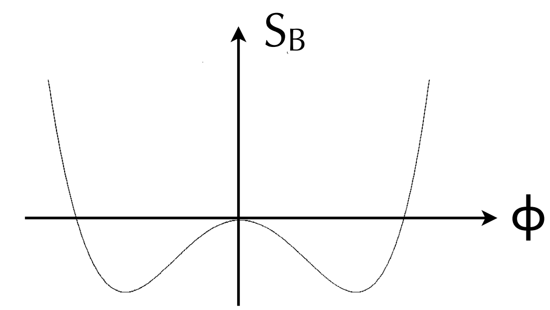
  - in principle, one needs its whole probability distribution,
  - or equivalently, the infinite set of its disorder-averaged cumulants (recovers translational invariance):

  $$W_1[J] = \overline{W_h[J]},\ W_2[J_1, J_2] = \overline{W_h[J_1]W_h[J_2]}|_c,\ \cdots$$

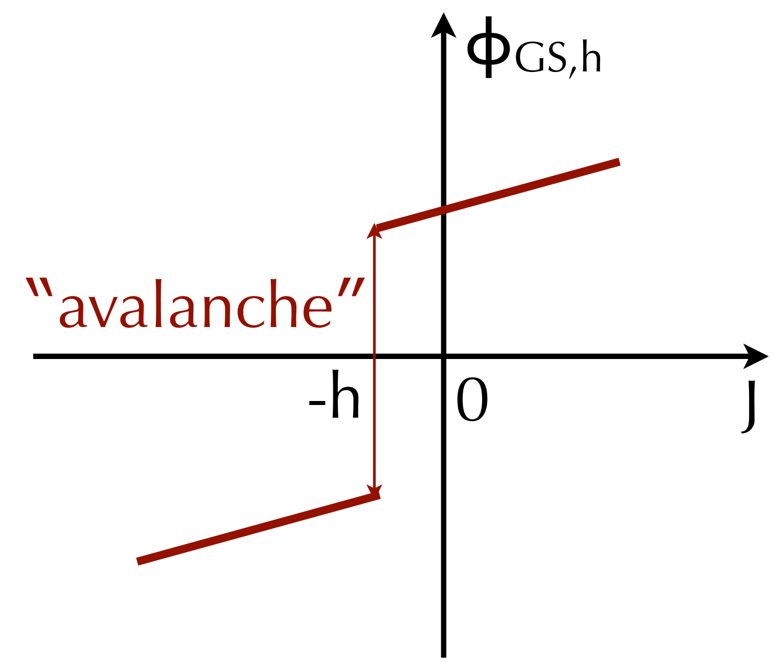
- Possible influence of rare events, rare spatial regions or rare samples: here, at T=0, **avalanches**!

Effect of avalanches : toy model (d=0 RFIM)

At T=0, stochastic equation: $\frac{\delta S_B(\phi)}{\delta \phi} = J + h$ with:



- For equilibrium, select the ground state:



The pair correlation function for slightly different sources,

$$\overline{\phi_{GS,h}(J+\delta J)\phi_{GS,h}(J-\delta J)} = \overline{\phi_{GS,h}(J)^2} + A(J)|\delta J| + O(\delta J^2)$$

has a nonanalytic behavior (a "**cusp**") when J->0 due to the ("static") avalanches.

The amplitude of the cusp ∝ second moment of the avalanche size.

[Note: the above p.c.f is given by $\partial_{J1}\partial_{J2}W_2(J_1,J_2)$=> need the full functional dependence]

- Similar for the out-of-equilibrium hysteresis behavior.

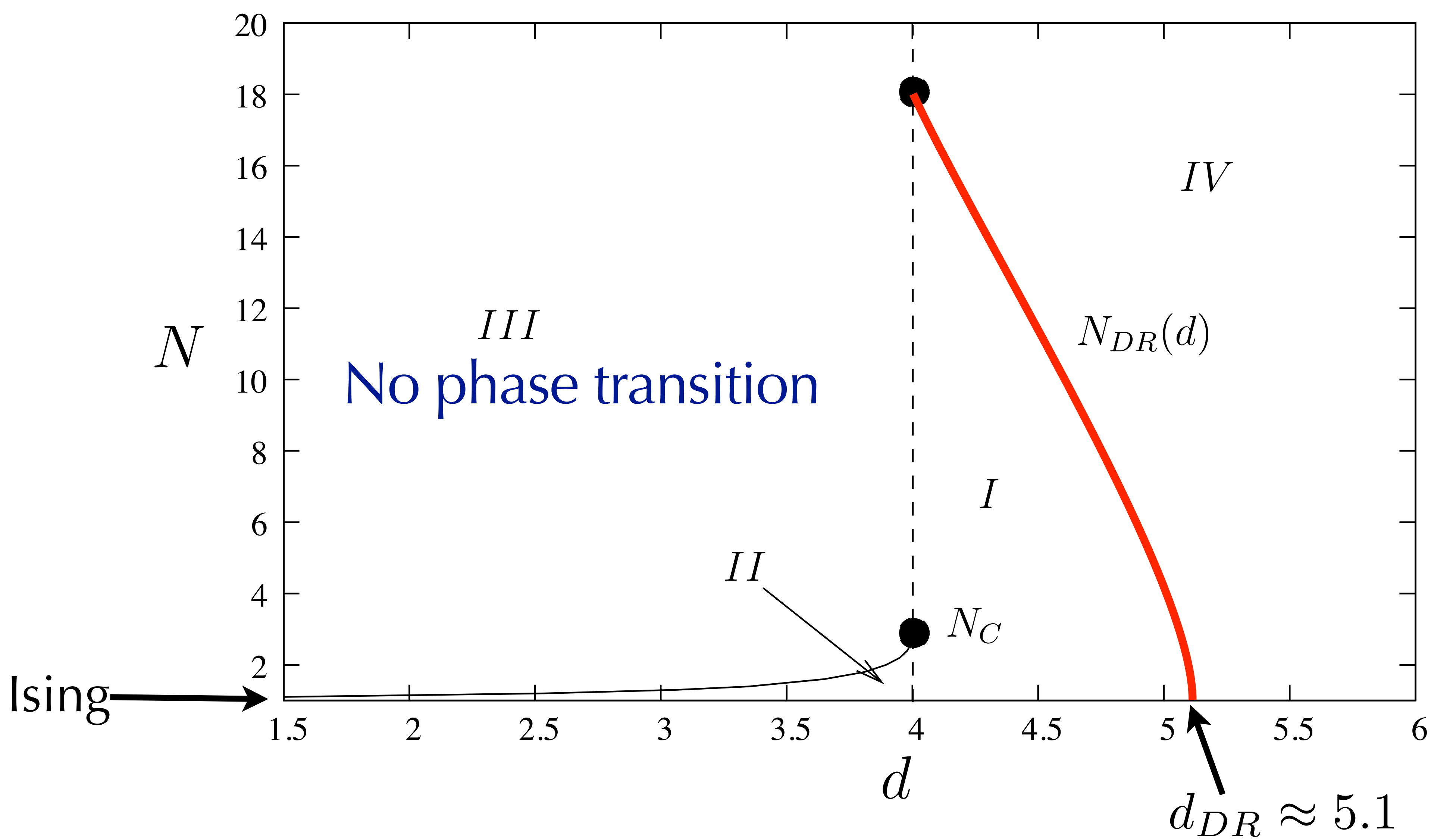
Our conclusion

- Dimensional reduction breaks down because of the appearance of a **cusp** in the functional dependence of the dimensionless cumulants(s) of the renormalized random field: same as in the case of elastic systems in random media [D. Fisher, T. Nattermann, P. Le Doussal, K. Wiese].

- This is associated with a spontaneous breaking of the (Parisi-Sourlas) supersymmetry.

- All of this takes place as a consequence of the presence of avalanches.

Result: *N*-*d* phase diagram of the RFO(*N*)M



Region IV: Dim. red. predictions O.K. (weak non-analyticity at the fixed pt.)

Regions I and II: Breakdown of dimensional reduction (Spontaneous SUSY breaking at finite RG scale; cusp in renormalized second cumulant).

Why a critical dimension d_{DR} if avalanches are always present?

- Due to avalanches at T=0, there is always a cusp in the cumulants of the renormalized random field (and the associated correlation functions).

- However, at criticality, all these functions diverge and under the RG flow the cusp may be subdominant (irrelevant) near the fixed point: d_{DR} precisely separates a region ($d > d_{DR}$) where the cusp is present but irrelevant from one ($d < d_{DR}$) where it persists at the fixed point (in the "dimensionless" quantities).

Why a critical dimension d_{DR} if avalanches are always present?

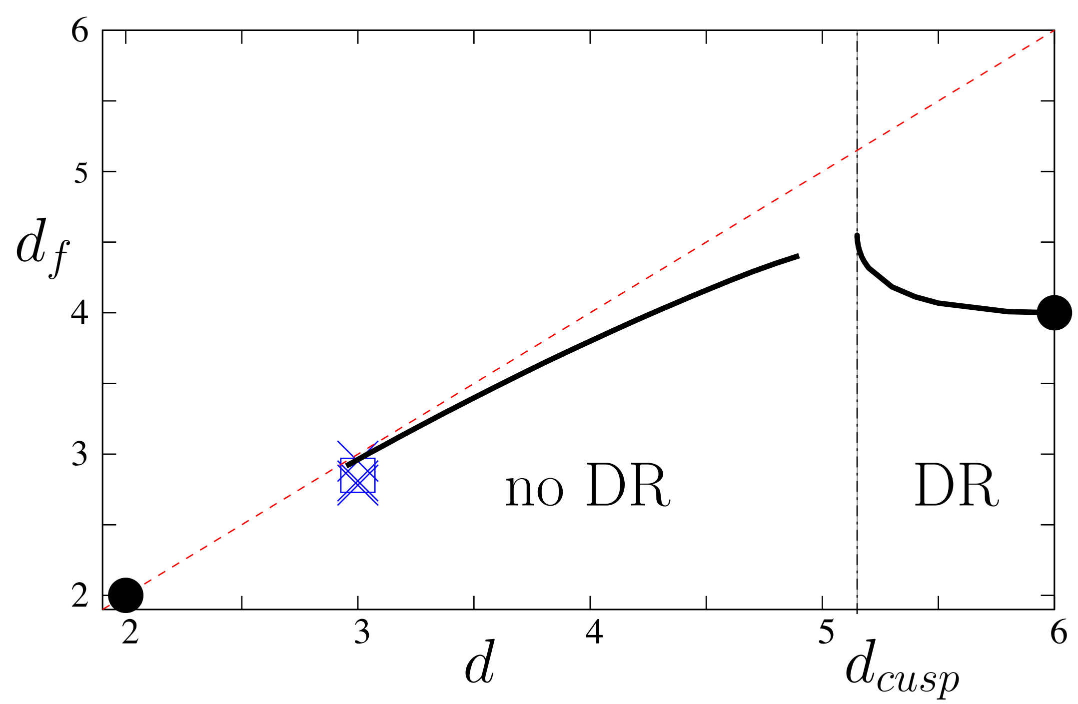
Simple scaling argument at criticality in a system of finite size L

- Avalanche size distribution $\propto S^{-\tau}\mathcal{D}(\frac{S}{S_L})$, with $S_L \propto L^{d_f}$ the size of the typical critical avalanches and d_f their fractal dimension.
- On the other hand, the total magnetization scales as $L^{d-\frac{d-4+\overline{\eta}}{2}}$

=> Avalanches have an effect at criticality *iff* $d_f = \frac{d+4-\overline{\eta}}{2}$

From the NP-FRG, we can compute the eigenvalue λ associated with a "cuspy" perturbation around the fixed point. Then, in general,

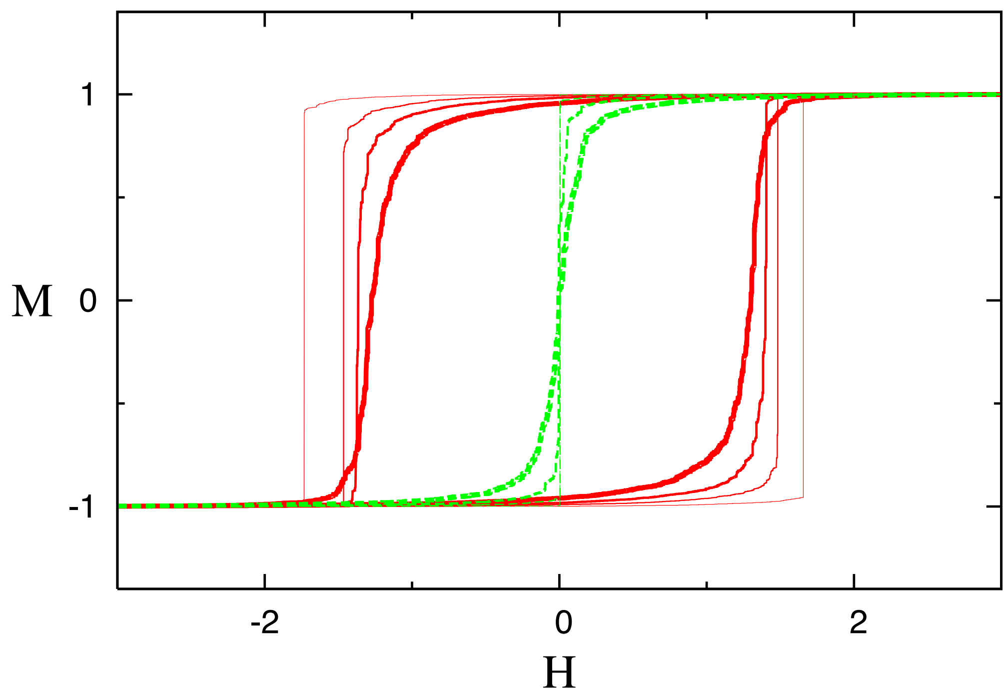
$$d_f = \frac{d+4-\overline{\eta}}{2} - \lambda$$



Problem 2

- From numerical studies of the d=3 slowly driven RFIM at T=0: The out-of equilibrium critical point on the hysteresis curve and the equilibrium critical point have very similar exponents and scaling functions. [PerezReche-Vives04,Colaiori et al 04, Liu-Dahmen07,09]

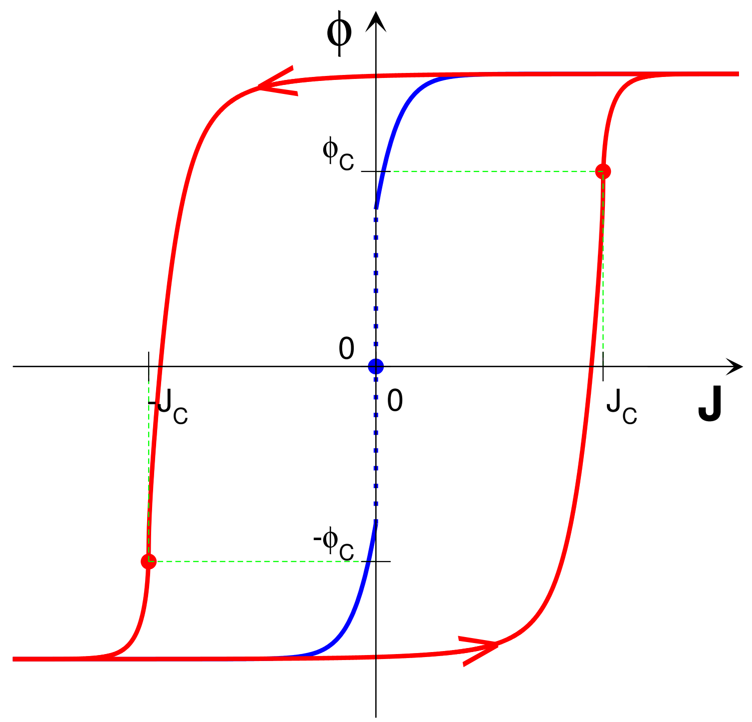
  => Suggests same critical behavior, i.e., same universality class



Dynamic versus static avalanches:
Magnetization vs applied field at T=0
[Liu-Dahmen '06]
Green: Equilibrium curves (ground state)
Red: Ascending and descending branches of the hysteresis loop

- **However**: in- versus out-of-equilibrium, not the same value of the critical disorder, not the same symmetry!!!
- To answer, need RG analysis: Are the two critical phenomena controlled by the **same fixed point of the RG flow**?

Replacing the dynamical problem by a static one: statistics of metastable states



Magnetization vs applied field at T=0 at the value of the disorder Δ_B for which the hysteresis curve has critical point(s).

- All inside the hysteresis loop: **Metastable states** (well defined at T=0).
- In blue: **Ground state**, relevant for equilibrium.
- In red: **Extremal states** with either max (descending branch) or min (ascending branch) magnetization at a given applied field J correspond to the rate-independent hysteresis loop.

=> Describe the out-of-equilibrium criticality by a stat-mech. treatment of extremal states with no reference to dynamics and history.

Superfield theory for the statistics of (metastable) states at T=0

- Metastable states are solution of the stochastic field equation:

$$\frac{\delta S[\varphi; h+J]}{\delta \varphi(x)} = \frac{\delta S_B[\varphi]}{\delta \varphi(x)} - h(x) - J(x) = 0$$

- Build the generating functional of the correlation functions of the solutions with chosen properties (β_a^{-1}=auxiliary temp.):

$$\mathcal{Z}_h[J, \hat{J}] = \int \mathcal{D}\varphi \, \delta\left[\frac{\delta S[\varphi; h+J]}{\delta \varphi}\right] \det\left[\frac{\delta^2 S[\varphi; h+J]}{\delta \varphi \delta \varphi}\right] e^{-\beta_a S[\varphi; h+J] + \int_x \hat{J}(x)\varphi(x)}$$

  - Select the **ground state**, with lowest energy (action)

$$\beta_a \to \infty, \; \hat{J} = 0 : \quad e.g., \quad \overline{\varphi^{GS}(x)} = \frac{\delta}{\delta \hat{J}(x)} \log \overline{\mathcal{Z}_h[J, \hat{J}]}\Big|_{\beta_a \to \infty, \hat{J}=0}$$

- Select the **extremal states**, e.g. with max magnetization

$$\beta_a = 0, \; \hat{J} \to \pm\infty : \; e.g., \quad \overline{\varphi^{Max}(x)} = \frac{\delta}{\delta \hat{J}(x)} \log \overline{\mathcal{Z}_h[J, \hat{J}]}\Big|_{\beta_a = 0, \hat{J} \to +\infty}$$

Superfield theory for the statistics of metastable states (cont'd)

- Introduce auxiliary fields $\hat{\varphi}$ (bosonic) and ψ, $\bar{\psi}$ (fermionic):

$$\delta\left[\frac{\delta S[\varphi; h+J]}{\delta\varphi}\right] \to \int \mathcal{D}\hat{\varphi} e^{-\int_x \hat{\varphi}(x)\frac{\delta S[\varphi;h+J]}{\delta\varphi(x)}}$$

$$\det\left[\frac{\delta^2 S_B[\varphi]}{\delta\varphi\delta\varphi}\right] \to \int\int \mathcal{D}\psi\mathcal{D}\bar{\psi} e^{\int_x\int_{x'}\bar{\psi}(x)\frac{\delta^2 S_B[\varphi]}{\delta\varphi(x)\delta\varphi(x')}\psi(x')}$$

- Group all fields $\varphi, \hat{\varphi}, \psi, \bar{\psi}$ in a "superfield" Φ leaving in a "superspace" $\underline{\mathbf{x}}$

- Introduce copies a=1,...,n of the disordered sample and average over the random field => Generating functional of a **superfield theory**:

$$\mathcal{Z}[\{\mathcal{J}_a\}] = \int \prod_a \mathcal{D}\Phi_a\, e^{-S_{super}[\{\Phi_a\}] + \sum_a \int_{\underline{x}} \mathcal{J}_a(\underline{x})\Phi_a(\underline{x})}$$

where the conjugate sources are grouped in "supersources" and the theory has a large group of symmetries and supersymmetries.

The theoretical approach: Why does one need a nonperturbative functional RG ?

- **RG**, because one is interested in the long-distance properties near to the critical point(s) and in the fixed points that control them.

- **Functional**, because the influence of the avalanches can only be described through a singular dependence of the cumulants of the renormalized disorder on their arguments.

- **Nonperturbative**, because standard perturbation theory completely fails (dimensional reduction), and because the behavior changes at a nontrivial critical dimension d_{DR}.

Nonperturbative (functional, exact) RG

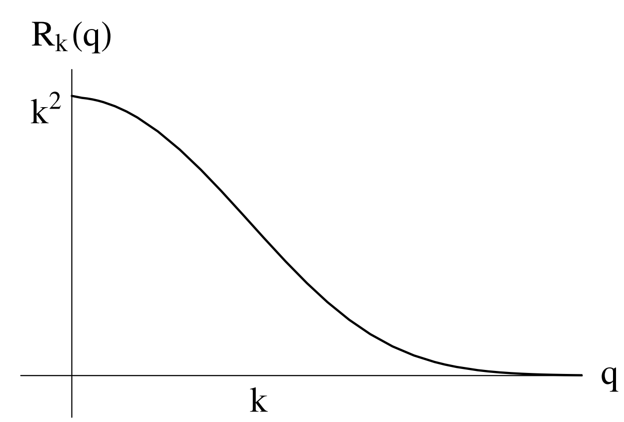
Sketch for the equilibrium φ⁴ theory

[Wilson, Polshinski, Wetterich]

- Progressive account of the fluctuations on longer length scales (shorter momenta) through the introduction of an infrared regulator in the generating functional of the connected correlation functions:

$$S_k[\varphi] = S_B[\varphi] + \frac{1}{2}\int_q R_k(q^2)\varphi(-q)\varphi(q)$$

$$W_k[J] = \log \int \mathcal{D}\varphi \, e^{-S_k[\varphi]+\int_x J(x)\varphi(x)}$$



- Via Legendre transform: **Effective average action** (Gibbs free energy at the IR scale k)

$$\Gamma_k[\phi] = -W_k[J] + \int_x J(x)\phi(x) - \frac{1}{2}\int_q R_k(q^2)\phi(-q)\phi(q); \;\; \phi =< \varphi >= \frac{\delta W_k}{\delta J}$$

with: $\Gamma_{k=\Lambda}[\phi] \simeq S_B[\phi] \;\; (\text{mean}-\text{field}) \;\; \rightarrow \;\; \Gamma_{\mathrm{k}=0}[\phi] = \Gamma[\phi] \;\; (\text{exact})$

Nonperturbative, functional, exact RG (cont'd)

- **Exact functional RG flow equation** for the effective average action:

$$\partial_k \Gamma_k[\phi] = \frac{1}{2}\int_q \partial_k R_k(q^2)\left(\Gamma_k^{(2)}[\phi] + R_k\right)^{-1}_{-q,q} \quad with \quad \Gamma^{(2)}_{k,qq'}[\phi] = \frac{\delta^2 \Gamma_k[\phi]}{\delta\phi(q)\delta\phi(q')}$$

- **Nonperturbative** approximation scheme:

$$\Gamma_k[\phi] = \int_x \{U_k(\phi(x)) + \frac{1}{2} Z_k(\phi(x))[\partial\phi(x)]^2 + \cdots\}$$

=> RG flow equations for the functions $U_k(\phi)$ and $Z_k(\phi)$

- Introduce **scaling dimensions** to cast the RG equations in a dimensionless form:

$$U_k \sim k^d\, u_k,\; Z_k \sim k^{-\eta}\, z_k,\; \phi \sim k^{(d-2+\eta)/2}\, \varphi,\; t = \log(k/\Lambda)$$

=> $\left| \begin{array}{l} \partial_t u_k(\varphi) = \beta_u[\varphi] \\ \partial_t z_k(\varphi) = \beta_z[\varphi] \end{array} \right.$ **Fixed point** describing criticality & scaling is solution with the "beta functions" = 0.

- Very efficient! (quantum field theory, stat phys, condensed matter)

NP-FRG in a superfield formalism

- **The program**: apply the NP-FRG to the multi-copy superfield theory, but recall...
  - ✳ Quenched disorder => heterogeneity and sample-dependent functionals => need to work with the **cumulants** of the random functionals: $\Gamma_k \;\to\; \Gamma_{k1}[\Phi_1], \Gamma_{k2}[\Phi_1, \Phi_2], \cdots$
  - ✳ Avalanches lead to a **nonanalytic** functional dependence in the cumulants of the renormalized disorder.
  - ✳ The outcome must be formulated for physical fields, not superfields: $\Phi_a \;\to\; \phi_a$

NP-FRG in a superfield formalism

- **Major simplification:** Asymptotically, only **one** state (extremal for hysteresis, ground state for equilibrium) is relevant for each copy.

  => Property of the superfield theory that allows one to derive exact RG equations for functionals of the physical fields only.

  => **Exact hierarchy of functional RG flow equations** for the (physical) cumulants: $\partial_k \Gamma_{k1}[\phi_1] = \cdots$, $\partial_k \Gamma_{k2}[\phi_1, \phi_2] = \cdots$, etc.

> These exact RG equations are **identical** for the equilibrium (ground state) and out-of-equilibrium (extremal states) cases.
>
> => Same set of fixed points!

- But, the **initial conditions of the flow** at the microscopic scale Λ (mean-field description) are different: Z_2 symmetry (ϕ -> -ϕ) for equilibrium ($\phi_c = J_c = 0$), no Z_2 symmetry for hysteresis (ϕ_c, $J_c \neq 0$).

  => Check that the flows go to the same (Z_2 symmetric) fixed point.

NP-FRG: approximation scheme

- **Nonperturbative SUSY-compatible truncation:**

$$\left|\begin{array}{l} \Gamma_{k1}[\phi] = \int_x \{U_k(\phi(x)) + \frac{1}{2} Z_k(\phi(x))[\partial\phi(x)]^2\} \\ \\ \Gamma_{k2}[\phi_1, \phi_2] = \int_x V_k(\phi_1(x), \phi_2(x)), \;\; \Gamma_{k,p>2} = 0 \end{array}\right.$$

- Scaling dimensions for a zero-temperature fixed point:

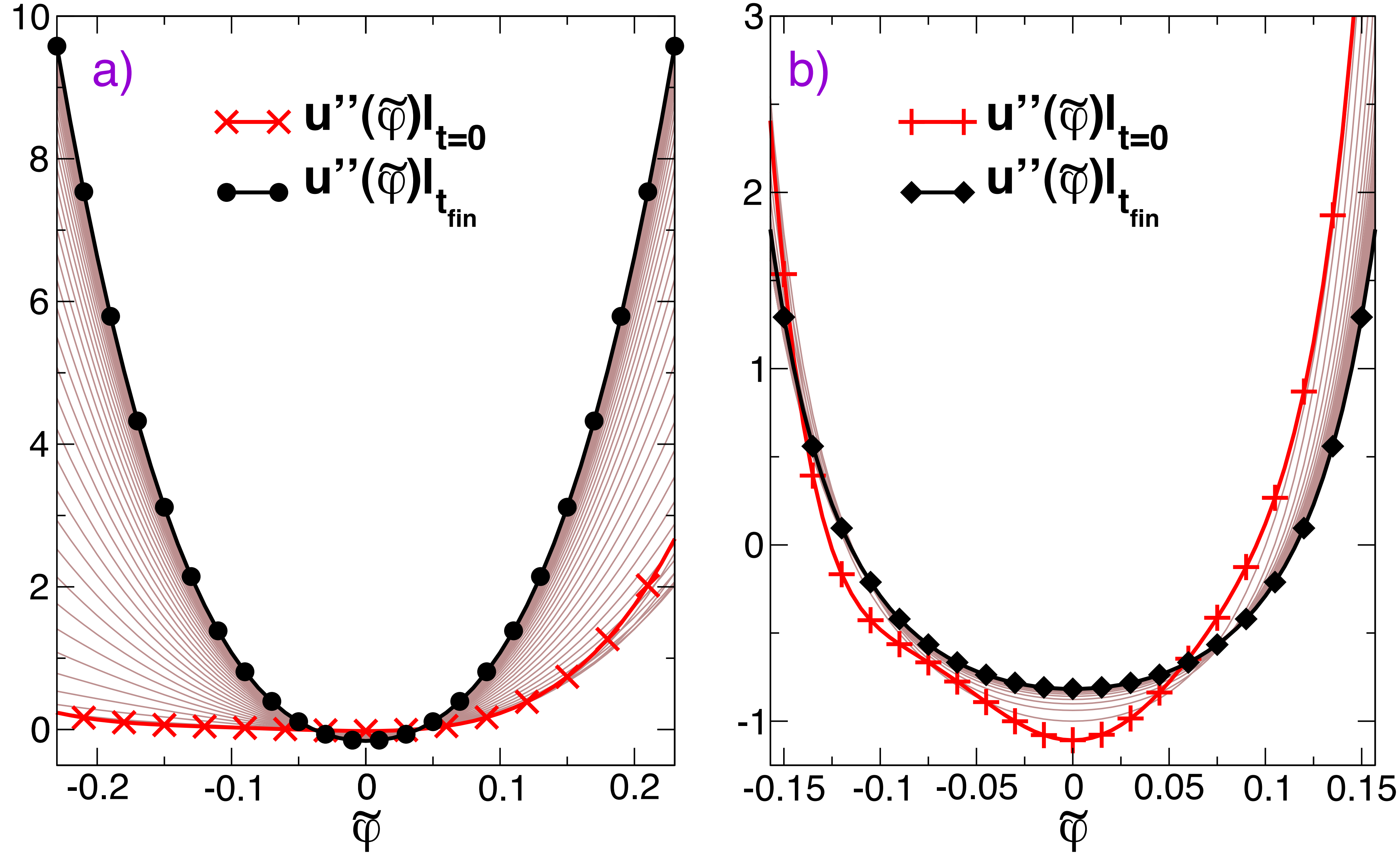
$$U_k' - J_c \sim k^{(d-2\eta+\bar\eta)/2}, \; Z_k \sim k^{-\eta}, \; \phi - \phi_c \sim k^{(d-4+\bar\eta)/2},$$
$$\Delta_k = \partial_{\phi_1}\partial_{\phi_2} V_k \sim k^{-2\eta+\bar\eta}, \; t = \log(k/\Lambda)$$

=> $$\left|\begin{array}{l} \partial_t u_k''(\varphi) = \beta_{u''}[\varphi], \; \partial_t z_k(\varphi) = \beta_z[\varphi] \\ \partial_t \delta_k(\varphi_1, \varphi_2) = \beta_\delta[\varphi_1, \varphi_2] \end{array}\right.$$

- Choose the initial conditions (dichotomy to find the critical manifold), the dimension **d,** and **SOLVE**!

Illustration: Results for d=4 and d=5.5

NP-FRG flow of the renormalized mass $u''_k(\varphi)$ in d=4 [left] and d=5.5 [right]

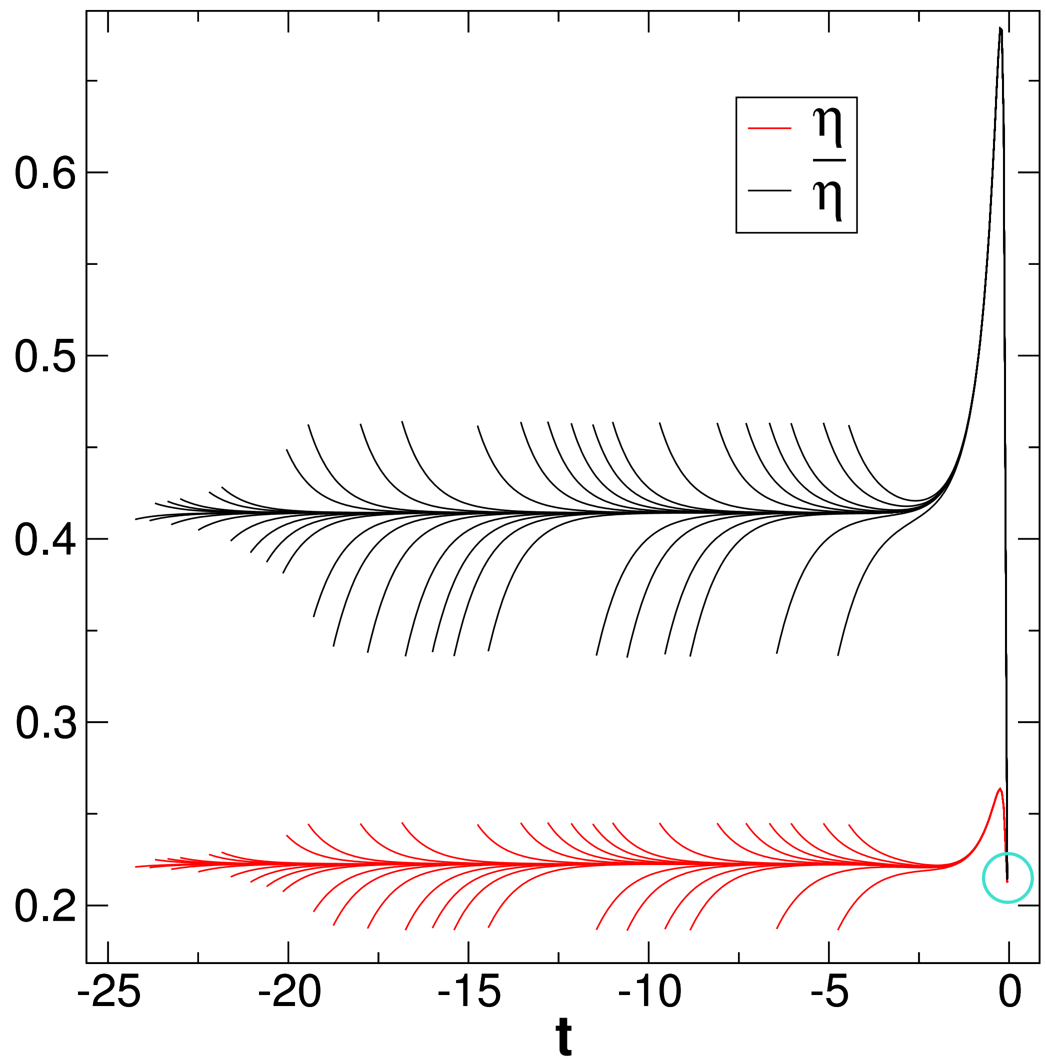


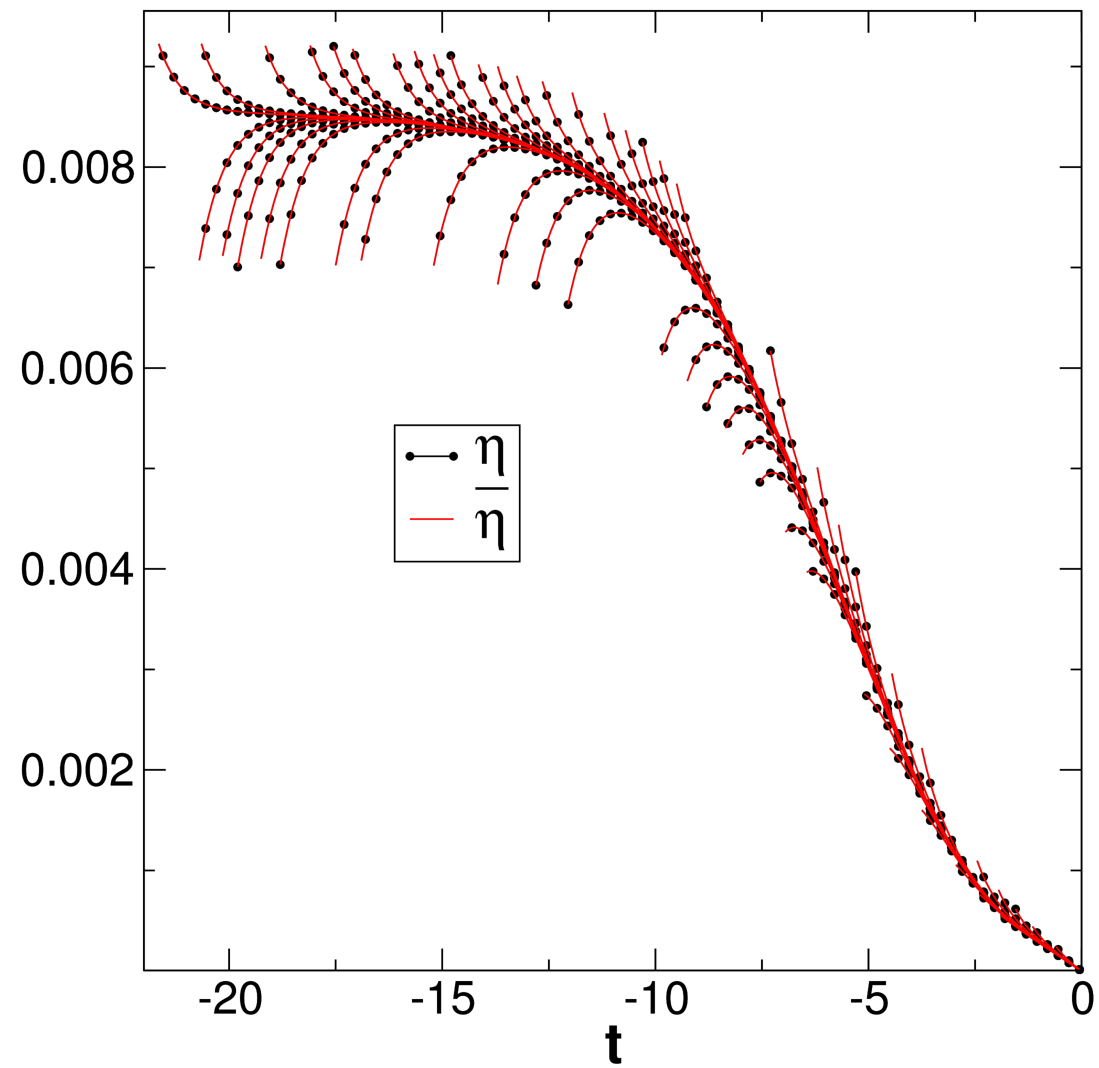
The initial condition is asymmetric (red) but it flows to a **symmetric fixed point** (black) that is **exactly the same one** obtained for the flow in the equilibrium case.

The same is found for the other functions $z_k(\varphi)$ and $\delta_k(\varphi_1,\varphi_2)$.

Illustration: Results for d=4 and d=5.5

Flow of the 2 "anomalous dimensions" with broken (d=4, left) or unbroken (d=5.5, right) dimensional reduction





Also: Good agreement with simulation results in d=3,4

Conclusion

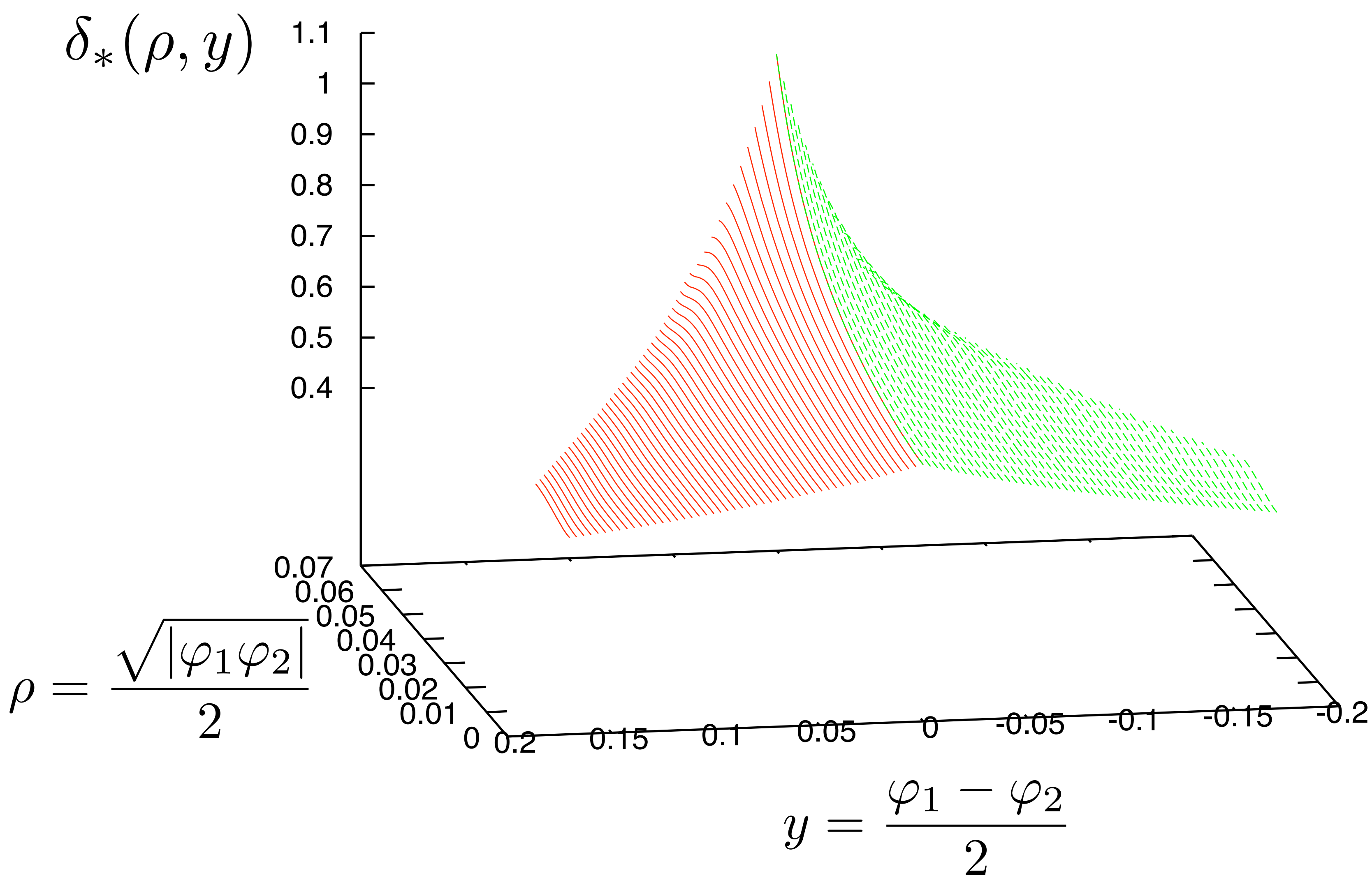
- Avalanches are crucial to explain the breakdown of dimensional reduction for the RFIM at criticality.
- For a theoretical description of scale-free avalanches, one needs a functional RG. In the case of the RFIM, the FRG must also be nonperturbative.
- The equilibrium and out-of-equilibrium (hysteresis) critical points of the RFIM are in the same universality class.

References: I. Balog, M. Tissier, G.T., PRB **89**, 104201(2014);
G.T., M. Baczyk, M. Tissier, PRL **110**, 135703 (2013);
M. Tissier, G.T., PRL **107**, 041601 (2011); PRB **85**, 104202 and 104203 (2012).

Results

Above $d_{DR} \simeq 5.15$: no cusp in $\delta_k(\varphi_1, \varphi_2)$ & dim.reduction
Below d_{DR}: cusp in $\delta_k(\varphi_1, \varphi_2)$ & breakdown of dim. reduction

Dimensionless cumulant of disorder at fixed point in d=3



Parisi-Sourlas supersymmetric approach

- At T=0, generating functional of the correlation functions:

$$\mathcal{Z}_h[J,\hat{J}] = \int \mathcal{D}\phi\, \delta\Big[\frac{\delta S_B[\phi]}{\delta\phi} - h - J\Big] \left|\frac{\delta^2 S_B[\phi]}{\delta\phi\delta\phi}\right| e^{\int_x \hat{J}(x)\phi(x)}$$

If unique solution of SFE, usual manipulations:

Introduce auxiliary fields $\hat{\phi}(x)$, $\psi(x)$, $\overline{\psi}(x)$, then average over disorder;
Introduce a superspace with 2 Grassmann coordinates $\underline{x} = (x, \overline{\theta}, \theta)$,
a superLaplacian $\Delta_{SS} = \partial_\mu^2 + \Delta_B \partial_\theta \partial_{\overline{\theta}}$,
a superfield $\Phi(\underline{x}) = \phi(x) + \overline{\theta}\psi(x) + \overline{\psi}(x)\theta + \overline{\theta}\theta\hat{\phi}(x)$, super-etc...

- Generating functional obtained from a superfield theory

$$S_{SUSY}[\Phi] = \int_{\underline{x}} \Big\{ -\frac{1}{2}\Phi(\underline{x})\Delta_{SS}\Phi(\underline{x}) + \frac{\tau}{2}\Phi(\underline{x})^2 + \frac{u}{4!}\Phi(\underline{x})^4 \Big\}$$

- Invariant under SUSY (super-rotations in superspace)
=> leads to "dimensional reduction": RFIM in *d* dim. is equivalent to pure theory in *d-2*. Beautiful, but wrong!!