Avalanches in Strained Amorphous Solids: Does Inertia Destroy Critical Behavior?

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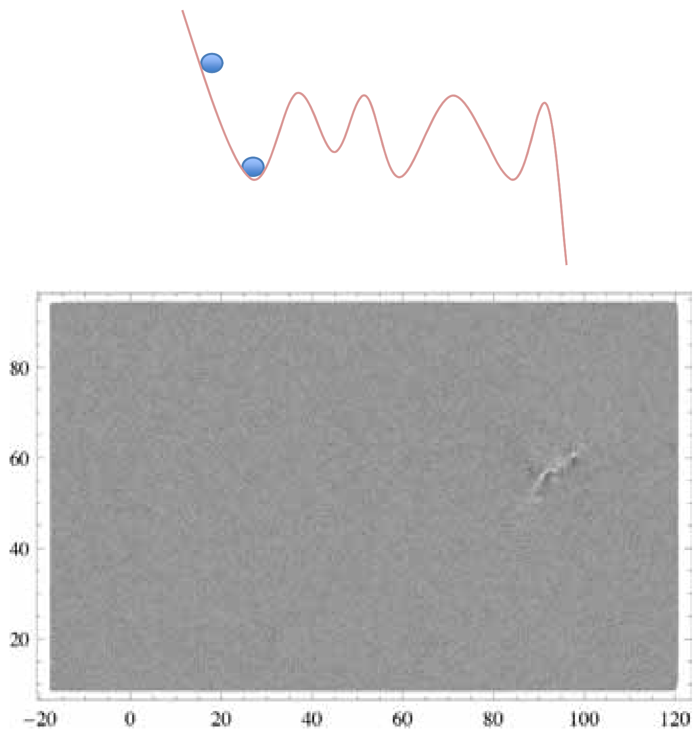
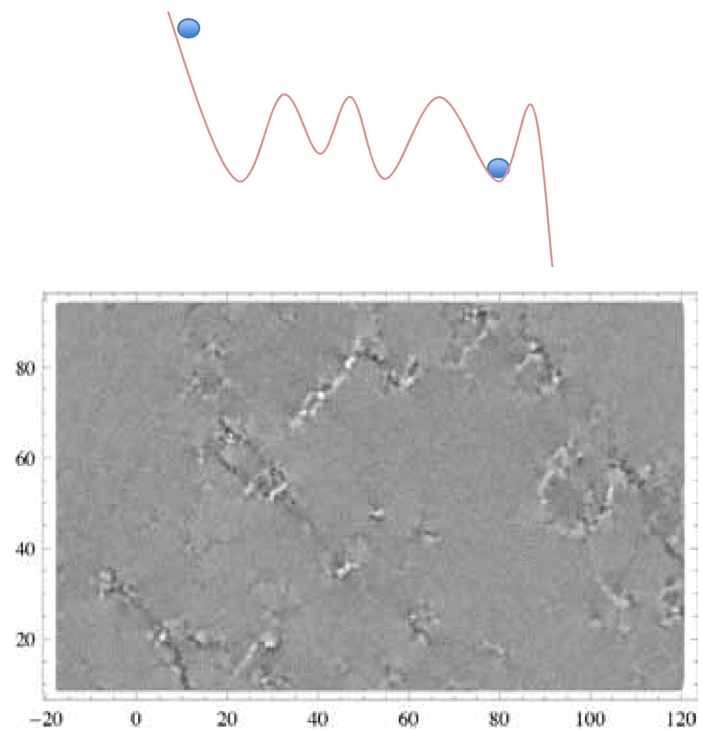




Supported by the National Science Foundation

Motivation

Find power law distribution of events, avalanches, earthquakes in a wide variety of systems and on wide range of scales as long as they are driven slowly ⇒ quasistatic

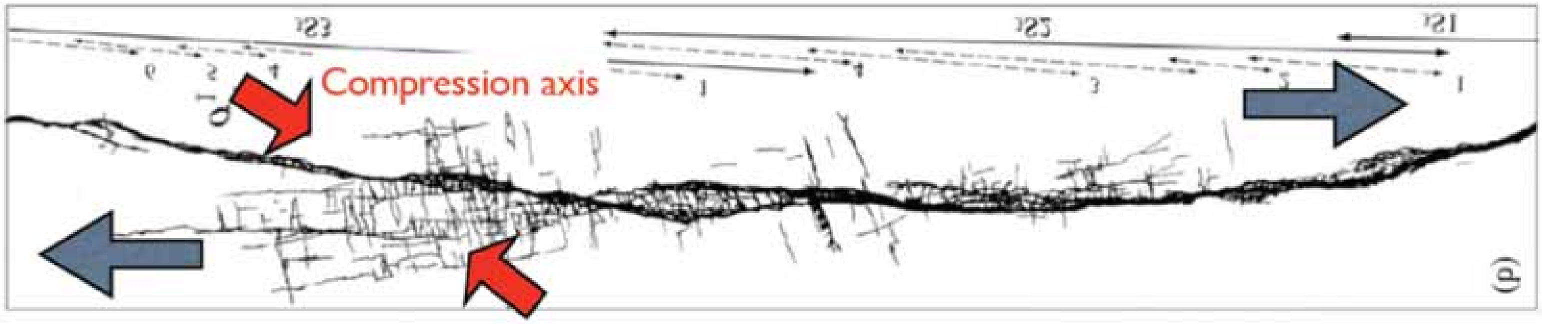
- Charge-density waves, fluid invasion, contact line motion, flux lattices, magnetic domains (Barkhausen noise), ...
- Deformation of solids, granular media, colloids, foams, ... (acoustic emission, dislocation bursts, cracks, earthquakes)

Above models usually studied with T=0, overdamped dynamics
⇒ Does T or inertia drive system away from criticality?

For large objects or events temperature may be irrelevant

But solids are generally underdamped at large scales

⇒Examine quasistatic, T=0 dynamics as vary damping/inertia



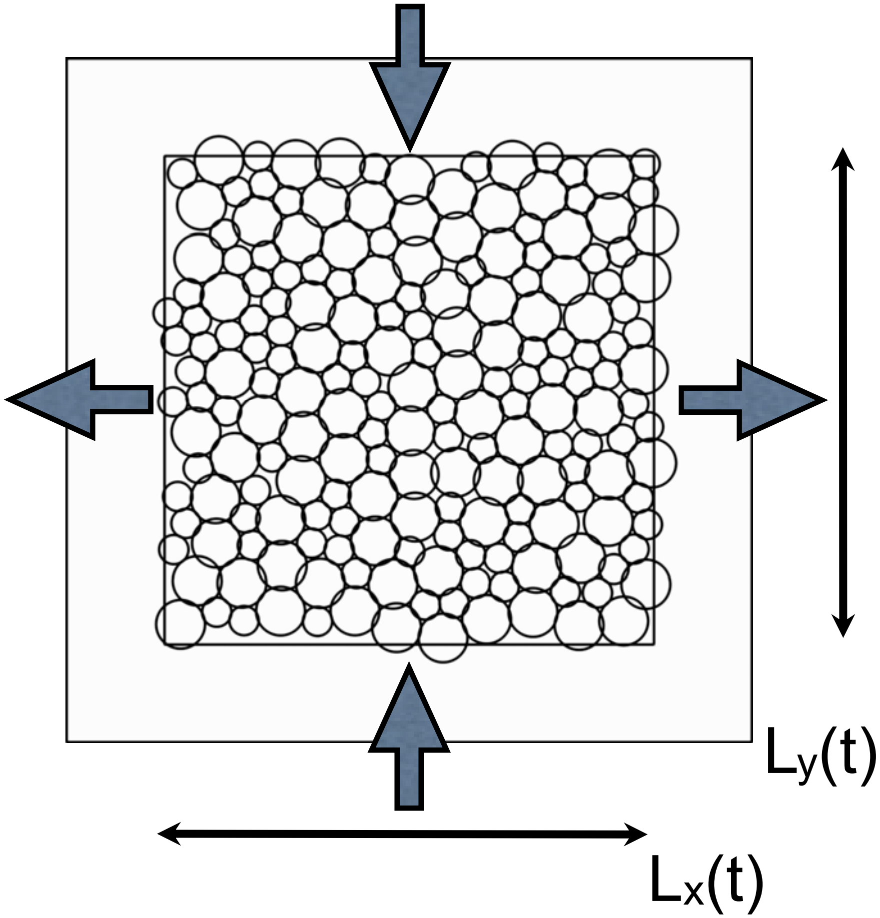
Quasistatic Athermal Simulations

Molecular Dynamics for 2D or 3D systems

- Binary Lennard-Jones to prevent crystallization
  Units - mean diameter a, binding energy u, time $t_{LJ}=a(m/u)^{1/2}$

$$V(r) = 4u\left[\left(\frac{a_{ij}}{r}\right)^{12} - \left(\frac{a_{ij}}{r}\right)^{6}\right]$$

- Quench at pressure p=0 to T=0
  - protocol not important
- Periodic boundaries
- Pure shear - fix area or vol.
  (also studied simple shear)
- Quasi-static limit ⇒ low rate
  ⇒ depends on $\Delta\gamma$ not Δt
  Different than saying motion of
  all atoms is always slow
- Either fix low rate or stop for event



Fix area $L_y(t)L_x(t)$

Damping

Dimensionless damping rate Γ, characteristic LJ time t_{LJ}

Kelvin damping or Diffusive Particle Dynamics (DPD)

- Drag force on particle i proportional to velocity differences from neighbors $-\Gamma\,(\mathbf{v}_i\text{-}\mathbf{v}_j)\ f(\mathbf{r}_i\text{-}\mathbf{r}_j)\ m/t_{LJ}$
- $\Rightarrow$ Gallilean invariant
- Lifetime of wavevector q ~ $t_{LJ}/\Gamma q^2$

Langevin (viscous) damping:

- Drag force $-\Gamma \mathbf{v}_i\, m/t_{LJ} \rightarrow$ not Gallilean invariant
- T=0 $\rightarrow$ Langevin thermostat with no noise
- Lifetime ~ t_{LJ} / Γ

Find same behavior for all types of thermostat and potential. Will mainly show Langevin

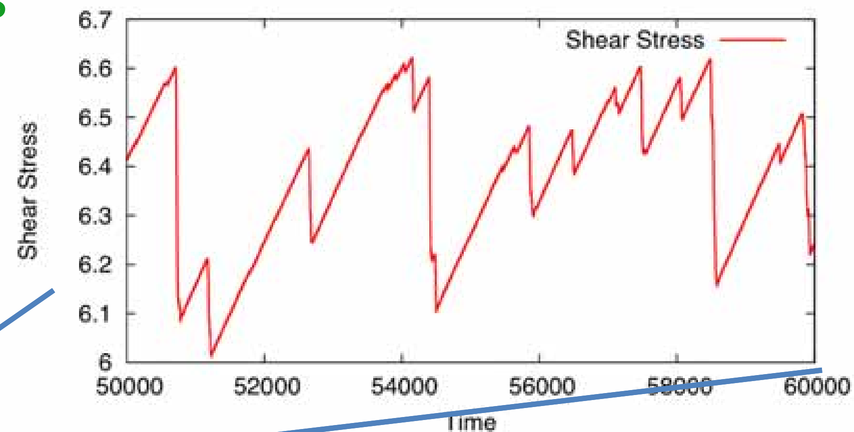
Quasistatic, Steady-State Shear

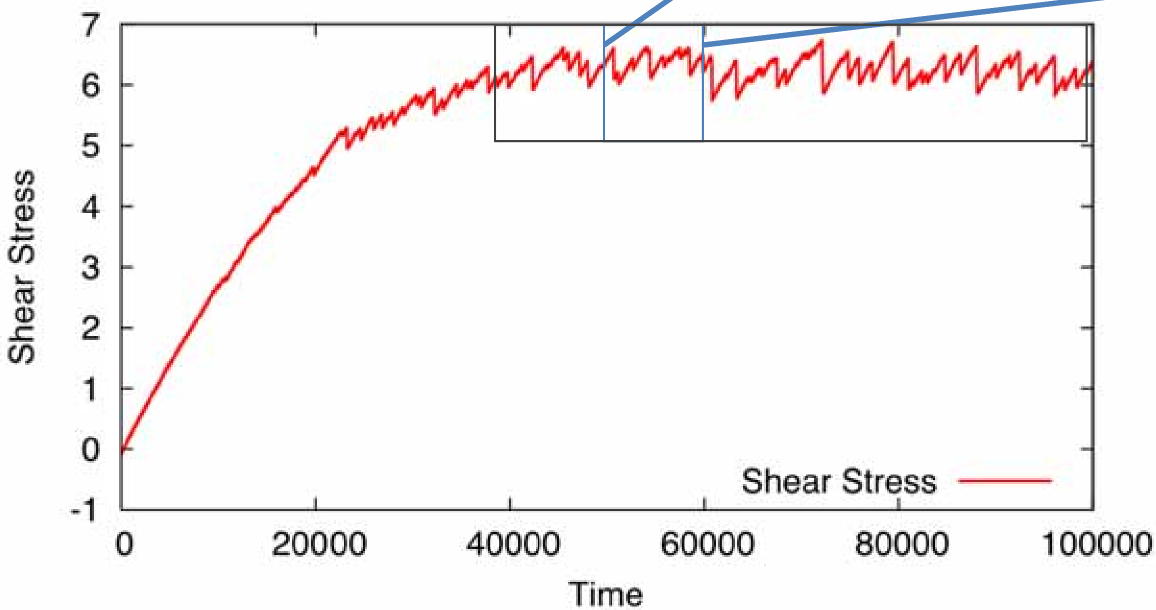
System deforms elastically, stress and energy build

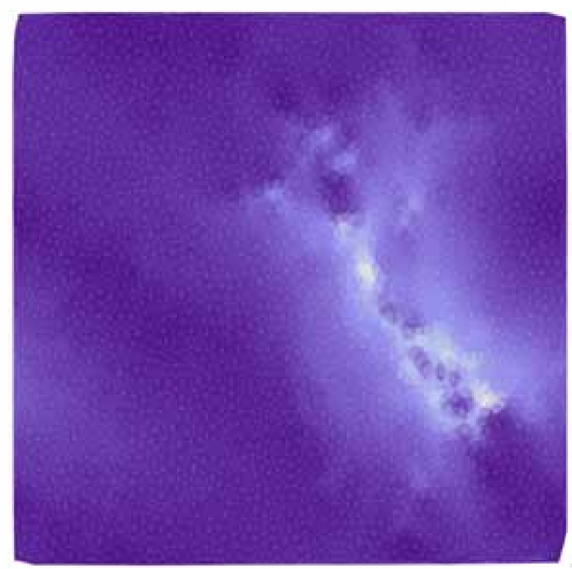
→ Local mechanical instability → energy and stress drop

Quasistatic – Independent events (i.e. not causally related) are separated in time $<10^{-6}\, t_{LJ}^{-1}$

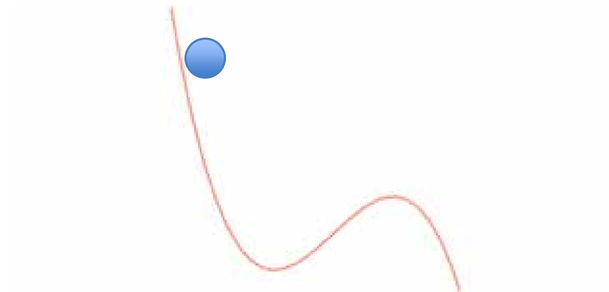
- Number independent of rate
- Statistics independent of initial configuration





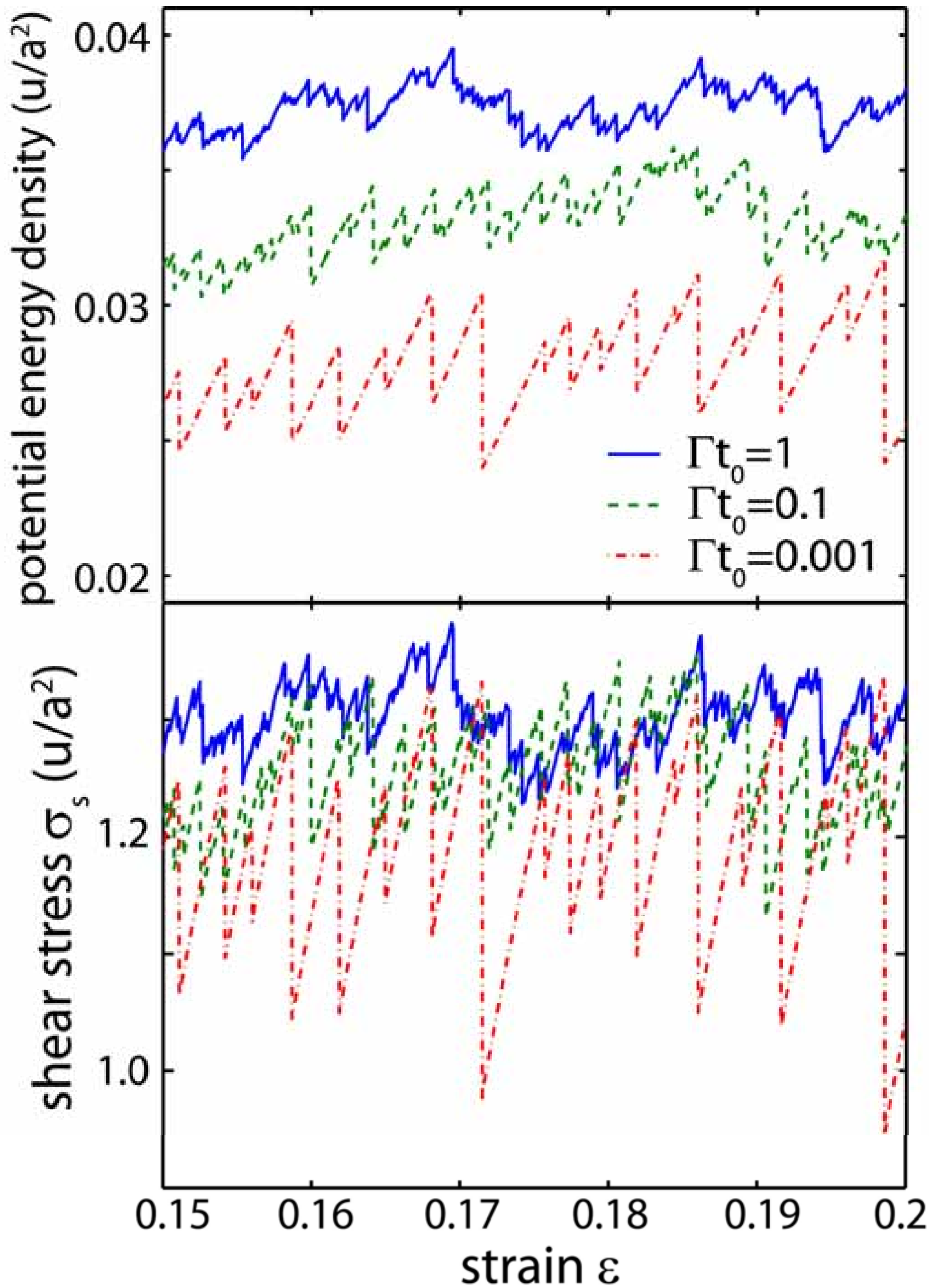


Inertia Matters

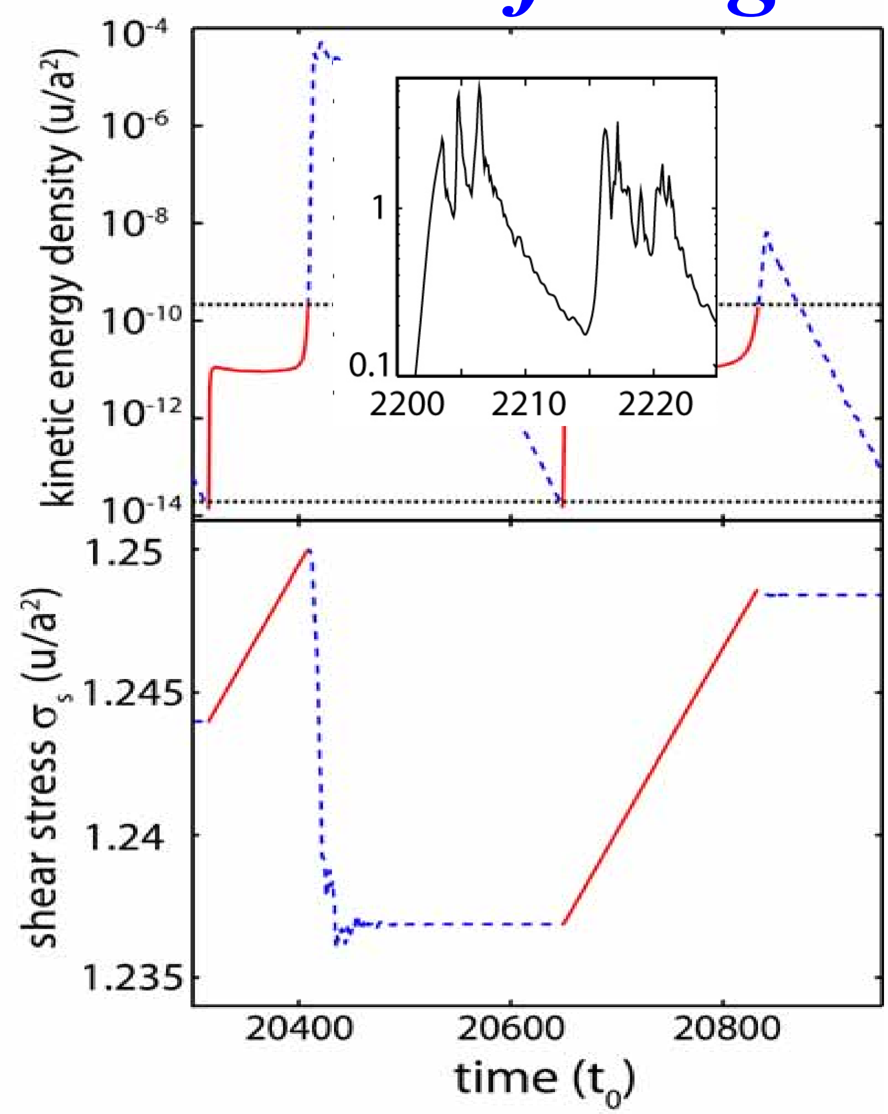


Sample totally different energy ranges
Damped – nearest min.
Inertia – lower mimima

Change distribution of drops in energy & stress
Damped – more, smaller
Inertia – larger events



Defining Avalanches



QUASISTATIC LIMIT:

- Shear at low rate ~10^{-7} (red) until detect kinetic energy rise
- Stop shearing (blue) and allow event to evolve until kinetic energy 10^{-4} of trigger

FIND:

- Drop E in potential energy
- Drop in shear stress $\Delta\sigma_s$

Define extensive stress drop:

$\boldsymbol{S} = (\Delta\sigma_s)L^d\langle\sigma_s\rangle/4\mu$

d=dimension, μ=shear mod.

Sum rule $S \propto E$ for large events

$$\int dE\,E\,R(E,L) = \int dS\,S\,R(S,L)$$

R(X,L)=Number of events of size X=E or S in system of size L per unit strain and X

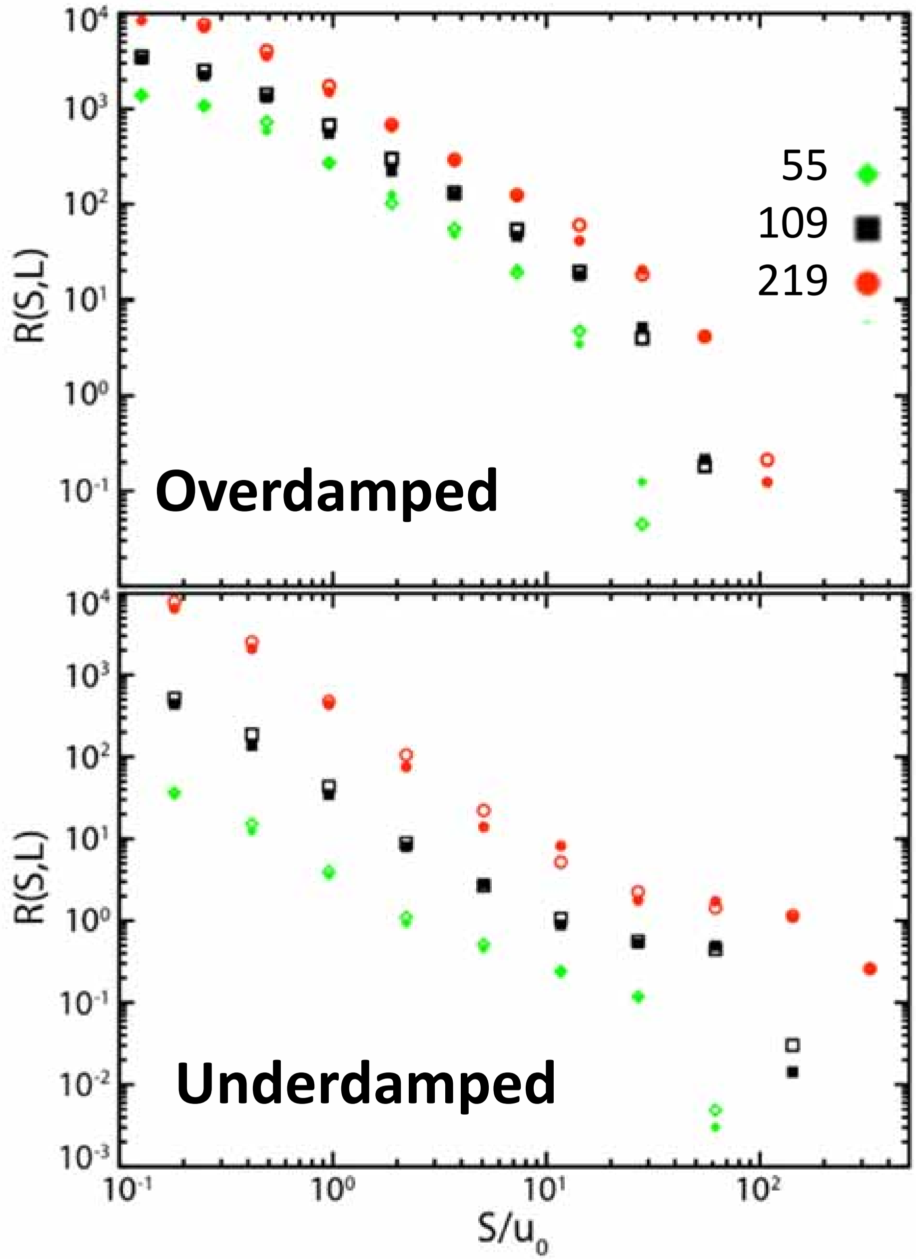
Results Independent of Thermostat

Langevin (open)
Galilean invariant "Kelvin" damping (closed)
$F_{i\alpha}=-\Gamma' m \Sigma_j (v_{i\alpha}-v_{j\alpha}) f(|r_{i\beta}-r_{j\beta}|)$

Scaling exponents and functions are independent of damping mechanism in overdamped and inertial (underdamped) limits

Also find same scaling for simple and pure shear

Number of small events does not scale with system size



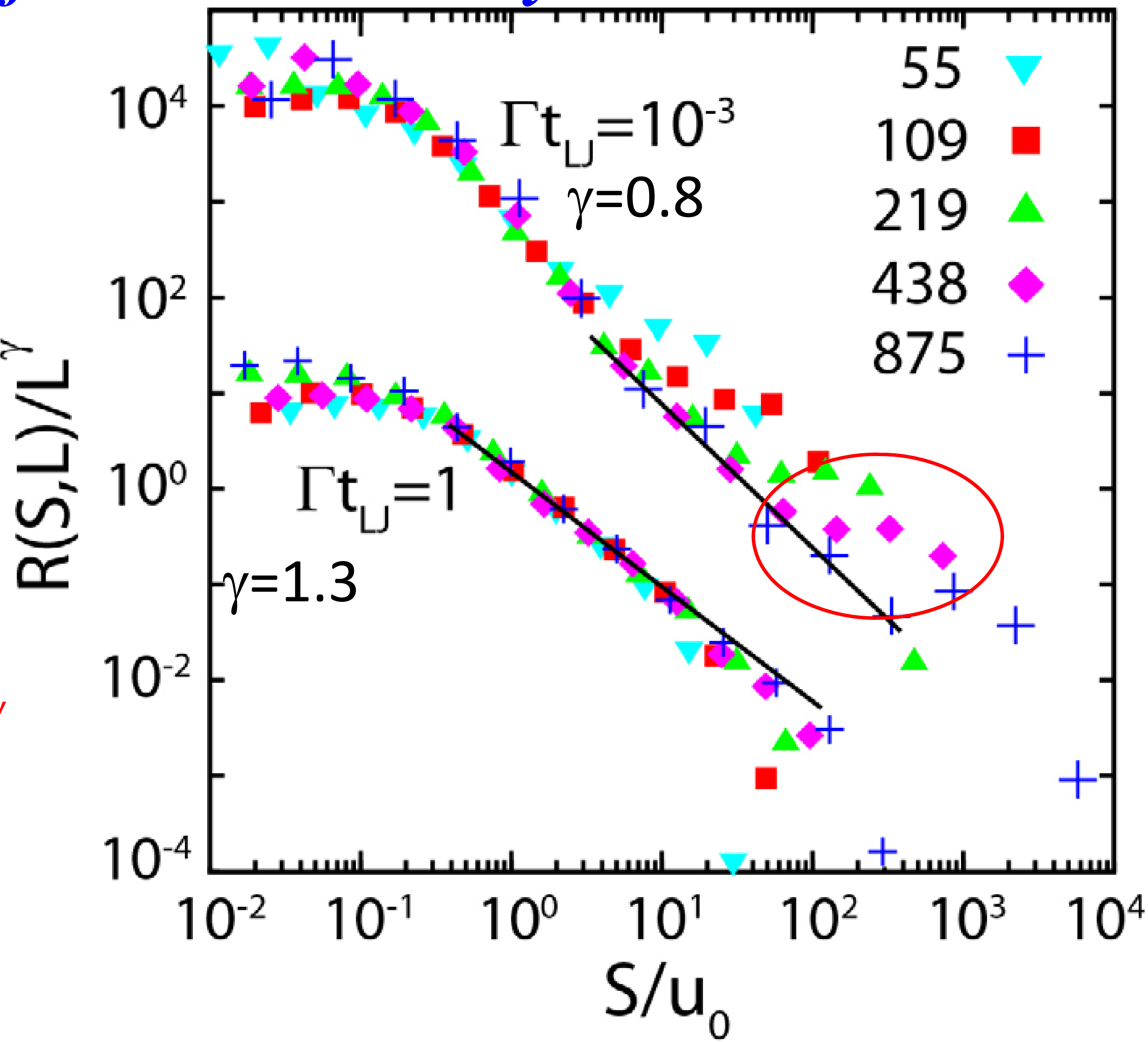
Overdamped and Underdamped Both Critical, Different Universality Classes

Range where $R \propto S^{-\tau}$

Largest event grows with increasing system size

Num. small events is not extensive: $\propto L^{\gamma}$ with $\gamma < d$

Inertia - bump at large events



Finite-Size Scaling Relations

Define R(E,L)=# of events of energy E in system of edge L per unit energy per unit strain

Expect: $R(E,L) = L^{\beta} g(E/L^{\alpha})$ – g universal scaling function

Find yield stress independent of L $\rightarrow$ dissipation $\sim L^d$

$$\int dE\,E\,R(E,L) = L^{\beta+2\alpha} \int dx\,x\,g(x) \sim L^d \quad \rightarrow \beta+2\alpha=d=2$$

For small $x=E/L^{\alpha}$, $g(x) \sim x^{-\tau} \rightarrow R(E,L) \sim L^{\beta+\alpha\tau} E^{-\tau} \longrightarrow \gamma=\beta+\alpha\tau$

Usually $R(E,L)\sim L^d$, i.e. is extensive$\rightarrow \beta+\alpha\tau=d \rightarrow \tau=2$ $(\alpha\neq 0)$

Find small events are strongly suppressed by large events and $\gamma\equiv\beta+\alpha\tau \sim 1.3 << 2$ in 2d ($\gamma\sim 2<<3$ in 3d)

Determine τ, α, β, γ and test scaling relations

$\gamma=\beta+\alpha\tau$, $\beta+2\alpha=d$

Finite-Size Scaling

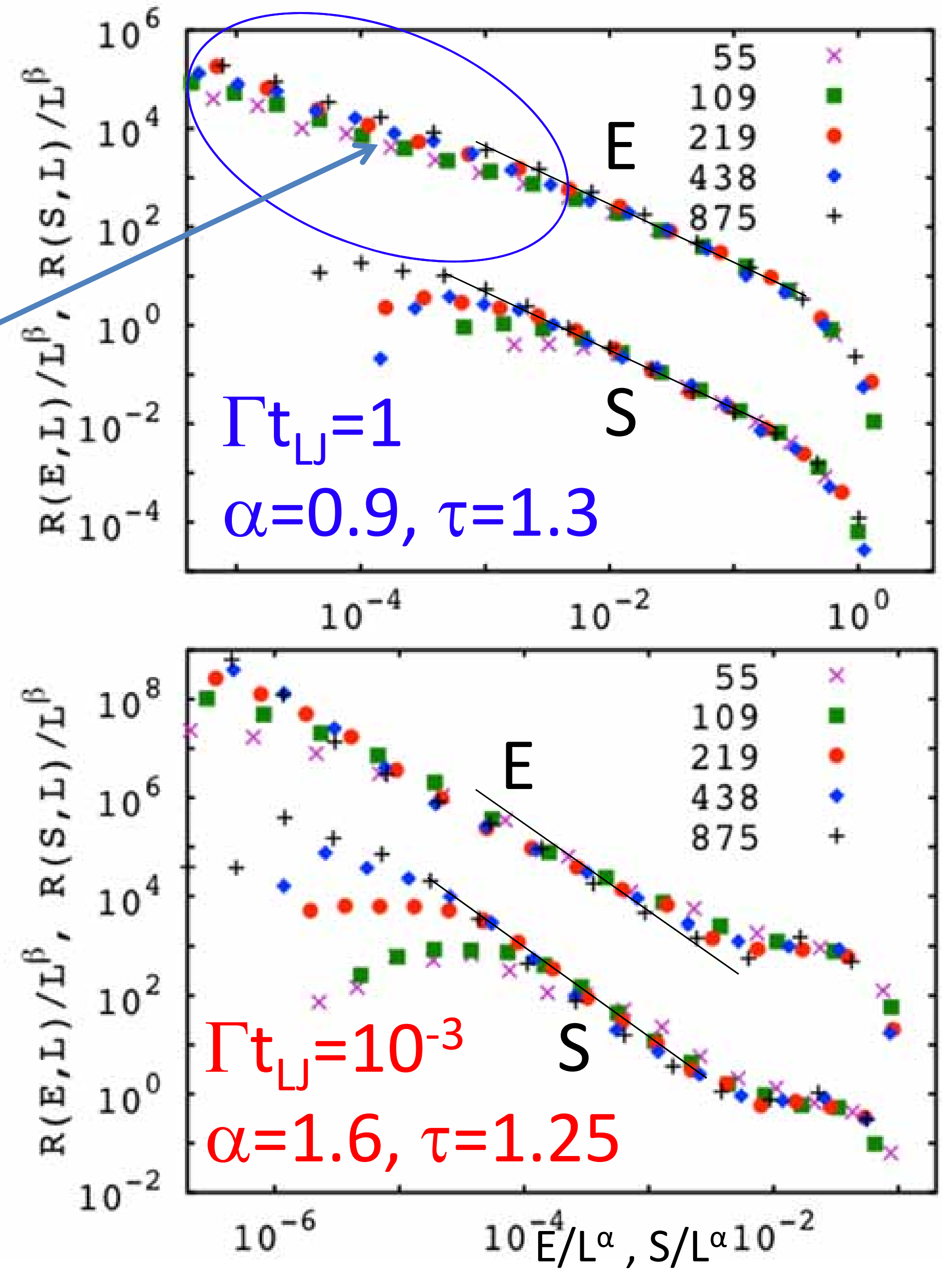
$\beta = d - 2\alpha$; $\gamma = \beta + \alpha\tau$

Overdamped

- E,S scale over similar range
- Large avalanches grow more slowly than system size~ $L^{0.9}$
- Noncritical power law in E dominated past studies

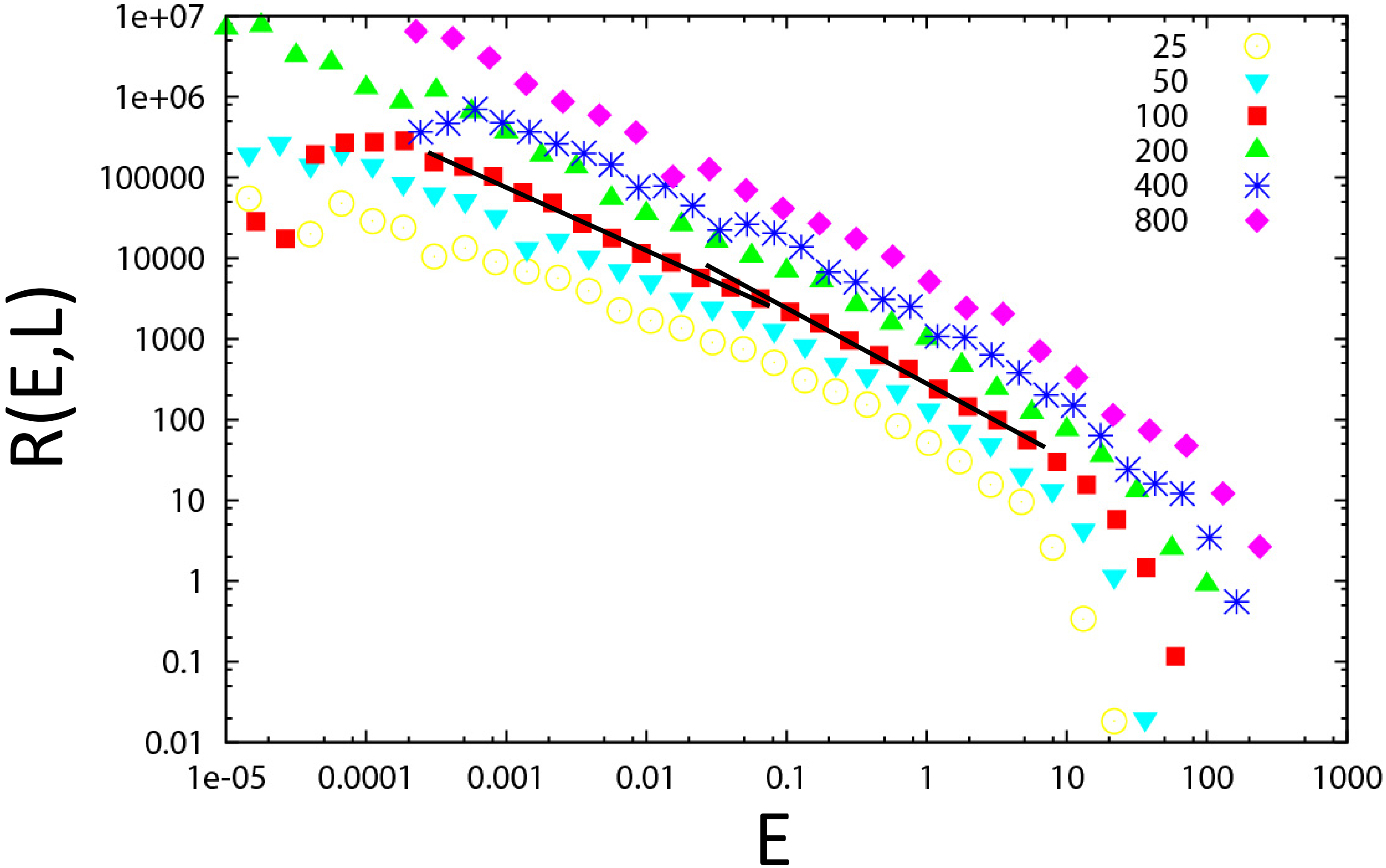
Underdamped

- Bump at large event size follows scaling law
- Large avalanches grow faster than system size ~ $L^{1.6}$



Overdamped Limit – Power Law Scaling

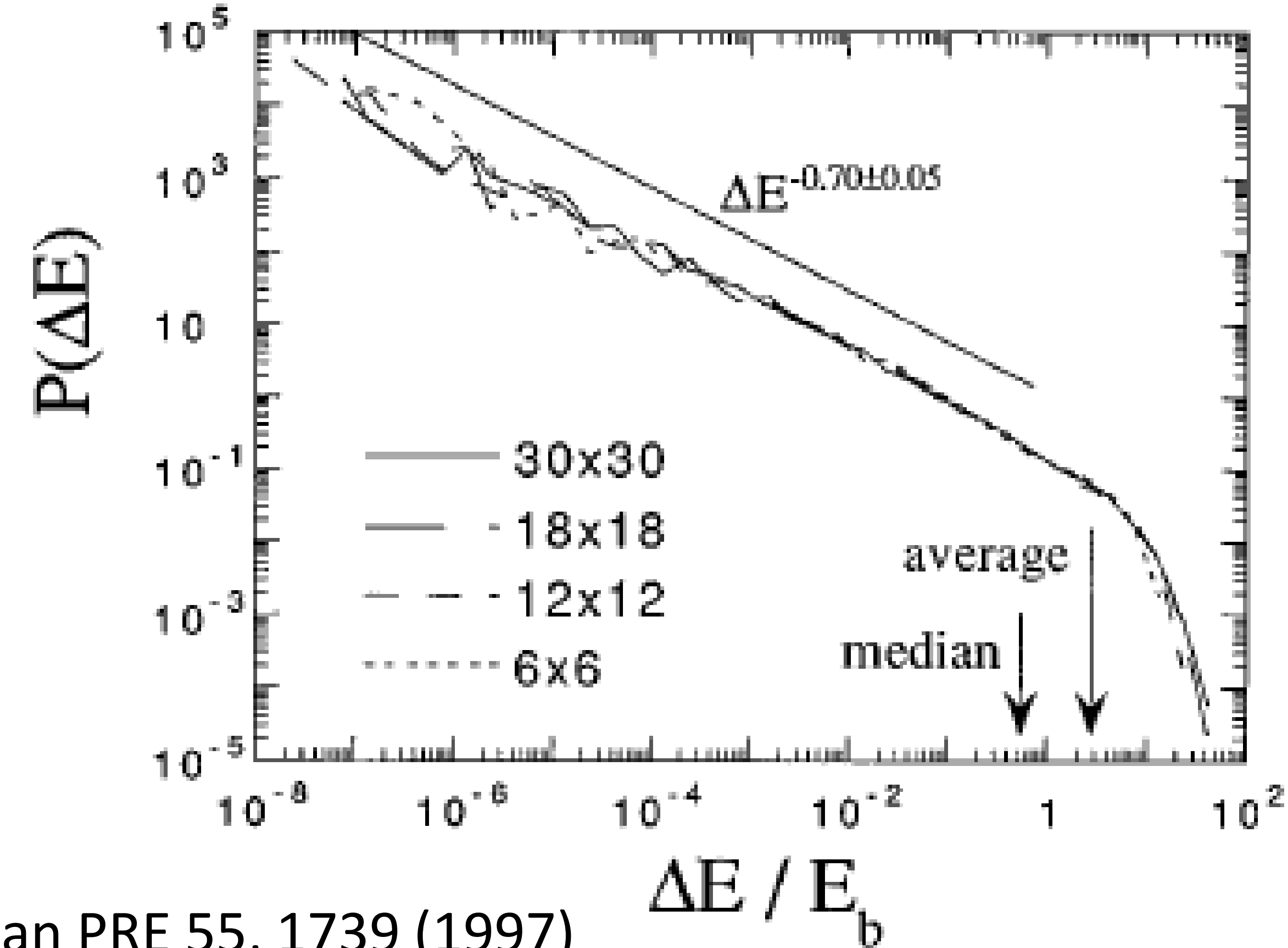
Apparently have >6 decades in E, cutoff increasing with L
BUT slope change at fixed E~0.1, not critical at low E



Foam Model – Power law, not critical

Find maximum E independent of L – small systems

$N(E) \sim (E)^{-\tau}$ with τ=0.7 ~ same as our noncritical regime



Doug Durian PRE 55, 1739 (1997)

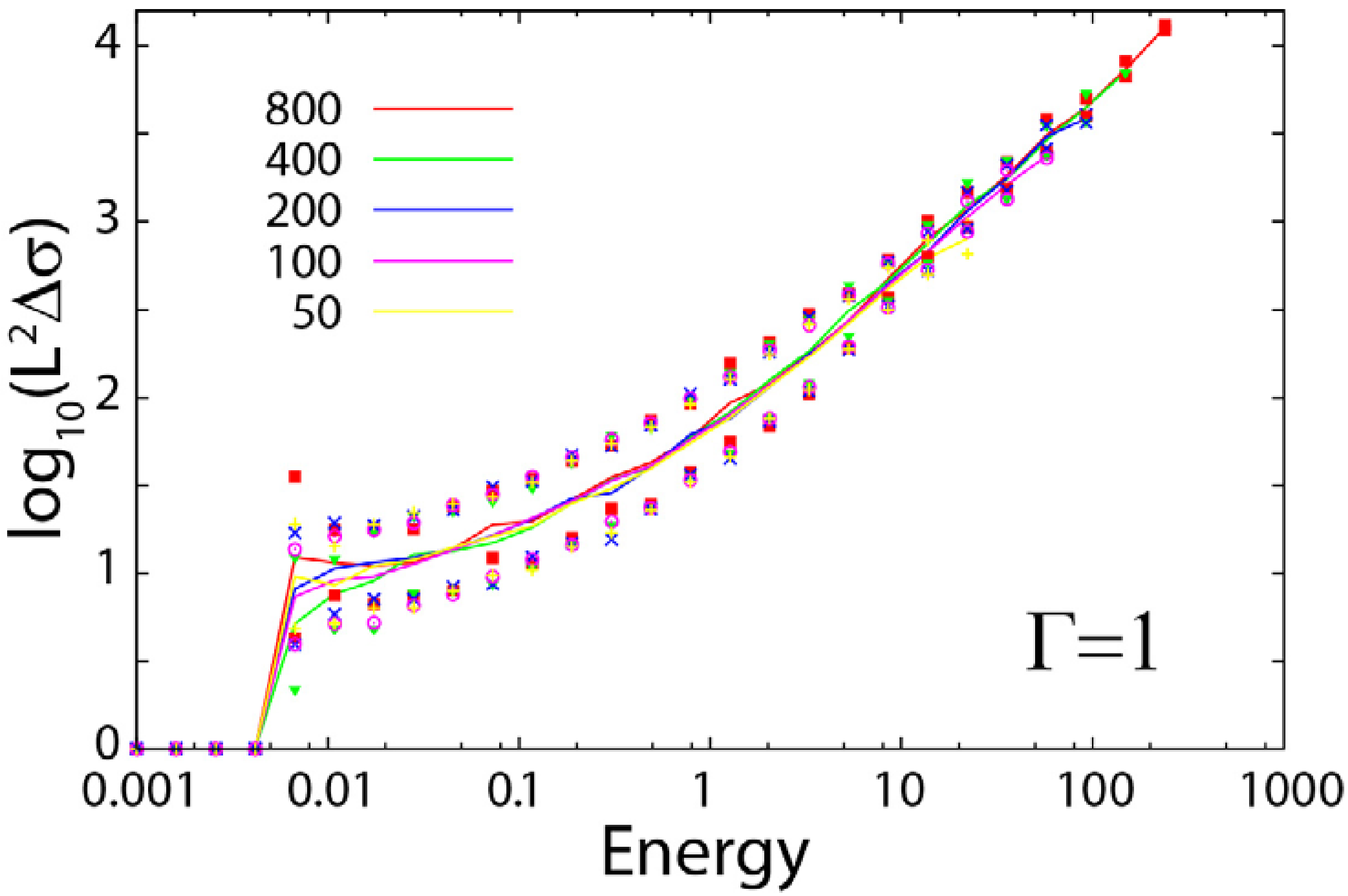
Relation Between Energy and Stress Drop Δσ

Large events $E \propto \Delta\sigma$ lose correlation for E<0.1, $L^2\Delta\sigma$<10

for E <0.1 some events have $\Delta\sigma < 0$

Sum rule – integral over $L^2\Delta\sigma$ and E are same

IF $<\sigma>$ indep of L, $\Delta\sigma < <\sigma>$ and mean modulus indep of L



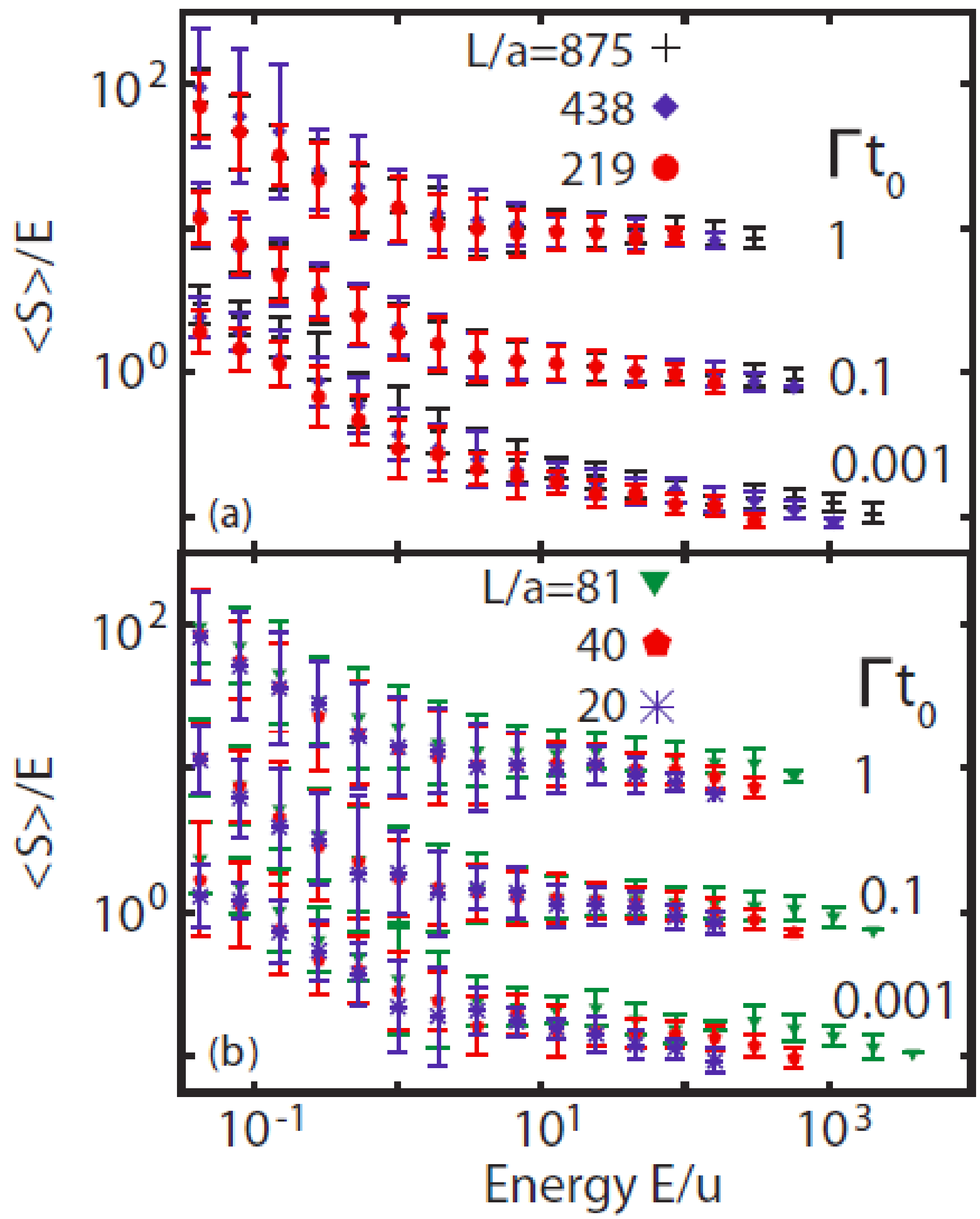
Relation Between Energy and Stress Drop Δσ

Ratio <S>/E constant for E/u>0.1 for overdamped

Continues to decrease to largest events for underdamped.

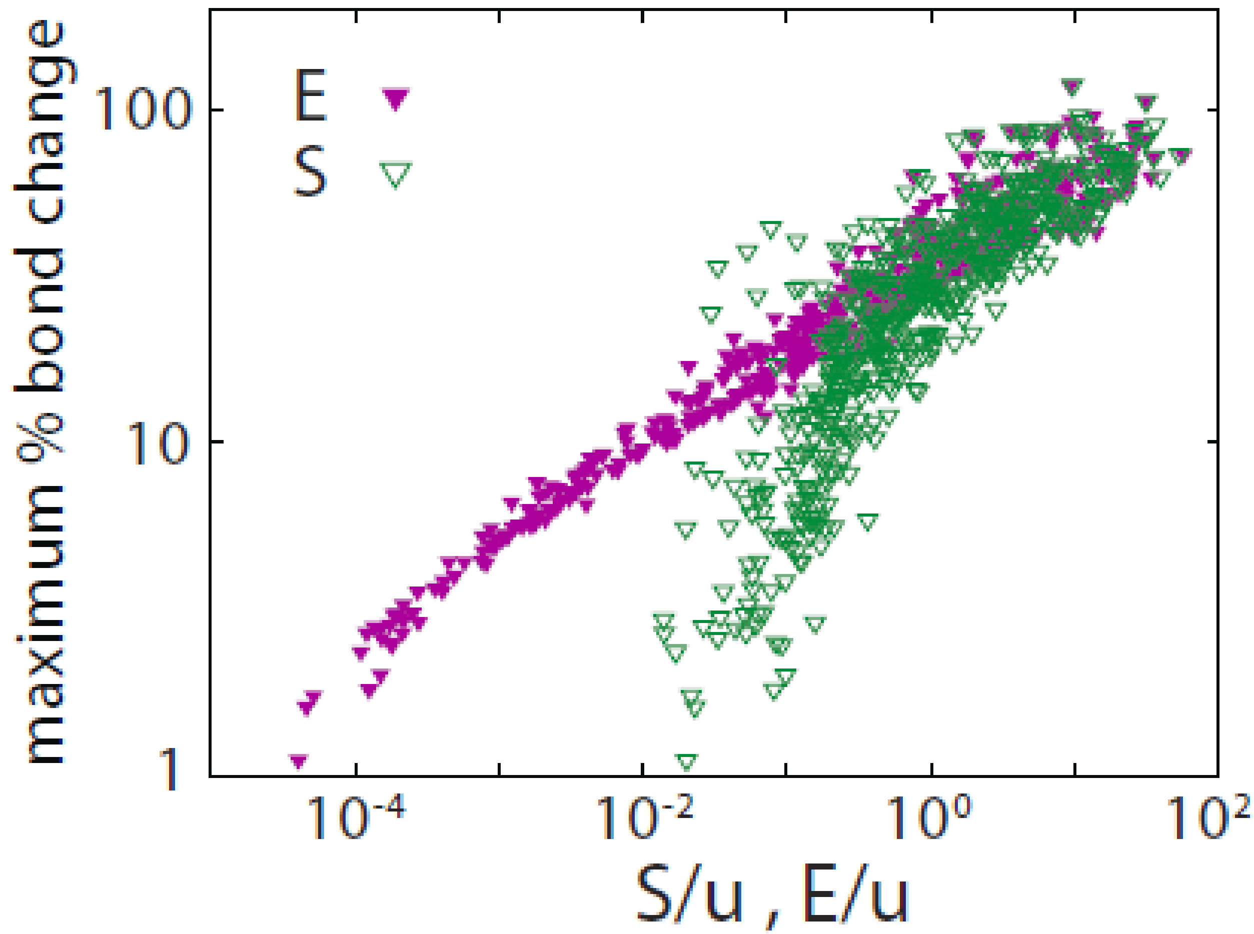
Break connection between E and S because inertia carries system over minima that are not on path that reduces shear stress

Find slightly better scaling for E in underdamped case and slightly different exponents.



Events at Small E are Different

Measure maximum % change in bond length during avalanche
Power law relation of energy and maximum change
Relation independent of Γ
For E<0.01, maximum change < elastic limit ~ 10%
See events for E~10^{-5}



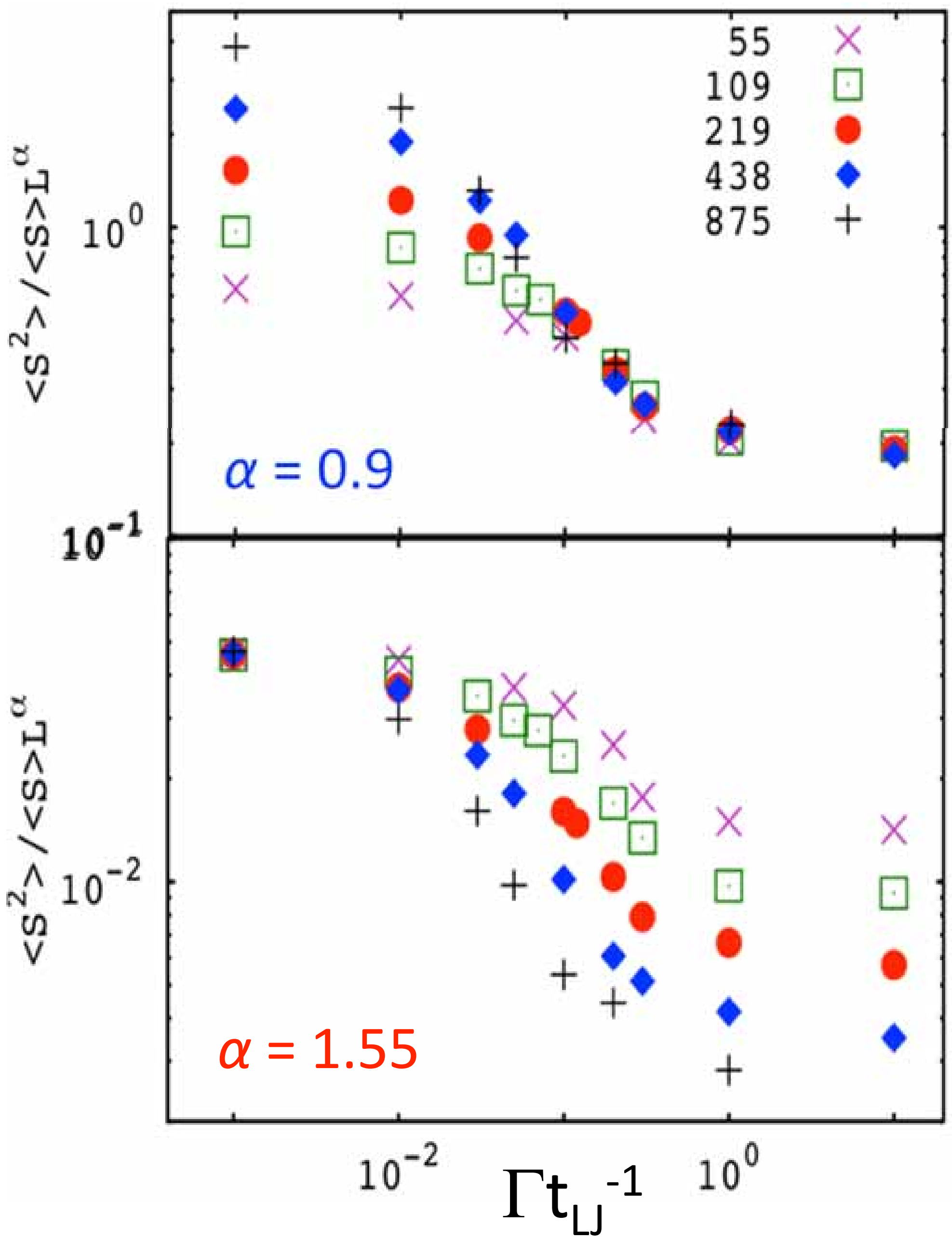
Transition Between Overdamped and Underdamped

If maximum energy $\sim L^{\alpha}$ then $\langle E^2\rangle / L^{\alpha}\langle E\rangle = \text{const}$

For $\Gamma > \Gamma_c = 0.1 t_{LJ}^{-1}$ all data scale with $\alpha = 0.9$ curves separate for lower damping

Small Γ, curves collapse for $\alpha = 1.55$

Multicritical point at $\Gamma_c = 0.1 t_{LJ}$

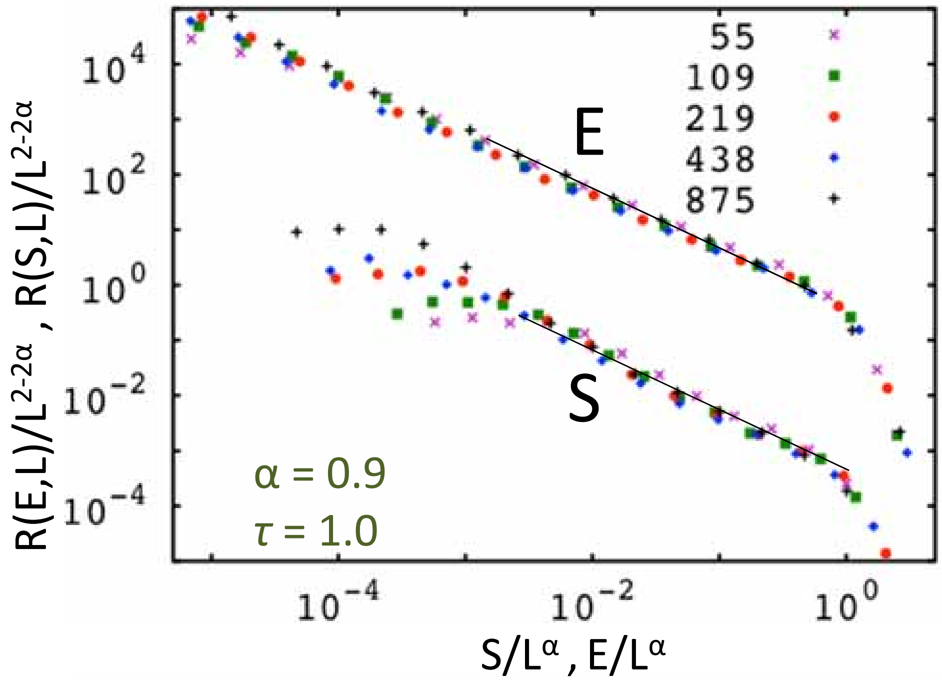


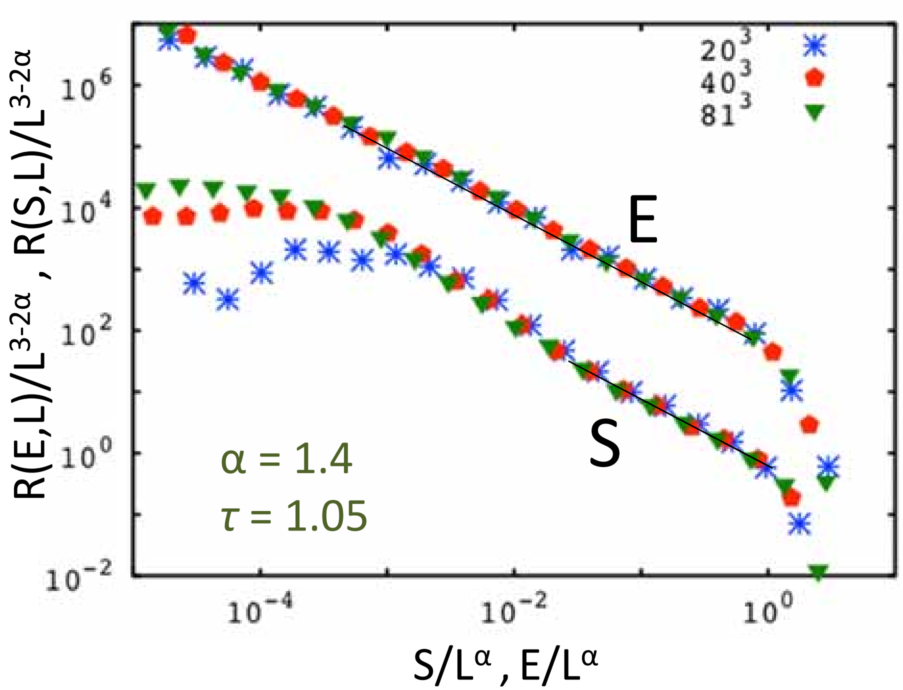
Scaling at Critical Damping

Near Γ=0.1 for 2D and 3D

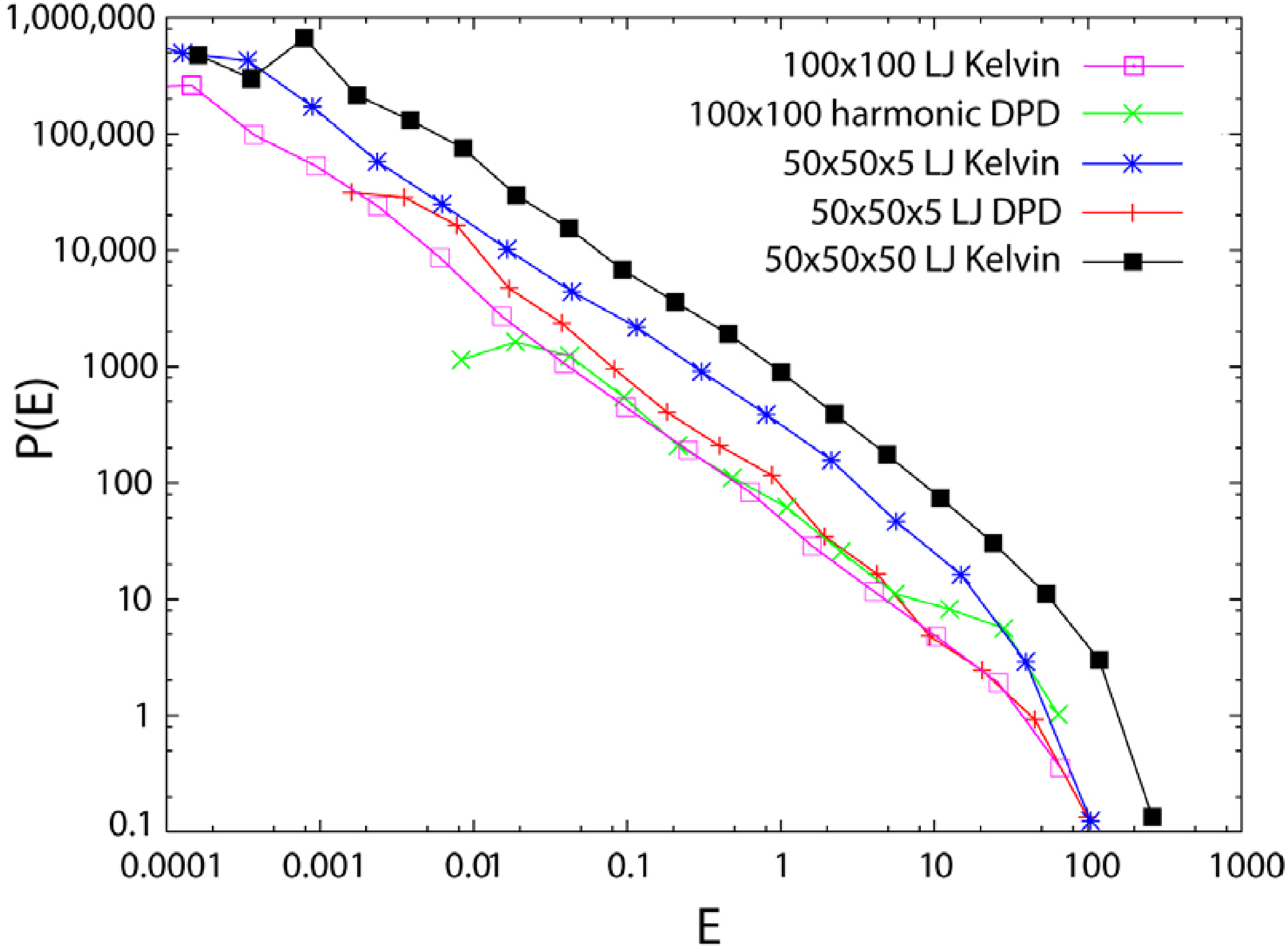
- Consistent τ for 2D and 3D
  - 2D exponent $\tau \sim 1.0$
  - 3D exponent $\tau \sim 1.05$

- Unstable multi-critical point?
  - Systems near $\Gamma = 0.1$ seem to flow away from critical point as system size grows

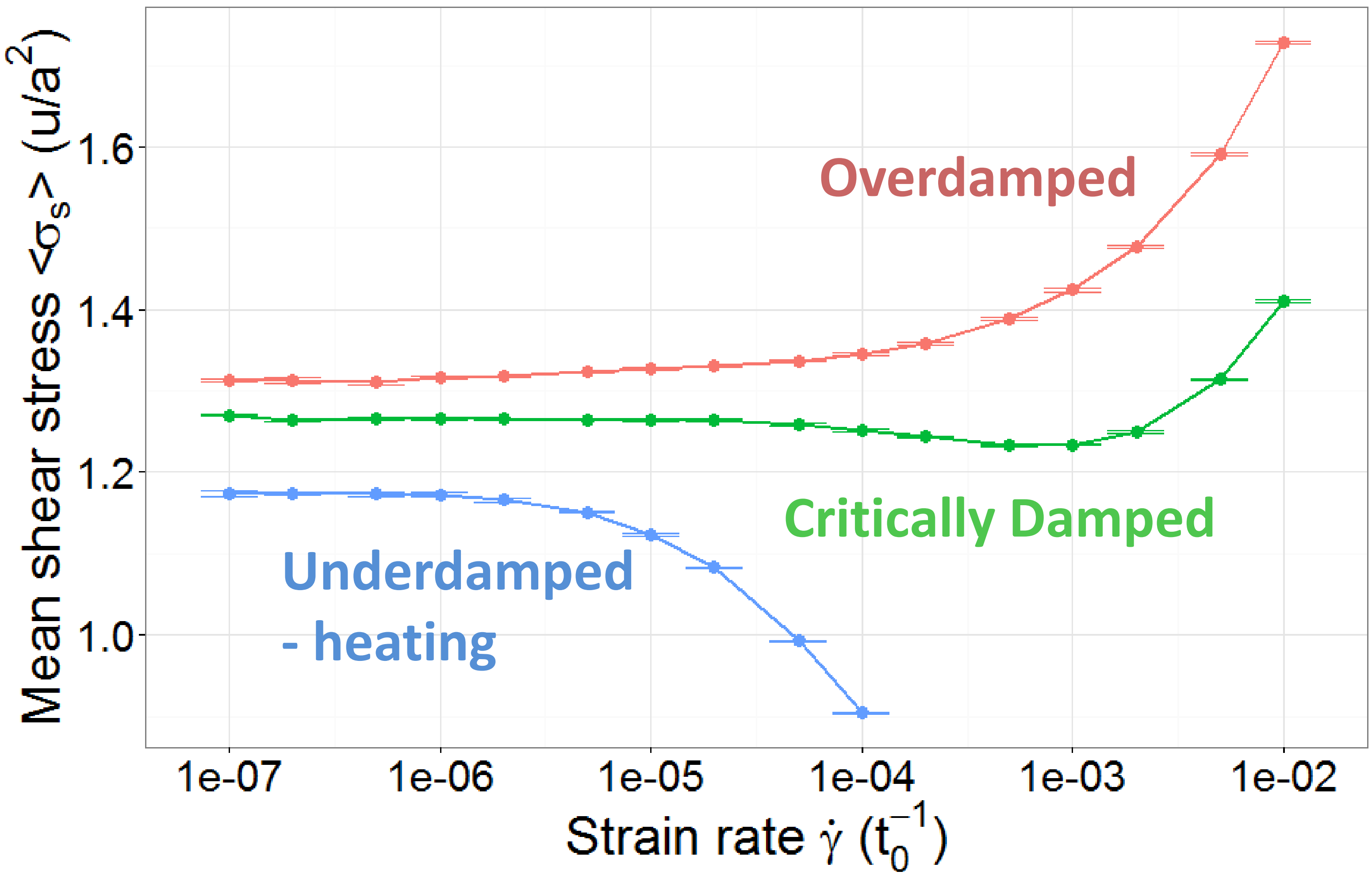




Scaling at Critical Damping is Independent of Potential, Geometry and Thermostat



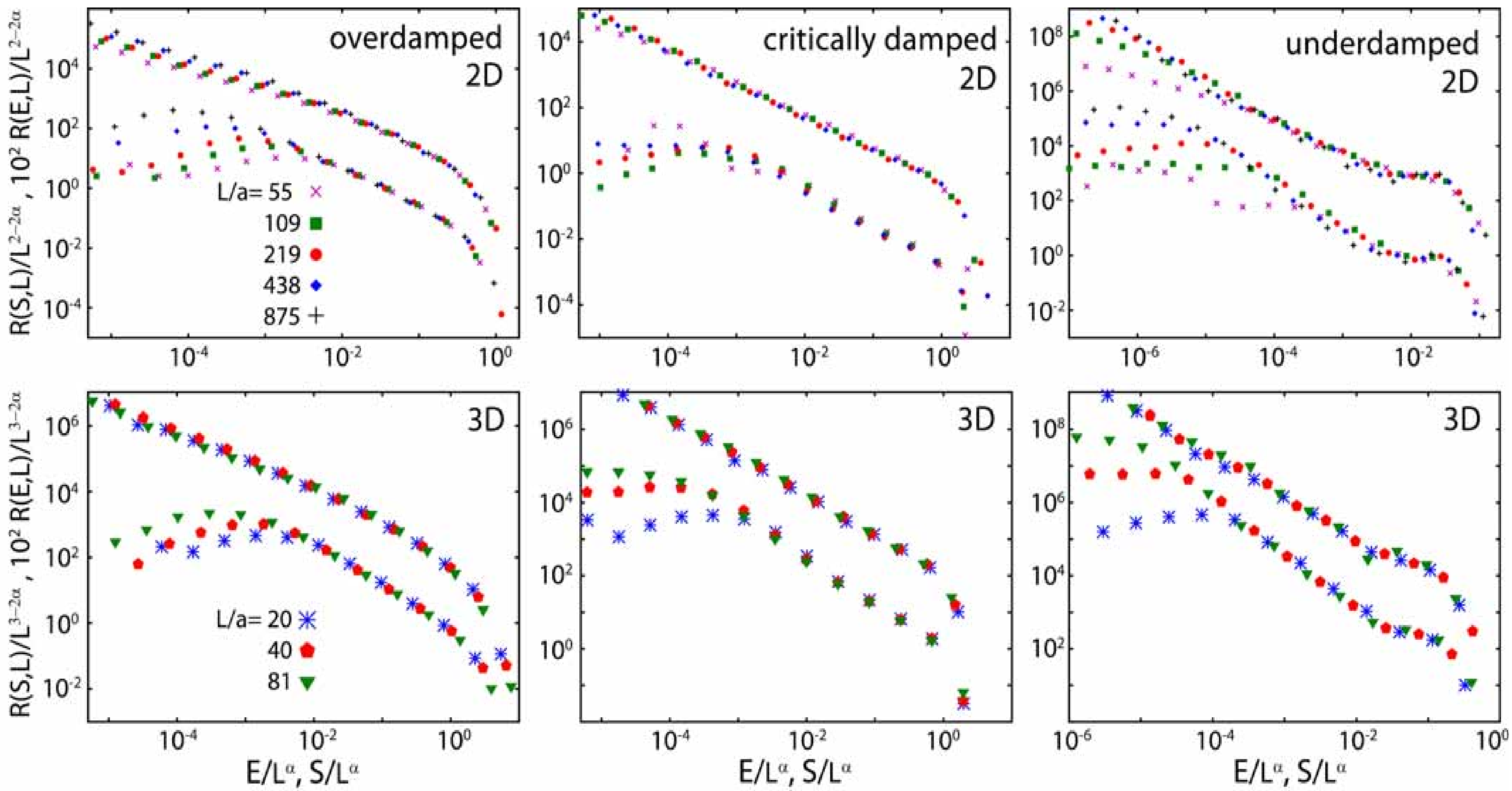
Stress vs. Constant Strain Rate



Critical damping takes energy out at same rate generated by plasticity
For Γ=1 see overdamped behavior but most modes underdamped

Finite Size Scaling Collapse for 2D & 3D

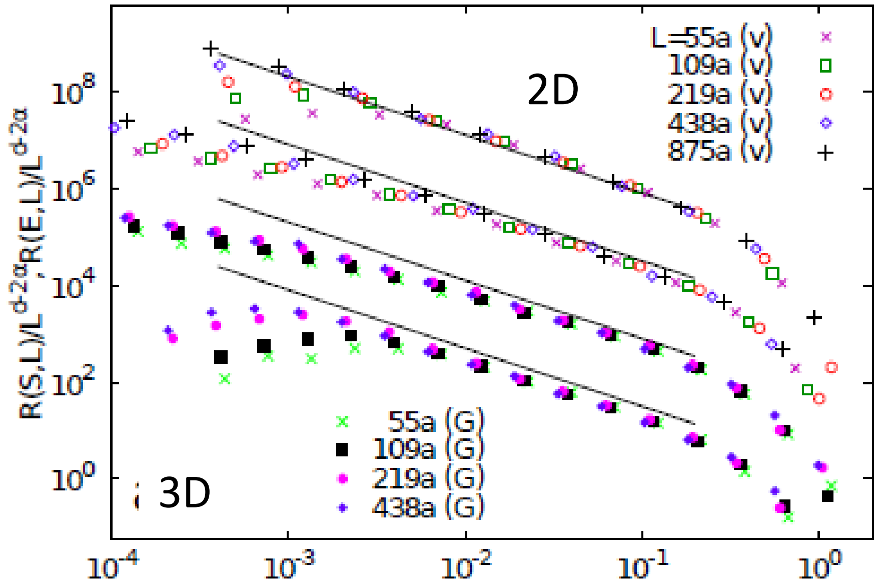
Overdamped and underdamped in different universality classes
Separated by critical damping with own exponents



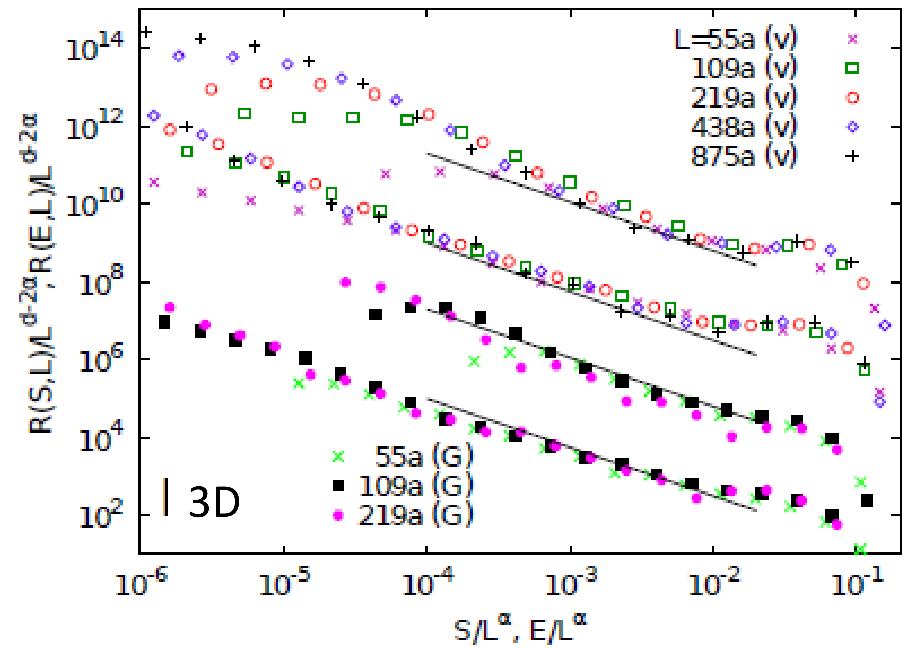
Galilean Invariant Damping

Results collapse with same scaling exponents in 2D & 3D

Overdamped →



Underdamped →



Scaling of Stress Variations with L

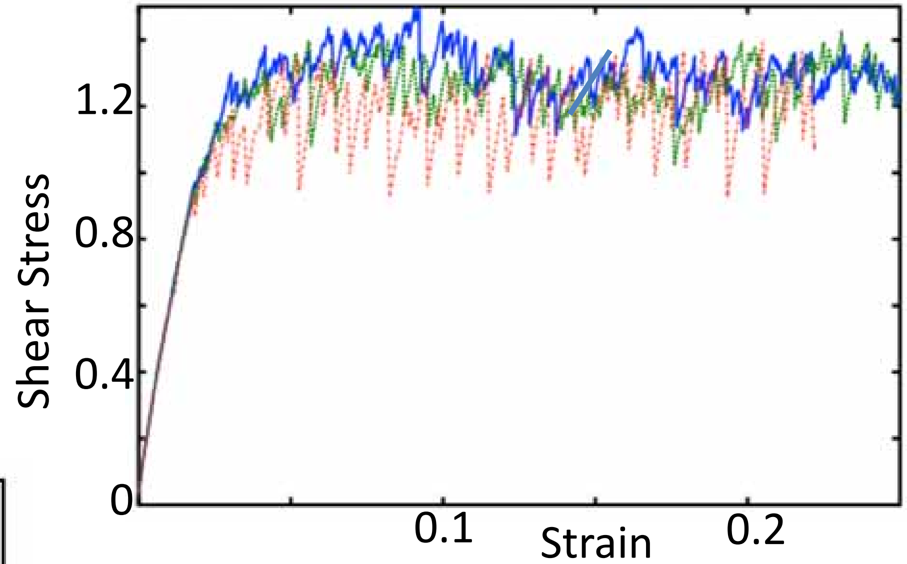
Does width narrow with rising L?

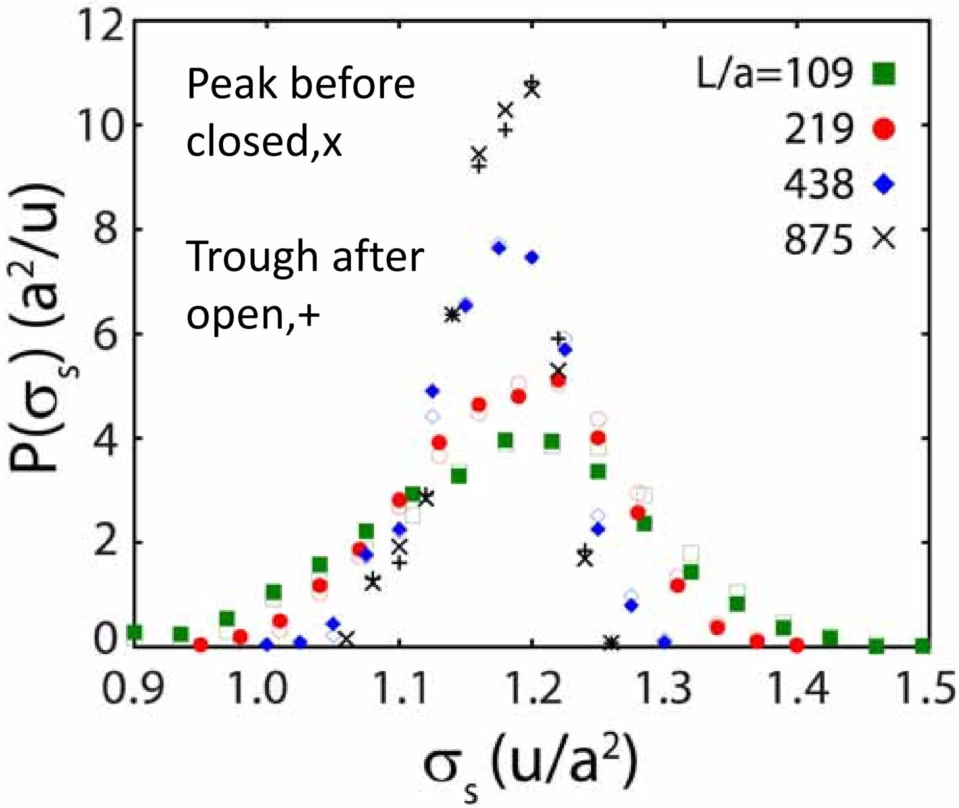
$\langle(\sigma_s - \langle\sigma_s\rangle)^2\rangle \sim L^{-2\phi}$

Or is there a gap like hysteresis in sand flow due to inertia?

Lower bound for stress variation

– largest events L^α/L^d , ϕ=d-α





Stress narrows in all cases

Even for underdamped, distribution of peaks before and troughs after avalanches overlap

No evidence of hysteresis

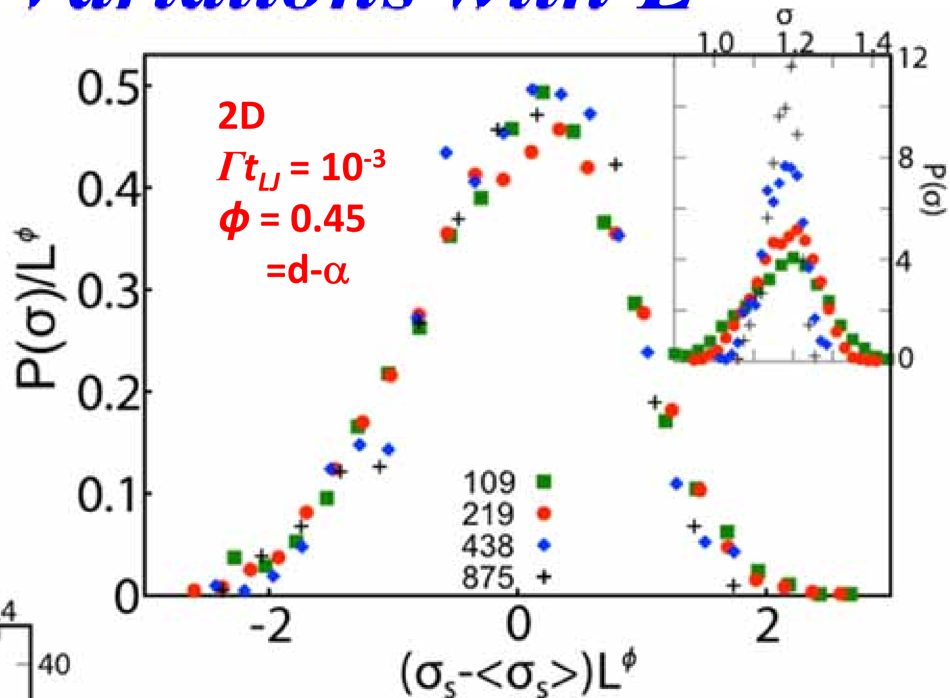
Scaling of Stress Variations with L

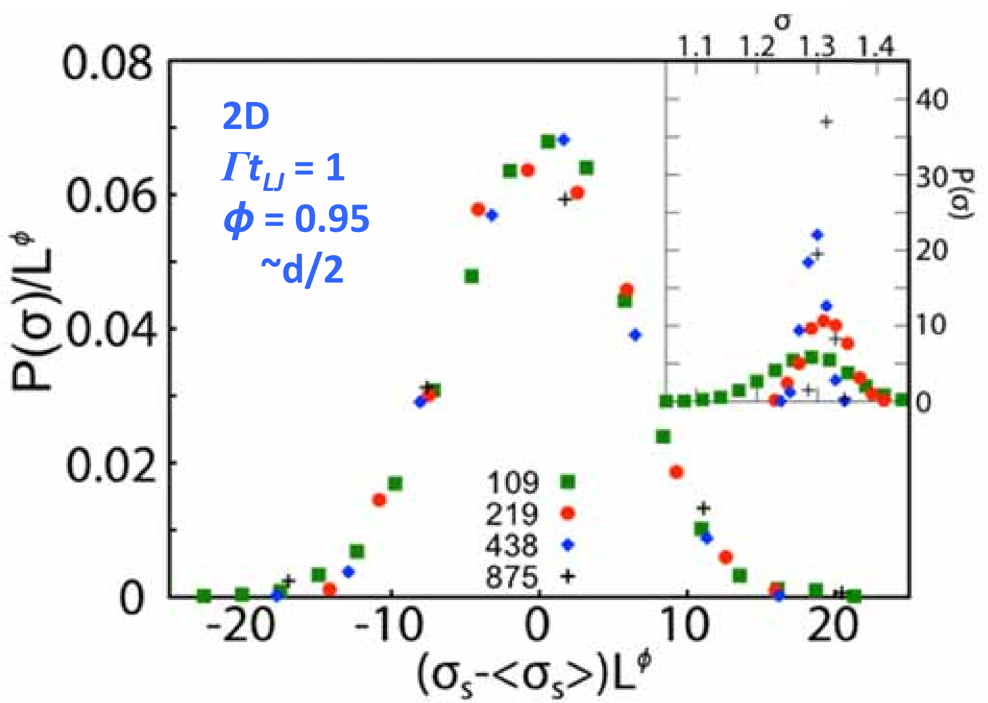
Does width narrow with rising L?

$\langle(\sigma_s - \langle\sigma_s\rangle)^2\rangle \sim L^{-2\phi}$

Lower bound for stress variation – largest events L^α/L^d, $\phi = d - \alpha$

But not quenched configuration, fluctuations in properties $\sim L^{-d/2}$





Find $\phi = \min(d - \alpha, d/2)$

For case where $\phi = d - \alpha$, reasonable that $\phi = 1/\nu$

correlation length $\xi \sim |\sigma - \sigma_c|^{-\nu}$

Finite Size Scaling Collapse for 2D & 3D

Overdamped and underdamped in different universality classes
Separated by critical damping with own exponents

Γ	d	τ	α	γ	ϕ
1.0	2	1.3 ± 0.1	0.9 ± 0.05	1.3 ± 0.1	1.00 ± 0.1
0.1	2	1.0 ± 0.05	0.8 ± 0.1	1.2 ± 0.1	0.9 ± 0.1
0.001	2	1.25 ± 0.1	1.6 ± 0.1	0.8 ± 0.1	0.5 ± 0.1
1.0	3	1.3 ± 0.1	1.1 ± 0.1	2.1 ± 0.1	1.5 ± 0.2
0.1	3	1.05 ± 0.05	1.5 ± 0.1	1.6 ± 0.1	1.30 ± 0.1
0.001	3	1.2 ± 0.1	2.1 ± 0.2	1.3 ± 0.2	0.9 ± 0.1

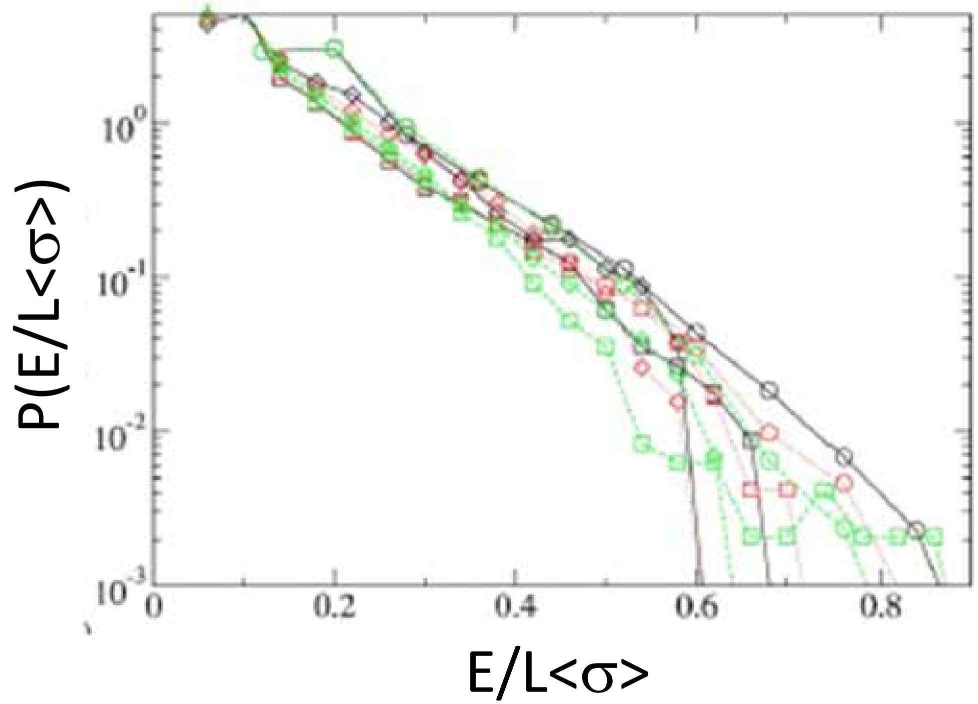
Past studies of this model only attempted to find α

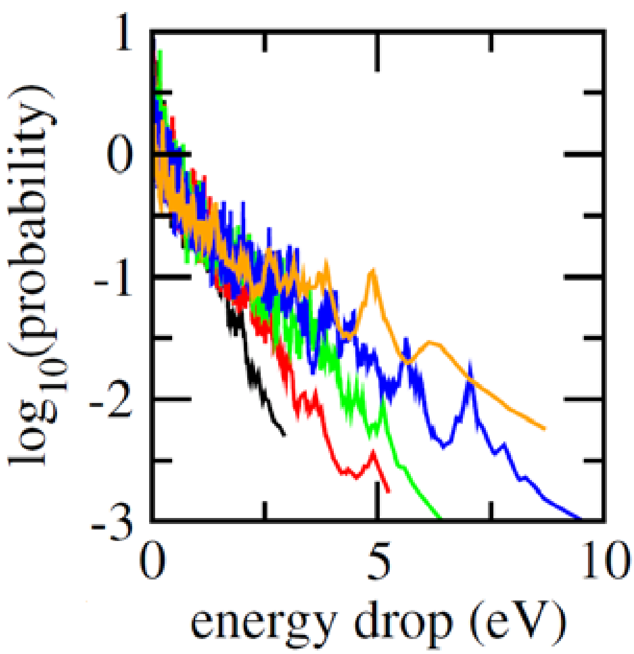
Past Studies of α

Same system fit α=1 (L≤50)

Largest event ~3ε vs. 200ε

(Maloney & Lemaître
PRE 74, 016118, '06)





3D amorphous metal
N(E)~exp[-E/L$^{\alpha}$] α=1.4
(Bailey, Schiøtz, Lemaître &
Jacobsen, PRL 98, 095501, '07).
Threw away small events –majority
Normalized – can't see if nonextensive

Past Studies of α

Same system (Lerner & Procaccia, PRE 79, 066109, 2009)

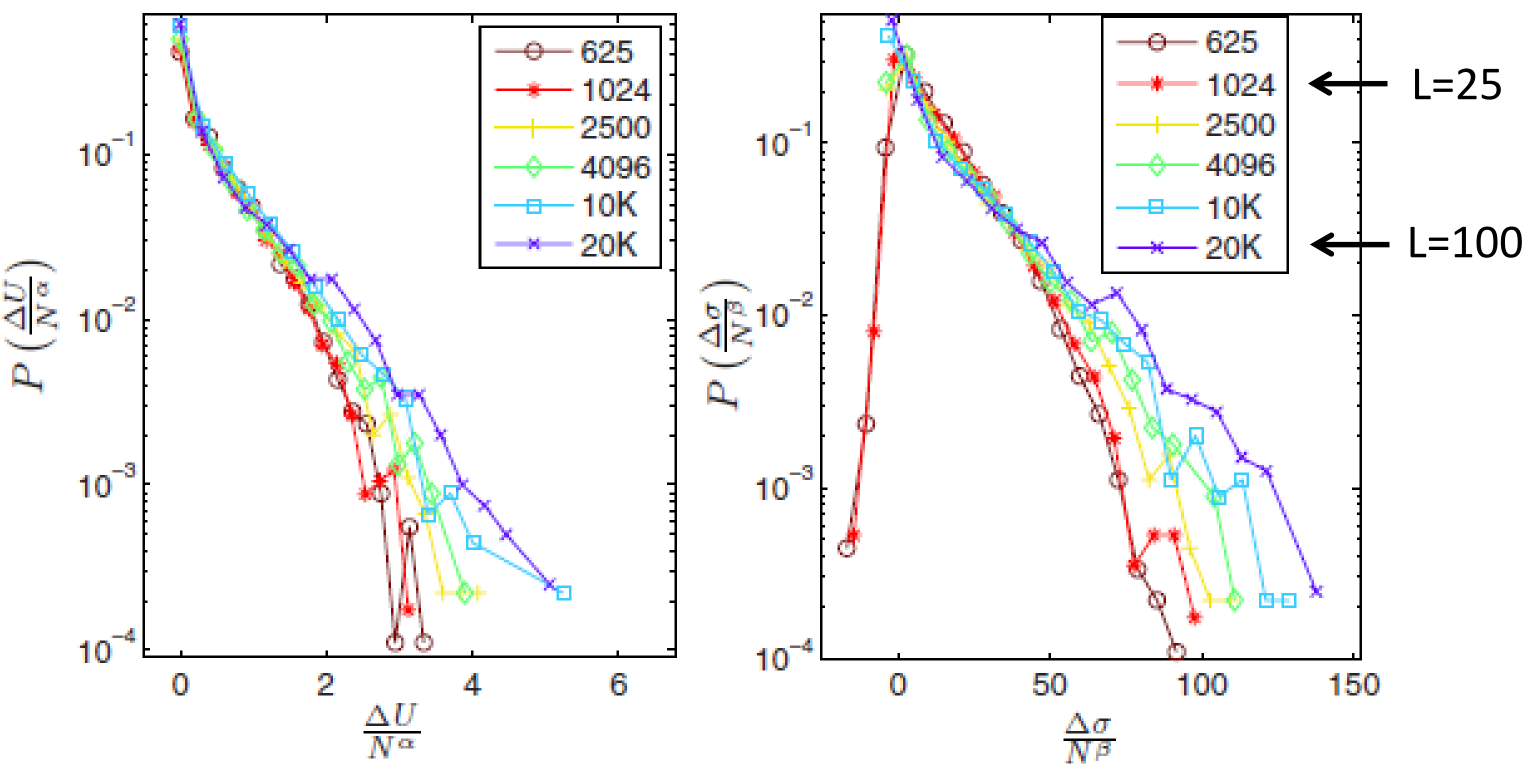
Rescaled $N \sim L^2$ by power corresponding to α=0.75

Fit intermediate range of exponential tail not largest events

Data consistent with larger α for large events

No information about scaling of #small events with L

Note $\Delta\sigma$<0 for some events



Lattice Models → Mean Field Scaling to 2D

K. Dahmen, Y. Ben-Zion, J. Uhl, Phys. Rev. Lett. 102 175501 (2009), Nature Physics 7, 554 (2011)

Many lattice models and experiments show scaling consistent with mean-field exponents, $\tau = 1.5$

BUT models are all overdamped where we find $\tau = 1.2$

Above models have positive definite elastic coupling

- advance in one spot makes all spots more unstable

$\Longrightarrow$ Obey no-passing rule – unique pinned states that flow to from different initial conditions

Shear produces quadrupolar field

– suppress instability at sites normal to shear plane

– no-passing rule does not apply

Lattice Models with Quadrupolar Coupling

Talamali, Petaja, Vandembroucq, and Roux PRE, **84**, 016115 (2011) found τ=1.25

Lin, Saade, Lerner, Rosso, Wyart Europhys Lett 105: 26003 (2014); Lin, Lerner, Rosso, Wyart, PNAS, (2014)

Random kicks from nearby avalanches cause states to diffuse toward instability

Probability that a kick of magnitude δ will cause instability goes to zero as $\delta^{\theta} \Longrightarrow$ leads to nonextensive scaling – γ<d

Find α=1.1 vs. 0.9 in 2D; 1.5 vs. 1.1 in 3D

τ=1.35 vs. 1.3 in 2D, 1.46 vs. 1.3 in 3d

Depinning models θ=0 – no information about coming instability (Martys, Robbins, Cieplak, Phys. Rev. B44, 12294 (1991))

Defining Local Deformation

Shear related to local rotation $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$, $\boldsymbol{u}$=displacement

Find Delaunay triangulation for initial particle centers

For each triangle:

$$\frac{\partial u_i}{\partial x_j} = F_{ij}$$

$$\epsilon_1 = \frac{F_{xx} - F_{yy}}{2}$$

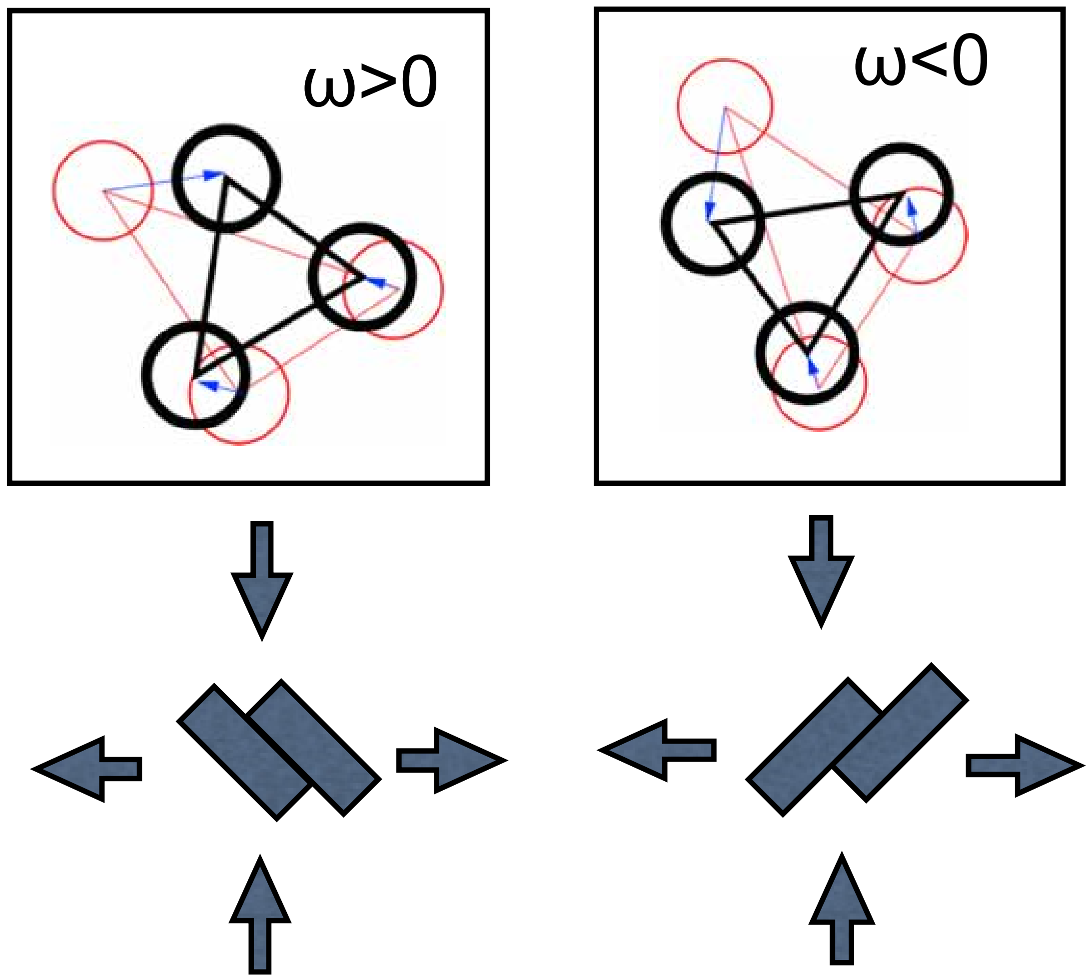
$$\epsilon_2 = \frac{F_{xy} + F_{yx}}{2}$$

Invariants:

$$\epsilon_d = \sqrt{\epsilon_1^2 + \epsilon_2^2}$$

$$\omega = F_{xy} - F_{yx}$$

$$\epsilon_I = (F_{xx} + F_{yy})/2$$



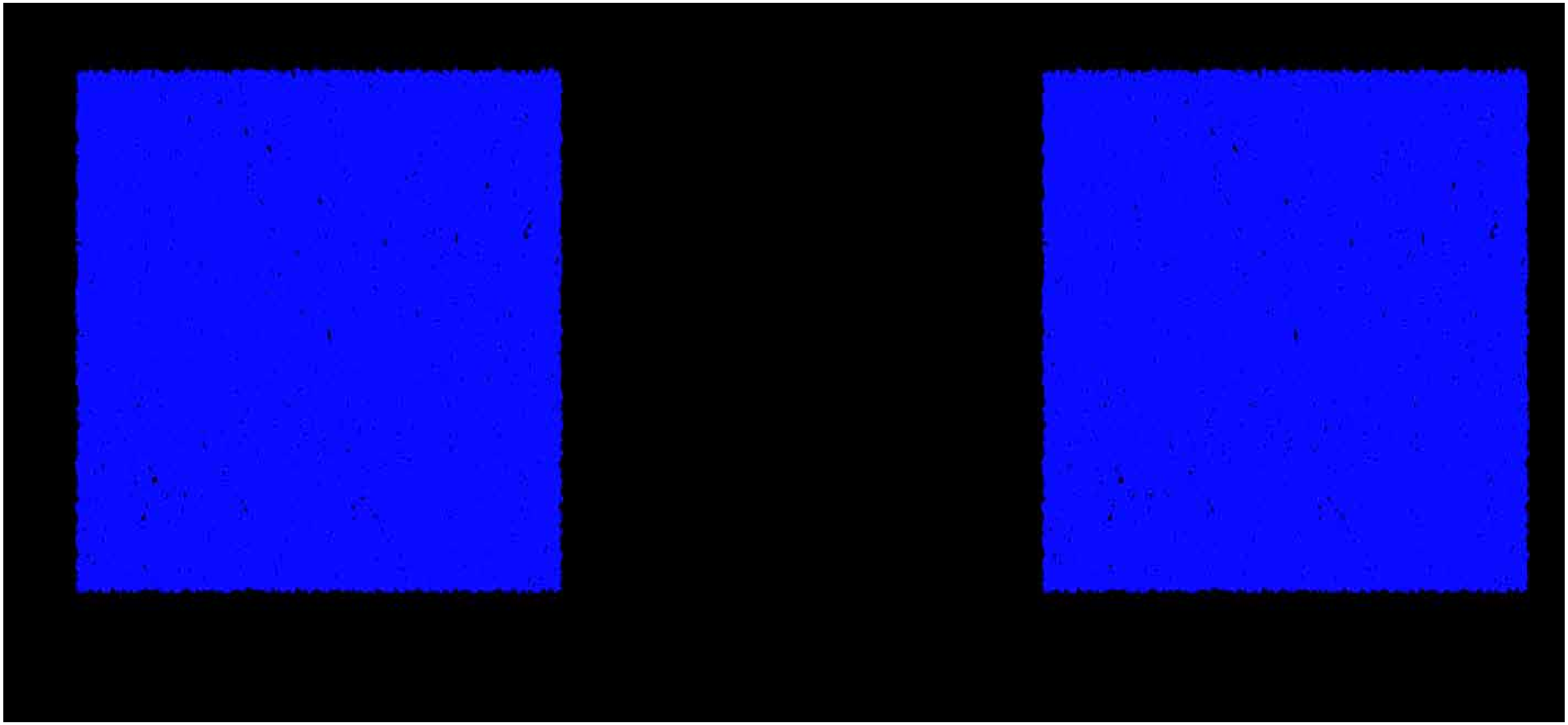
"Right Strain" "Left Strain"

Rich Event Dynamics

Critically damped system

Total curl

Local kinetic E

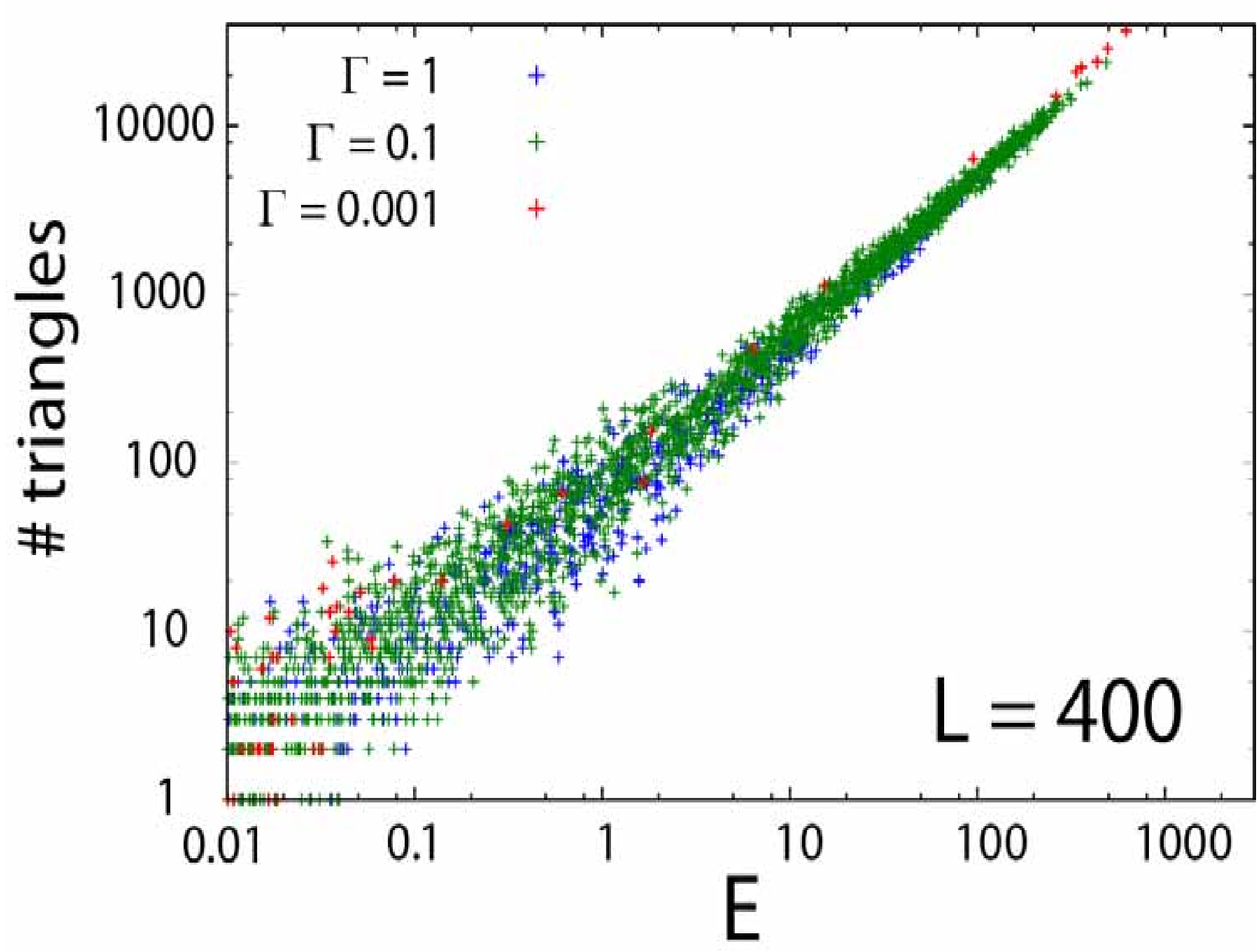


Relation Between Energy and Spatial Size

Size = # triangles where curl of displacement field has magnitude > 0.1 (limit of elastic deformations 5% strain)

Large E, E~# →spatial extent of plastic regions ~ E for all Γ

Less clear correlation with S, especially for underdamped



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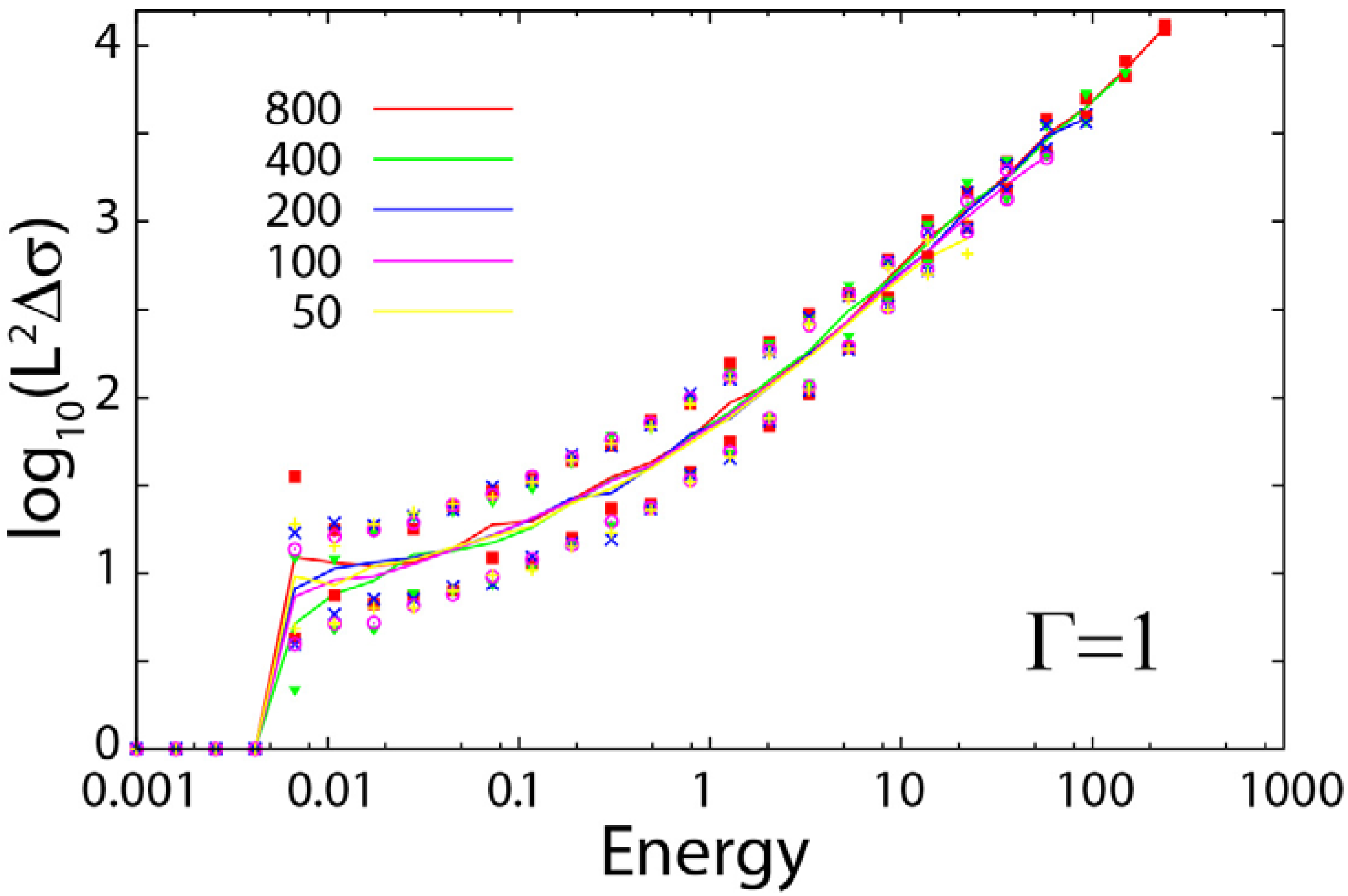
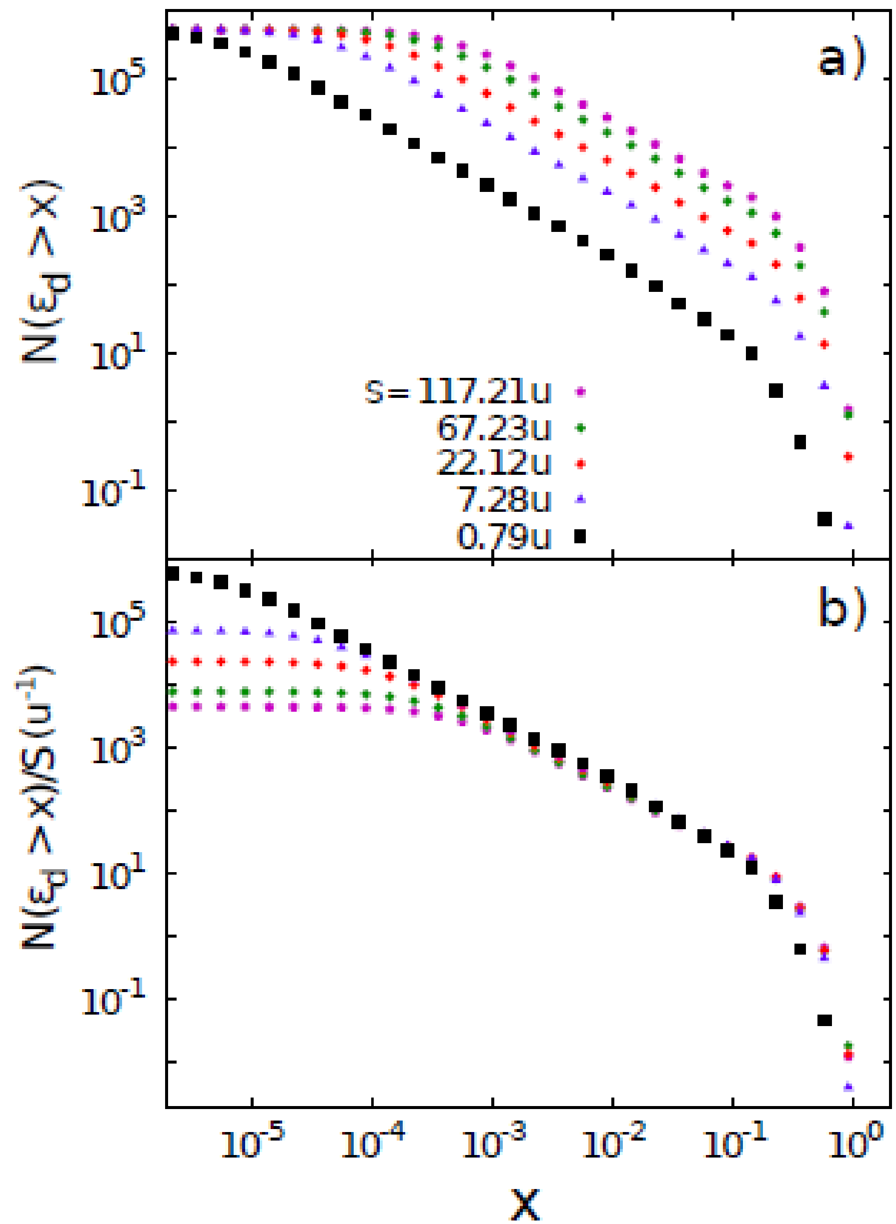


Relation Between Energy and Spatial Size

Quadrupolar elastic strain should give cumulative distribution of triangles strained more than x that scales as S/x.

Find clear cutoff marking transition to plastic region.

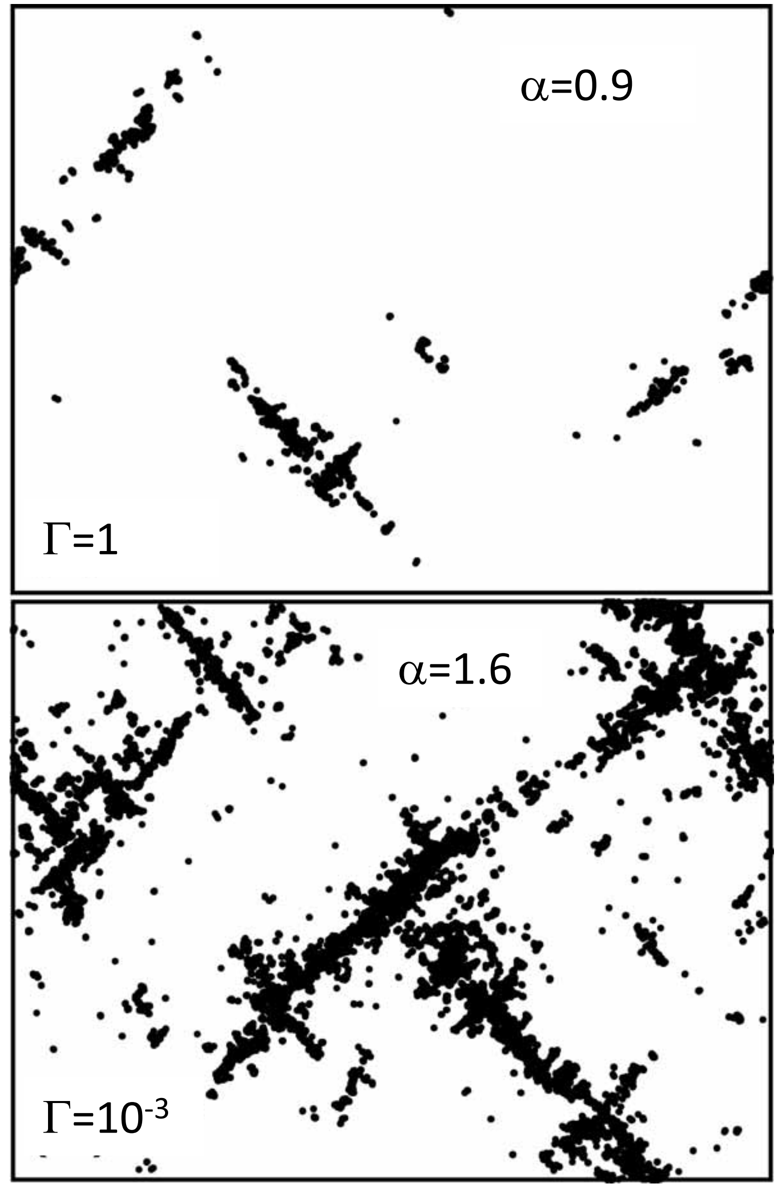


Relation Between Energy and Spatial Size

Linear relation between plastic area/volume and E
⇒Larger E deforms larger region but not with larger strains
Similar in lattice models but perhaps not for earthquakes

Largest size event ~ L^{α}
→is α a fractal dimension d_f?

Not clear that fractal object



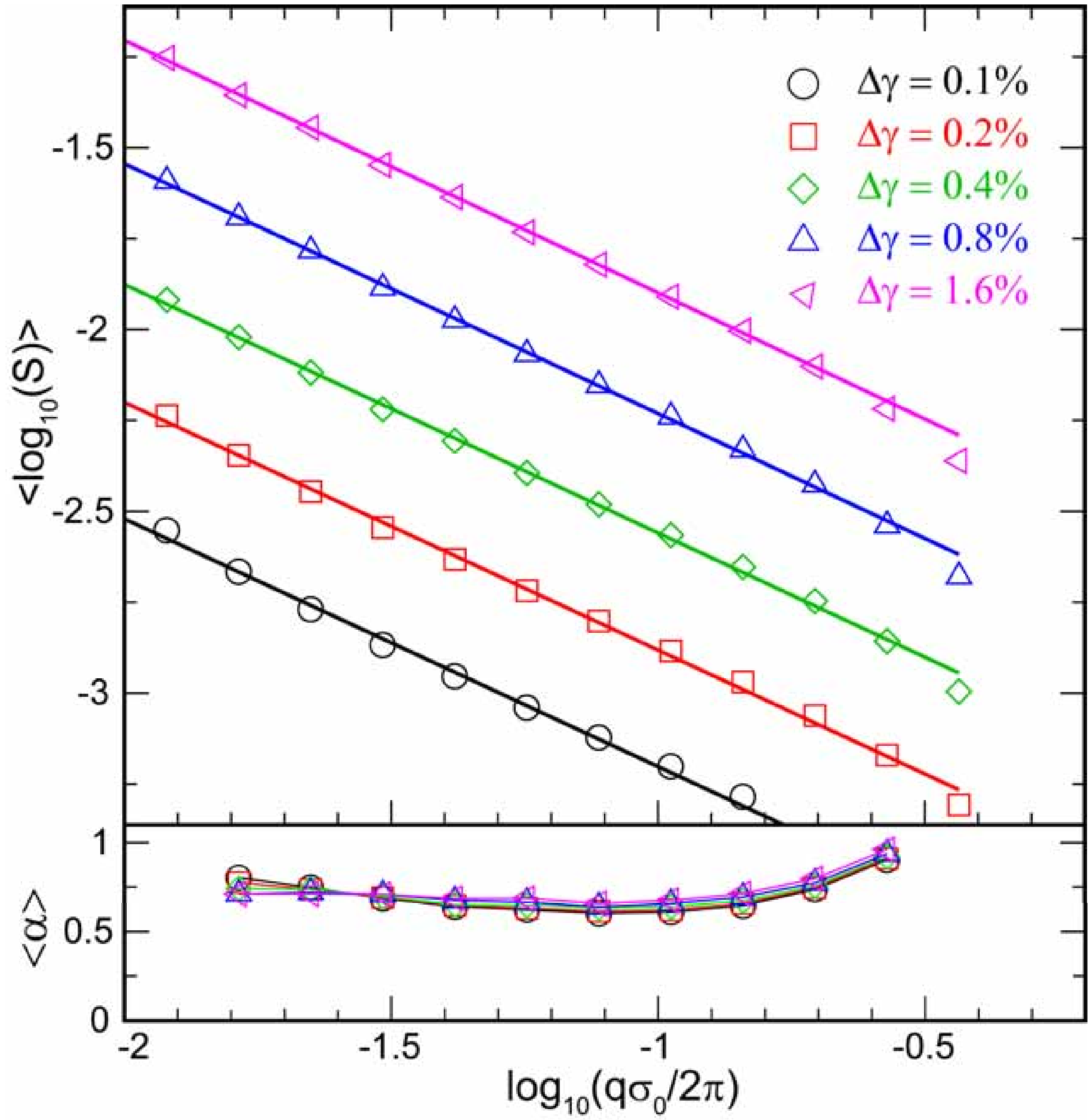
Vorticity Correlation Function

$$S(\vec{q}) = \left| \int \omega(\vec{r}) exp[i\vec{q} \cdot \vec{r}] dr \right|^2$$

Mean of log S scales as power of wave vector.

Prefactor linear in Δγ
→Incoherent addition of successive intervals

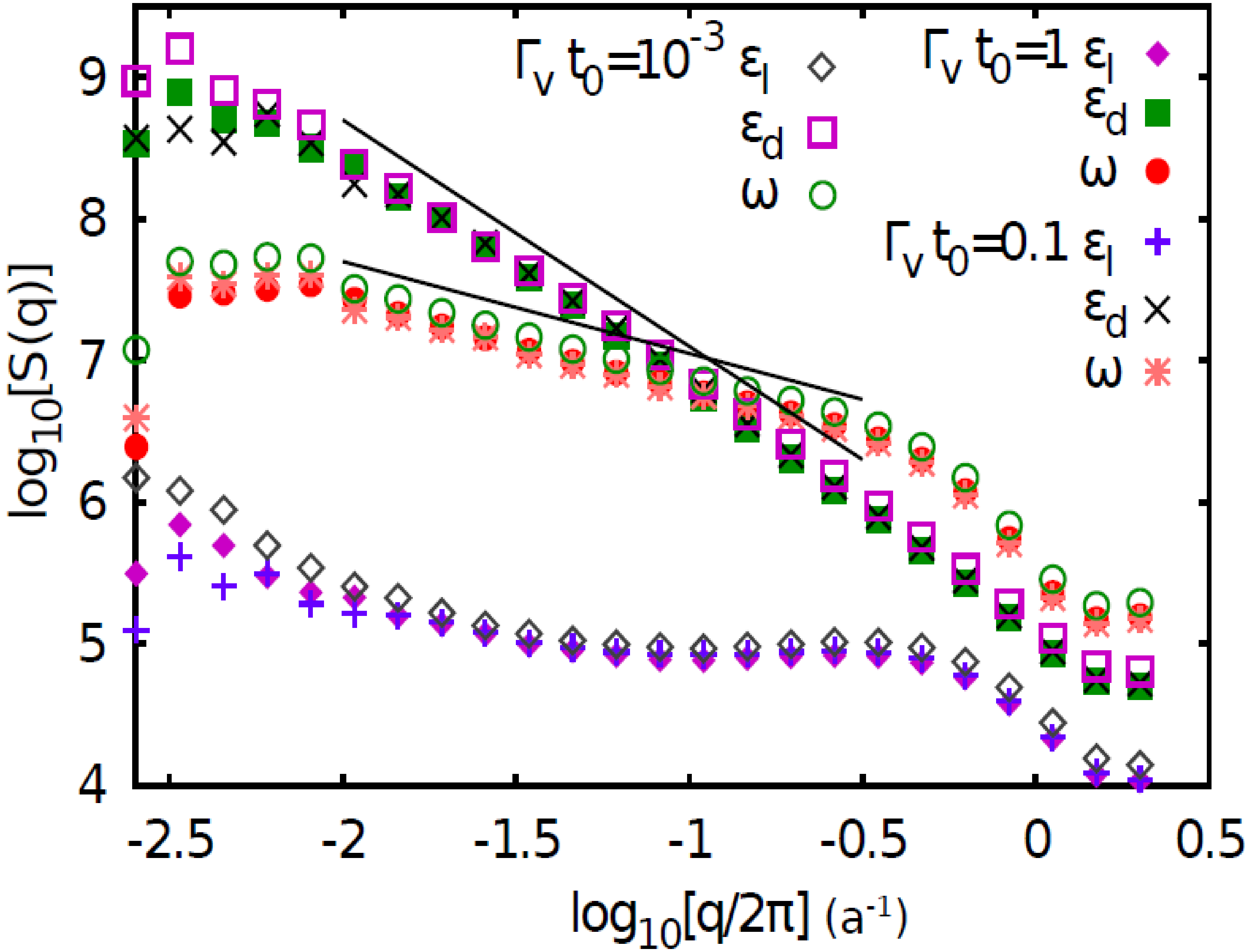
BUT scaling highly anisotropic



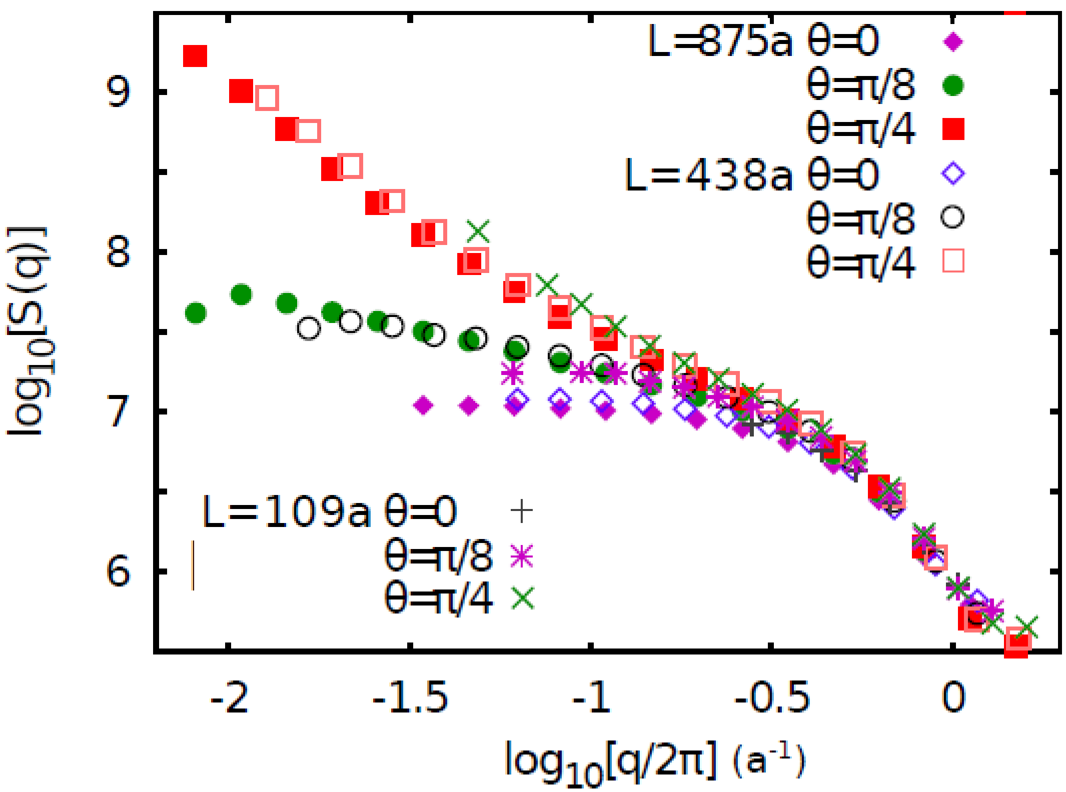
Power Spectrum for 3 Strain Measures

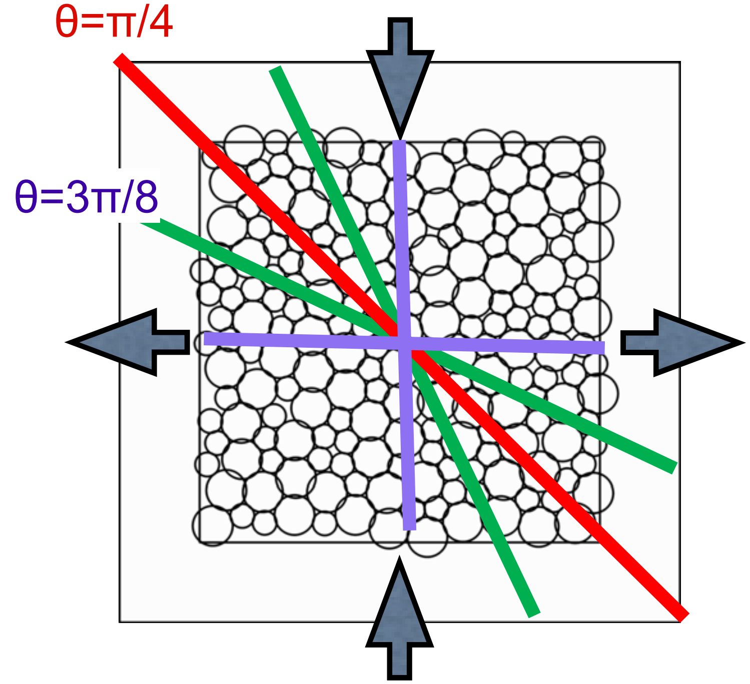
All Γ: Same scaling over a given strain interval (multiple avalanches)
No correlation in density fluctuations, power laws in shear measures

ε_I=divergence
ω=curl
ε_d=deviatoric
shear invar.
(not linear)



Angle Dependence of Structure Factor, Δγ=0.1%





Strongly anisotropic scaling $S(q,\theta)=A(\theta)q^{-\alpha(\theta)}$

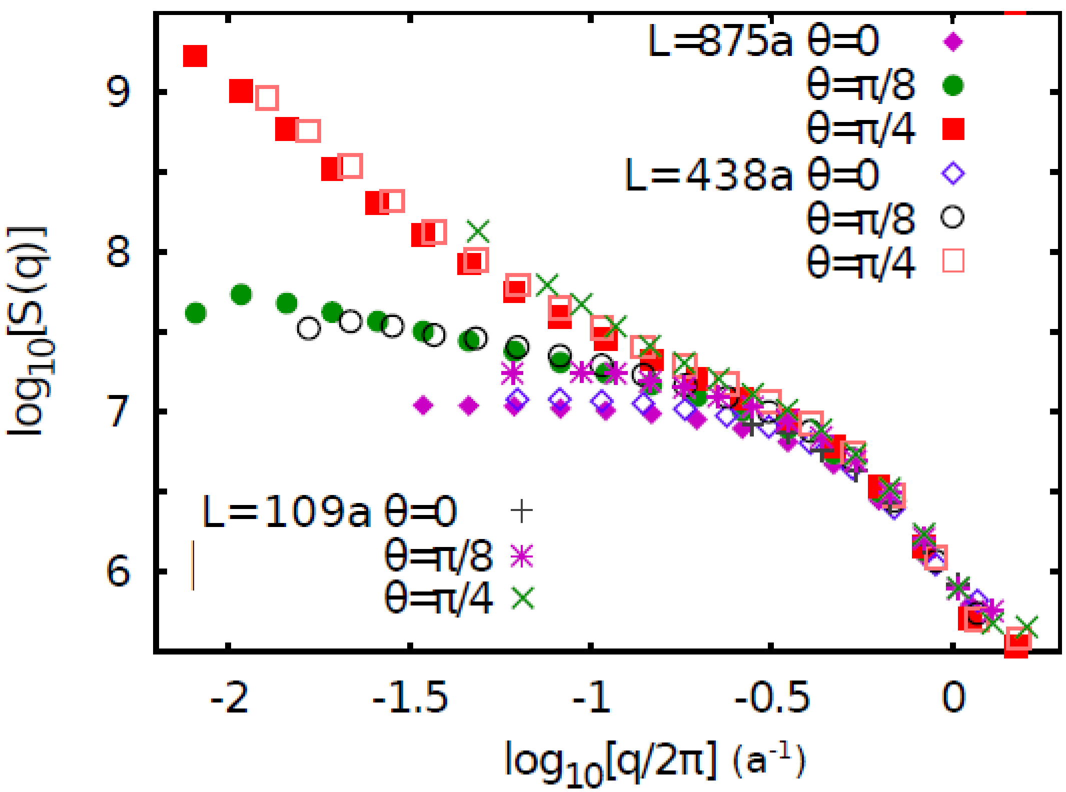
Not a typical fractal where a(q) related to D_f

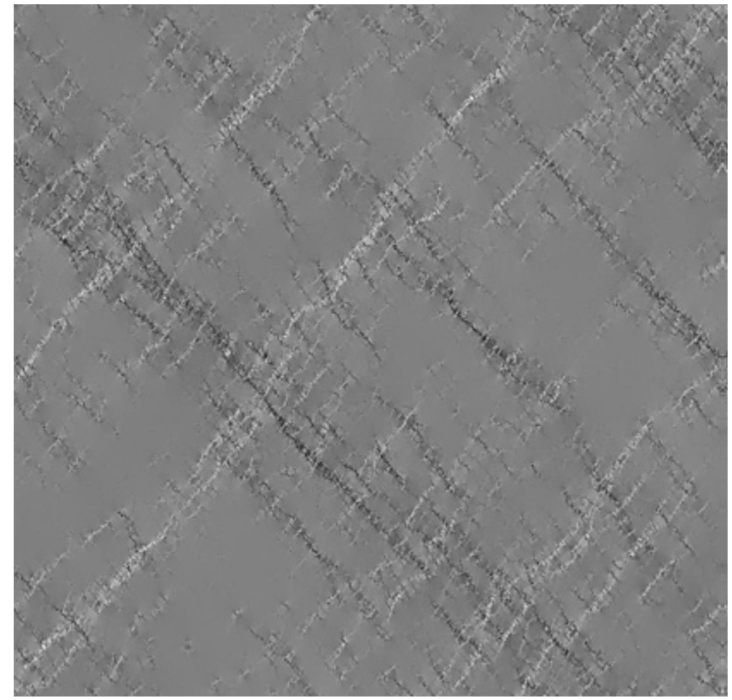
4mm symmetry

θ=π/8 and θ=3π/8 – same scaling

θ=0 and θ=π/2 – same scaling

Angle Dependence of Structure Factor, Δγ=0.1%





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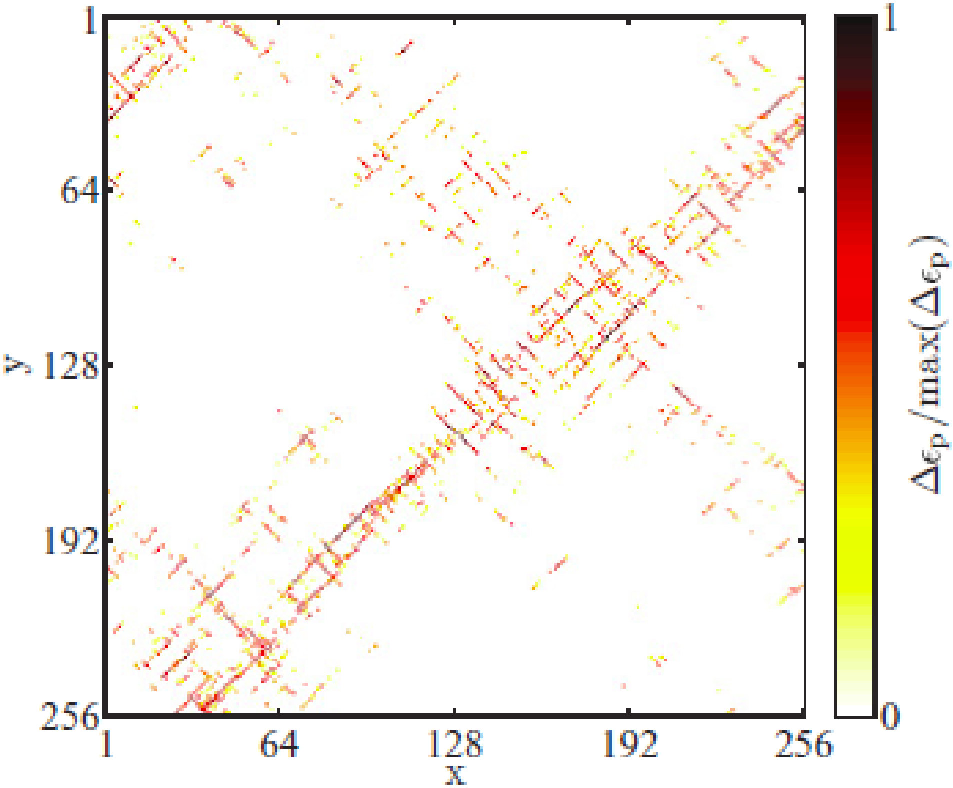
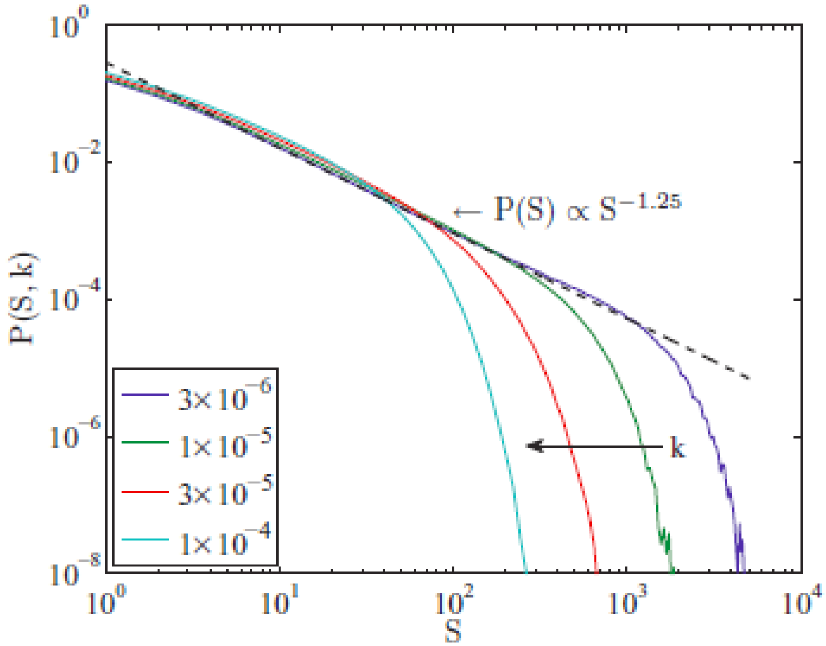
Lattice Model that Preserves Spatial Correlations

Talamali, Petaja, Vandembroucq, and Roux PRE, **84**, 016115 (2011)

Get anisotropic correlations and $\tau = 1.25$ close to ours

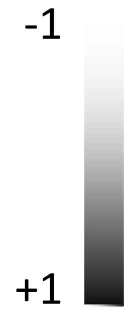


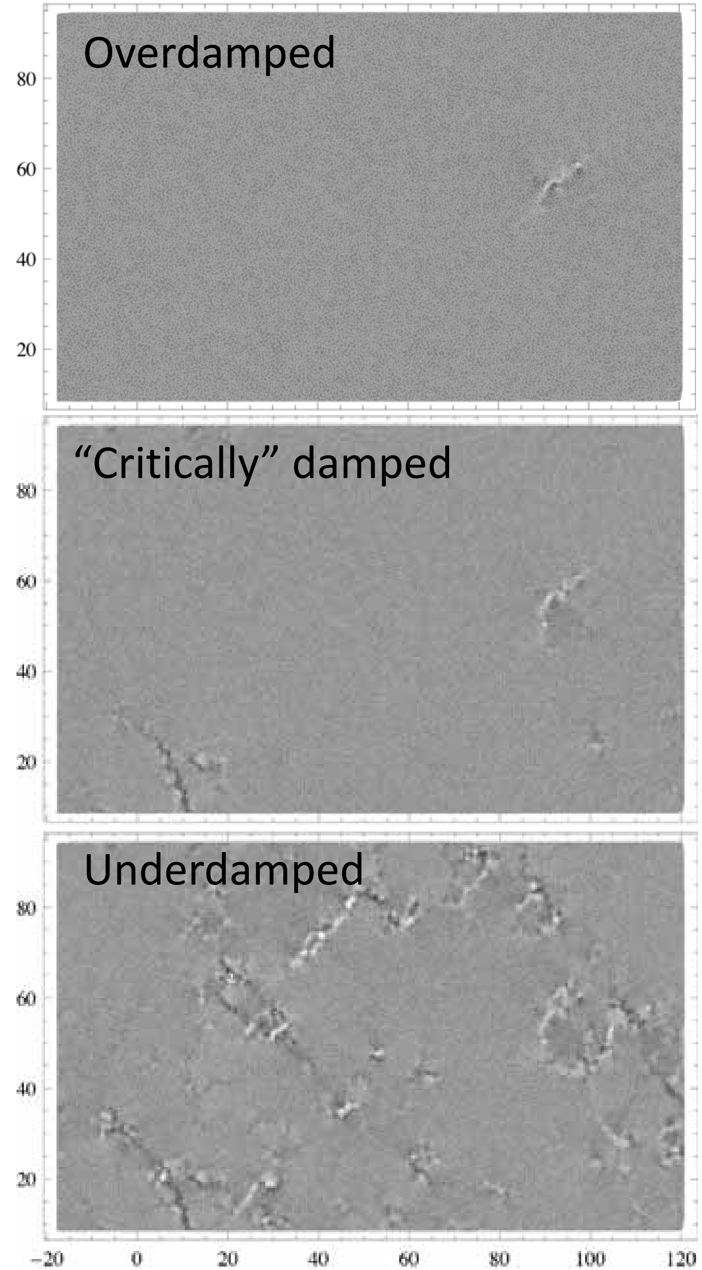
(b)



Plasticity From Same Initial State

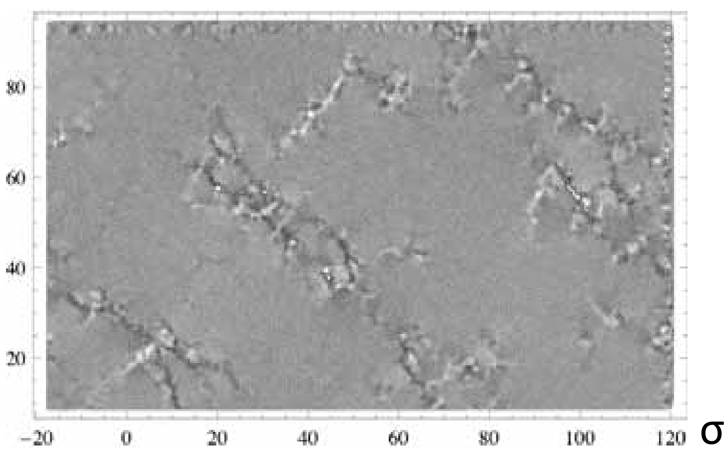
- Calculate curl of local displacement field $\omega = \frac{1}{2}\left(u_{x,y} - u_{y,x}\right)$
- Amount of plasticity matches energy during event
- Lower damping can activate weak "zones"
- Net effect over strain interval can be similar for different damping





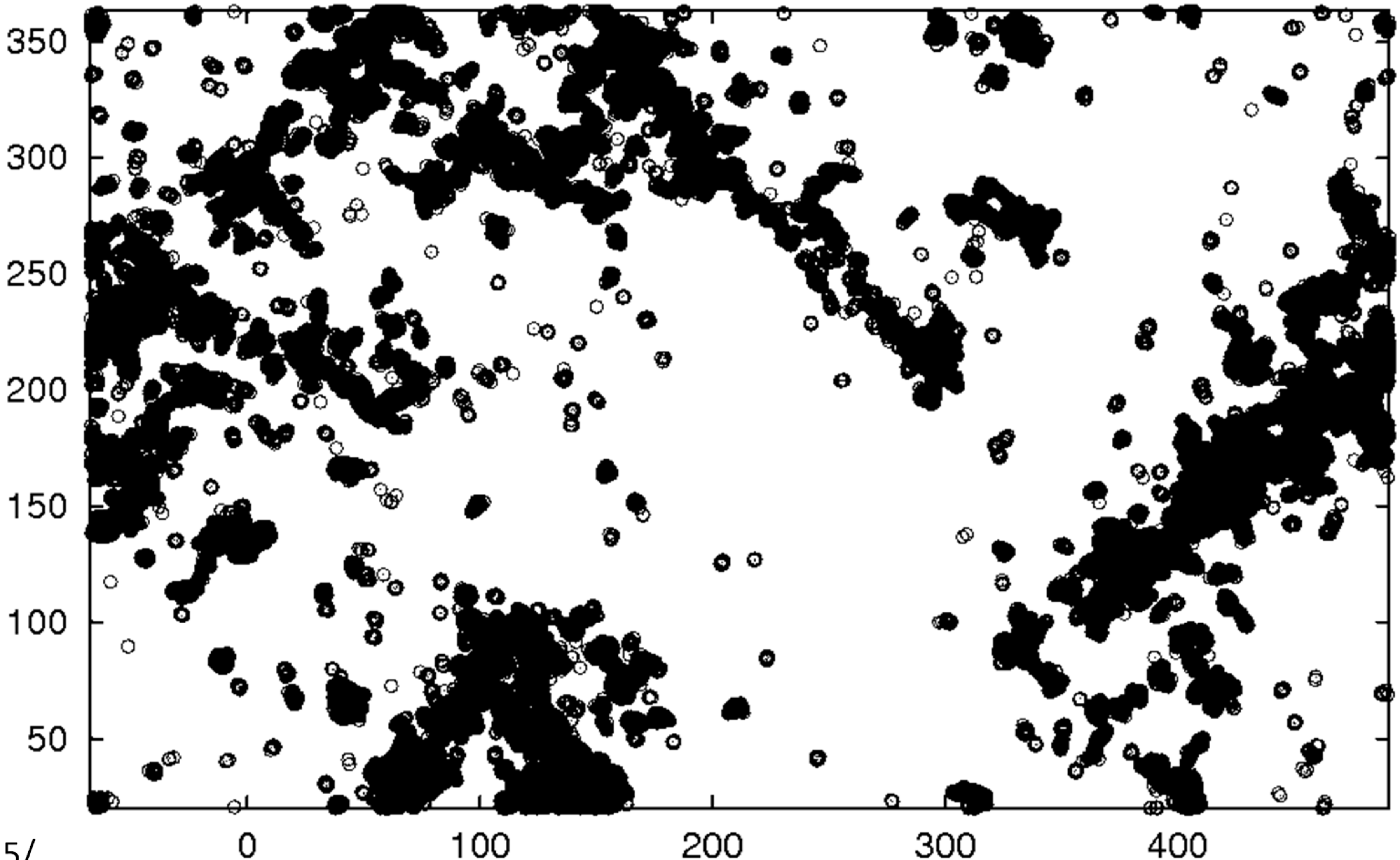
"Critically" damped

$$\Delta\gamma = 0.01$$



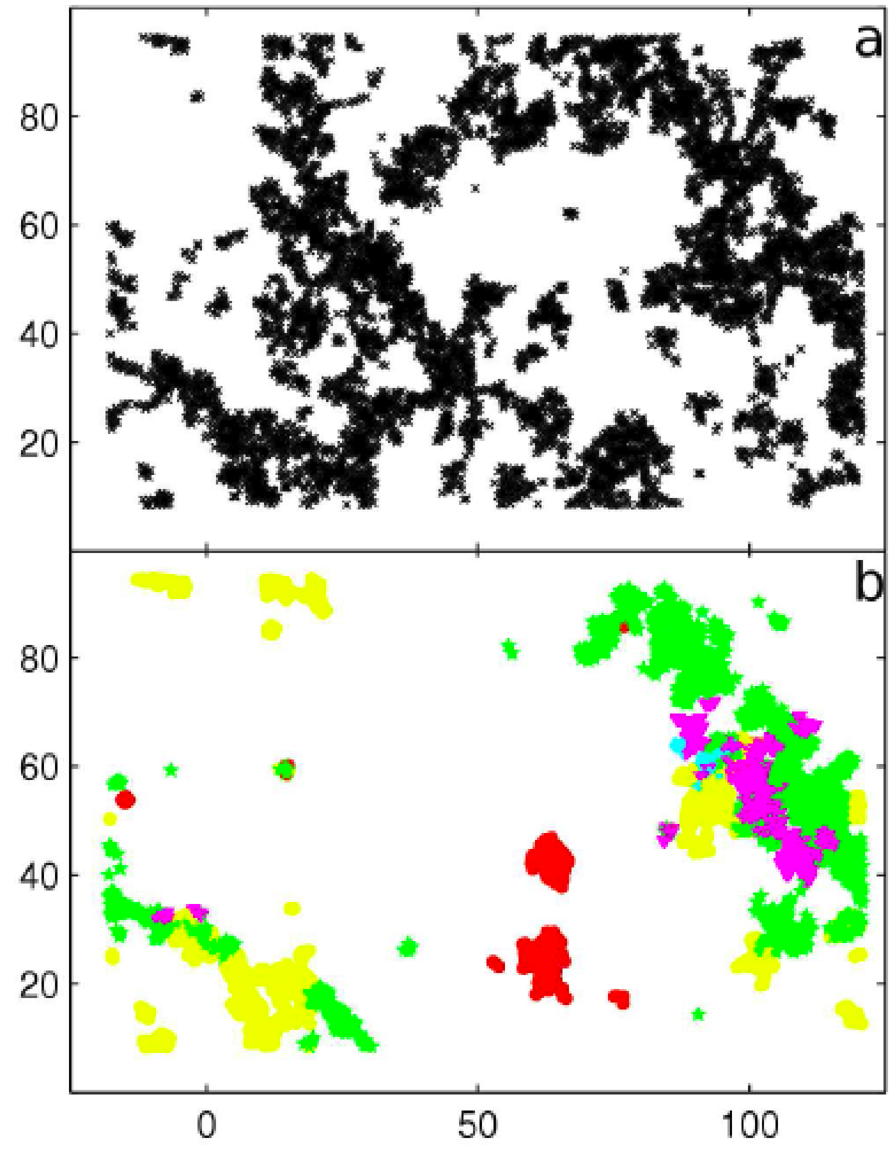
Plasticity From Same Initial State

Black-underdamped, green-overdamped earthquake, red-past earthquakes



Plasticity From Same Initial State

Black-underdamped,
colors different
ovedamped earthquakes



Conclusions for Critical Scaling

Define R(E,L)=# of events per unit energy per unit strain
Expect: $R(E,L) = L^{\beta}\, g(E/L^{\alpha})$ and $R(E,L) \sim L^{\beta+\alpha\tau}\, E^{-\tau}$ for small E
with scaling relations $\gamma=\beta+\alpha\tau$, $\beta+2\alpha=2$

Always find subextensive # of small events $\gamma < d$
$\rightarrow$Large events suppress small events

Overdamped – Small E non-critical power law, $\tau=0.7$
$\Gamma>1$ strange events with bond changes << 10%
Large E, S find $\tau=1.2$, $\alpha < 1$

Underdamped - Large E, $\Delta\sigma$ find $\tau=1.4$, $\alpha > d-1$
Γ L/c <1 Plateau of extra large events scales with cutoff

Critically damped $\longrightarrow$ Find large range with $\tau=1$ at damping between under and over damped. Same for all damping types