The Status Quo of Self-Organised Criticality

History, Models, Universality Classes, Tools

Gunnar Pruessner

Department of Mathematics
Imperial College London

Kavli Institute for Theoretical Physics, Santa Barbara, Nov 2014

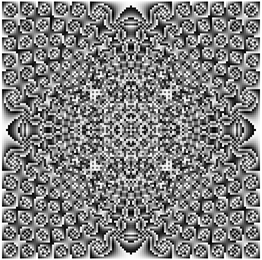
Imperial College London

Outline

1. SOC: Past and Present
2. Universality Classes
3. Theory of SOC
4. Summary: Any Answers?

Imperial College London

Prelude: The physics of fractals

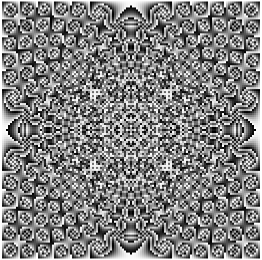


Question: Where does scale invariant behaviour in nature come from?

Answer: Due to a phase transition, self-organised to the critical point.

Imperial College London

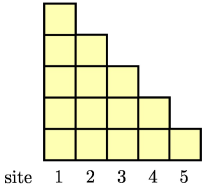
Prelude: The physics of fractals



- Anderson, 1972: *More is different*
  Correlation, cooperation, emergence
- $1/f$ noise "everywhere" (van der Ziel, 1950; Dutta and Horn, 1981)
- Kadanoff, 1986: *Fractals: Where's the Physics?*
- Bak, Tang and Wiesenfeld, 1987: *Self-Organized Criticality: An Explanation of $1/f$ Noise*

Imperial College London

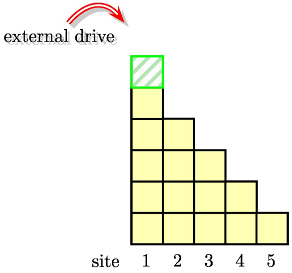
The BTW Model



The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics. (Exact results for correlation functions by Mahieu, Ruelle, Jeng *et al.*)
- **The physics of fractals.**

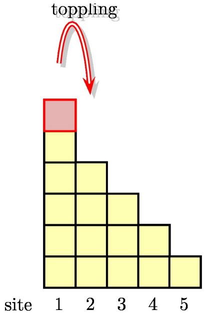
The BTW Model



The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics. (Exact results for correlation functions by Mahieu, Ruelle, Jeng *et al.*)
- **The physics of fractals.**

The BTW Model

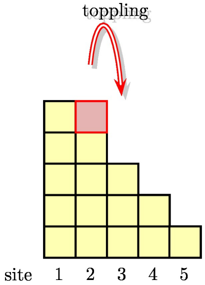


The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics. (Exact results for correlation functions by Mahieu, Ruelle, Jeng *et al.*)
- **The physics of fractals.**

Imperial College London

The BTW Model

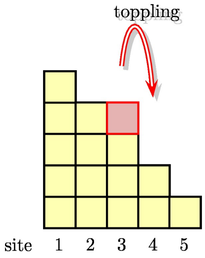


The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics. (Exact results for correlation functions by Mahieu, Ruelle, Jeng *et al.*)
- **The physics of fractals.**

Imperial College London

The BTW Model

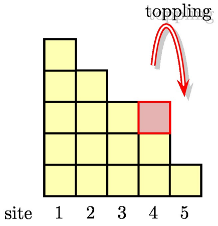


The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics. (Exact results for correlation functions by Mahieu, Ruelle, Jeng *et al.*)
- **The physics of fractals.**

Imperial College London

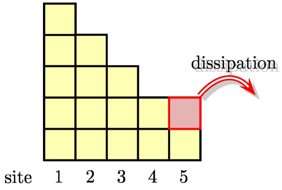
The BTW Model



The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics. (Exact results for correlation functions by Mahieu, Ruelle, Jeng *et al.*)
- **The physics of fractals.**

The BTW Model

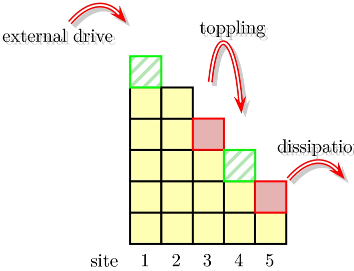


The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics. (Exact results for correlation functions by Mahieu, Ruelle, Jeng *et al.*)
- **The physics of fractals.**

Imperial College London

The BTW Model

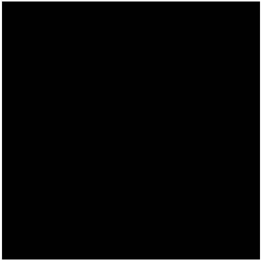


The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics. (Exact results for correlation functions by Mahieu, Ruelle, Jeng *et al.*)
- **The physics of fractals.**

Imperial College London

The BTW Model

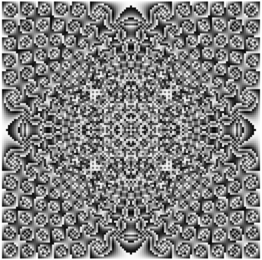


The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics. (Exact results for correlation functions by Mahieu, Ruelle, Jeng *et al.*)
- **The physics of fractals.**

Imperial College London

The BTW Model



Key ingredients for SOC models:

- Separation of time scales.
- Interaction.
- Thresholds (non-linearity).
- Observables: Avalanche sizes and durations.

Imperial College London

Why is SOC important?

SOC today: Slowly driven, avalanching (intermittent) systems with non-linear interactions, that display non-trivial power-law correlations (cutoff by the system size) as known from ordinary critical phenomena, but with internal, self-organised, rather than external tuning of a control parameter (to a non-trivial value).

Emergence!

- Explanation of emergent,
- . . . cooperative,
- . . . long time and length scale
- . . . phenomena,
- . . . as signalled by power laws.

Imperial College London

Why is SOC important?

SOC today: Slowly driven, avalanching (intermittent) systems with non-linear interactions, that display non-trivial power-law correlations (cutoff by the system size) as known from ordinary critical phenomena, but with internal, self-organised, rather than external tuning of a control parameter (to a non-trivial value).

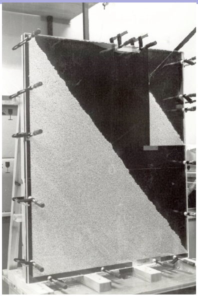
Universality!

- Understanding and classifying natural phenomena
- ...using *Micky Mouse Models*
- ...on a small scale (in the lab or on the computer).
- (Triggering critical points?)
- But: Where is the evidence for scale invariance in nature (dirty power laws)?

Imperial College London

Experiments:

Granular media, superconductors, rain. . .



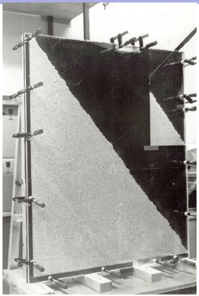
Photograph courtesy of V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang and P. Meakin.

- Large number of experiments and observations:
- Earthquakes suggested by Bak, Tang and Wiesenfeld.
- Sandpile experiments by Jaeger, Liu and Nagel (PRL, 1989).
- Superconductors experiments by Ling, *et al*. (Physica C, 1991).
- Ricepiles experiments by Frette *et al*. (Nature, 1996).
- Precipitation statistics by Peters and Christensen (PRL, 2002).

Imperial College London

Experiments:

Granular media, superconductors, rain. . .



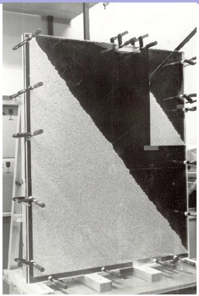
Photograph courtesy of V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang and P. Meakin.

- Large number of experiments and observations:
- Earthquakes suggested by Bak, Tang and Wiesenfeld.
- Sandpile experiments by Jaeger, Liu and Nagel (PRL, 1989).
- Superconductors experiments by Ling, *et al*. (Physica C, 1991).
- Ricepiles experiments by Frette *et al*. (Nature, 1996).
- Precipitation statistics by Peters and Christensen (PRL, 2002).

Imperial College London

Experiments:

Granular media, superconductors, rain. . .



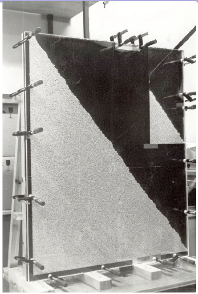
Photograph courtesy of V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang and P. Meakin.

- Large number of experiments and observations:
- Earthquakes suggested by Bak, Tang and Wiesenfeld.
- Sandpile experiments by Jaeger, Liu and Nagel (PRL, 1989).
- Superconductors experiments by Ling, *et al*. (Physica C, 1991).
- Ricepiles experiments by Frette *et al*. (Nature, 1996).
- Precipitation statistics by Peters and Christensen (PRL, 2002).

Imperial College London

Experiments:

Granular media, superconductors, rain. . .



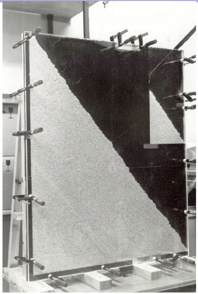
Photograph courtesy of V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang and P. Meakin.

- Large number of experiments and observations:
- Earthquakes suggested by Bak, Tang and Wiesenfeld.
- Sandpile experiments by Jaeger, Liu and Nagel (PRL, 1989).
- Superconductors experiments by Ling, *et al*. (Physica C, 1991).
- Ricepiles experiments by Frette *et al*. (Nature, 1996).
- Precipitation statistics by Peters and Christensen (PRL, 2002).

Imperial College London

Experiments:

Granular media, superconductors, rain. . .



Photograph courtesy of V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang and P. Meakin.

- Large number of experiments and observations:
- Earthquakes suggested by Bak, Tang and Wiesenfeld.
- Sandpile experiments by Jaeger, Liu and Nagel (PRL, 1989).
- Superconductors experiments by Ling, *et al*. (Physica C, 1991).
- Ricepiles experiments by Frette *et al*. (Nature, 1996).
- Precipitation statistics by Peters and Christensen (PRL, 2002).

Imperial College London

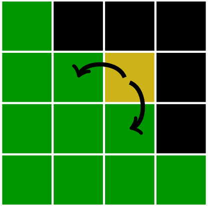
More models

- Initial intention for more models: Expand BTW universality class.
- Later: Provide more evidence for SOC as a whole.
- More models…

More models

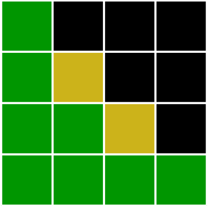
- Zhang Model (1989) [scaling questioned]
- Dhar-Ramaswamy Model (1989) [solved, directed]
- Forest Fire Model (1990, 1992) [no proper scaling]
- Manna Model (1991) [solid!]
- Olami-Feder-Christensen Model (1992) [scaling questioned, $\alpha \approx 0.05$ (localisation), $\alpha = 0.22$ (jump)]
- Bak-Sneppen Model (1993) [scaling questioned]
- Zaitsev Model (1992)
- Sneppen Model (1992)
- Oslo Model (1996) [solid!]

The Bak-Chen-Tang Forest Fire Model



- Originally by Bak, Chen and Tang (1990).
- Intended as a model of turbulence.
- Sites empty, occupied (by tree) or on fire.
- Slow regrowth at rate p.
- Occasional re-lighting.
- Grassberger and Kantz (1991):
  Deterministic pattern, scale given by $1/p$.

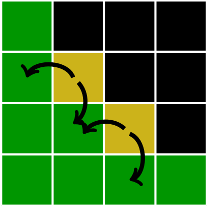
The Bak-Chen-Tang Forest Fire Model



- Originally by Bak, Chen and Tang (1990).
- Intended as a model of turbulence.
- Sites empty, occupied (by tree) or on fire.
- Slow regrowth at rate p.
- Occasional re-lighting.
- Grassberger and Kantz (1991):
  Deterministic pattern, scale given by $1/p$.

Imperial College London

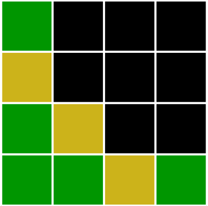
The Bak-Chen-Tang Forest Fire Model



- Originally by Bak, Chen and Tang (1990).
- Intended as a model of turbulence.
- Sites empty, occupied (by tree) or on fire.
- Slow regrowth at rate p.
- Occasional re-lighting.
- Grassberger and Kantz (1991):
  Deterministic pattern, scale given by $1/p$.

Imperial College London

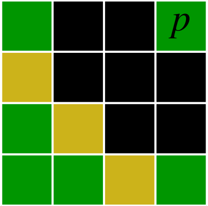
The Bak-Chen-Tang Forest Fire Model



- Originally by Bak, Chen and Tang (1990).
- Intended as a model of turbulence.
- Sites empty, occupied (by tree) or on fire.
- Slow regrowth at rate p.
- Occasional re-lighting.
- Grassberger and Kantz (1991):
  Deterministic pattern, scale given by $1/p$.

Imperial College London

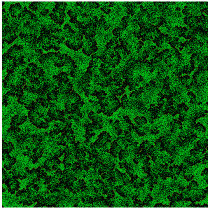
The Bak-Chen-Tang Forest Fire Model



- Originally by Bak, Chen and Tang (1990).
- Intended as a model of turbulence.
- Sites empty, occupied (by tree) or on fire.
- Slow regrowth at rate p.
- Occasional re-lighting.
- Grassberger and Kantz (1991):
  Deterministic pattern, scale given by $1/p$.

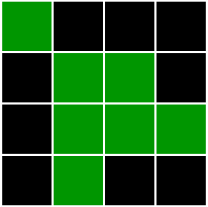
Imperial College London

The Bak-Chen-Tang Forest Fire Model



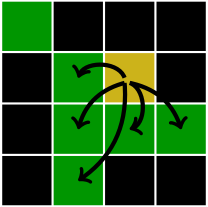
- Originally by Bak, Chen and Tang (1990).
- Intended as a model of turbulence.
- Sites empty, occupied (by tree) or on fire.
- Slow regrowth at rate p.
- Occasional re-lighting.
- Grassberger and Kantz (1991):
  Deterministic pattern, scale given by $1/p$.

The Drossel-Schwabl Forest Fire Model



- Originally by Henley (1989) and independently by Drossel and Schwabl (1992).
- Fires instantaneous, explicit lightning mechanism with θ trees grown between two lighntnings attempts.
- Grassberger (2002) and Pruessner and Jensen (2002): Not scale invariant.

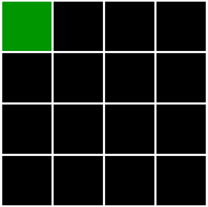
The Drossel-Schwabl Forest Fire Model



- Originally by Henley (1989) and independently by Drossel and Schwabl (1992).
- Fires instantaneous, explicit lightning mechanism with θ trees grown between two lighntnings attempts.
- Grassberger (2002) and Pruessner and Jensen (2002): Not scale invariant.

Imperial College London

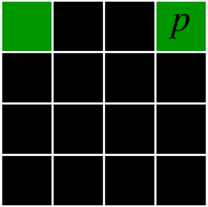
The Drossel-Schwabl Forest Fire Model



- Originally by Henley (1989) and independently by Drossel and Schwabl (1992).
- Fires instantaneous, explicit lightning mechanism with θ trees grown between two lighntnings attempts.
- Grassberger (2002) and Pruessner and Jensen (2002): Not scale invariant.

Imperial College London

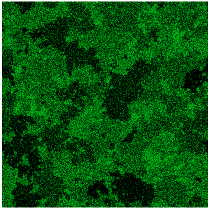
The Drossel-Schwabl Forest Fire Model



- Originally by Henley (1989) and independently by Drossel and Schwabl (1992).
- Fires instantaneous, explicit lightning mechanism with θ trees grown between two lighntnings attempts.
- Grassberger (2002) and Pruessner and Jensen (2002): Not scale invariant.

Imperial College London

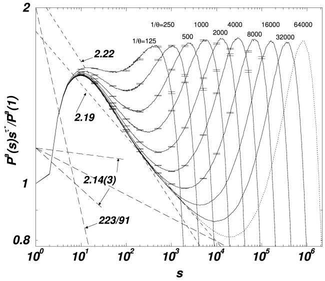
The Drossel-Schwabl Forest Fire Model



- Originally by Henley (1989) and independently by Drossel and Schwabl (1992).
- Fires instantaneous, explicit lightning mechanism with θ trees grown between two lighntnings attempts.
- Grassberger (2002) and Pruessner and Jensen (2002): Not scale invariant.

The Drossel-Schwabl Forest Fire Model

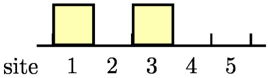
Lack of scaling



- Finite size not the only scale.
- Scale invariance possible only in the limit of $\theta \to \infty$.
- Lower cutoff moves as well.

Imperial College London

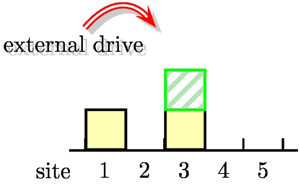
Manna Model



Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Envisaged to be in the same universality class as BTW.
- Robust, solid, universal, reproducible.
- Defines a universality class.

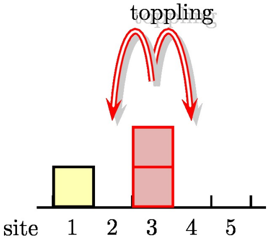
Manna Model



Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Envisaged to be in the same universality class as BTW.
- Robust, solid, universal, reproducible.
- Defines a universality class.

Manna Model

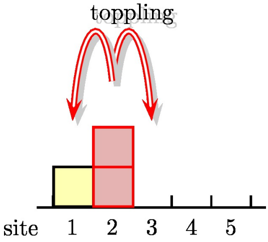


Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Envisaged to be in the same universality class as BTW.
- Robust, solid, universal, reproducible.
- Defines a universality class.

Imperial College London

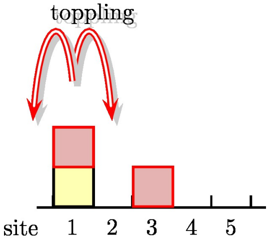
Manna Model



Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Envisaged to be in the same universality class as BTW.
- Robust, solid, universal, reproducible.
- Defines a universality class.

Manna Model

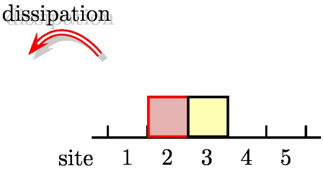


Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Envisaged to be in the same universality class as BTW.
- Robust, solid, universal, reproducible.
- Defines a universality class.

Imperial College London

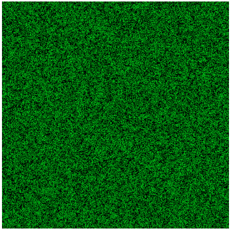
Manna Model



Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Envisaged to be in the same universality class as BTW.
- Robust, solid, universal, reproducible.
- Defines a universality class.

Manna Model

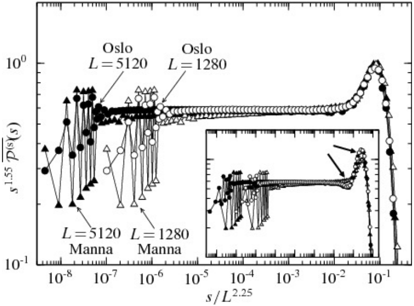


Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Envisaged to be in the same universality class as BTW.
- Robust, solid, universal, reproducible.
- Defines a universality class.

Imperial College London

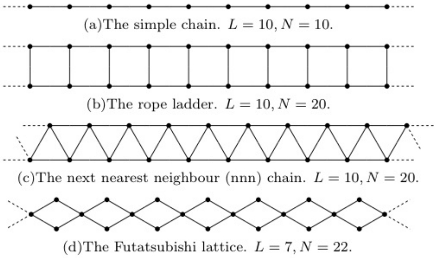
Collapse with Oslo



The Manna Model is in the same universality class as the Oslo model.

Manna on different lattices

One and two dimensions



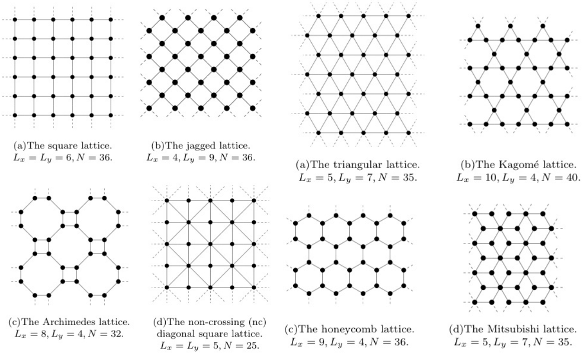
From: Huynh, G P, Chew, 2011

The Manna Model has been investigated numerically in great detail.

Imperial College London

Manna on different lattices

One and two dimensions



From: Huynh, G P, Chew, 2011

The Manna Model has been investigated numerically in great detail.

Manna on different lattices

One and two dimensions

lattice	d	D	τ	z	α	D_a	τ_a	$\mu_1^{(s)}$	$-\Sigma_s$	$-\Sigma_t$	$-\Sigma_a$
simple chain	1	2.27(2)	1.117(8)	1.450(12)	1.19(2)	0.998(4)	1.260(13)	2.000(4)	0.27(2)	0.27(3)	0.259(14)
rope ladder	1	2.24(2)	1.108(9)	1.44(2)	1.18(3)	0.998(7)	1.26(2)	1.989(5)	0.24(2)	0.26(5)	0.26(2)
nnn chain	1	2.33(11)	1.14(4)	1.48(11)	1.22(14)	0.997(15)	1.27(5)	1.991(11)	0.33(11)	0.3(2)	0.27(5)
Futatsubishi	1	2.24(3)	1.105(14)	1.43(3)	1.16(6)	0.999(15)	1.24(5)	2.008(11)	0.24(3)	0.23(9)	0.24(5)
square	2	2.748(13)	1.272(3)	1.52(2)	1.48(2)	1.992(8)	1.380(8)	1.9975(11)	0.748(13)	0.73(4)	0.76(2)
jagged	2	2.764(15)	1.276(4)	1.54(2)	1.49(3)	1.995(7)	1.384(8)	2.0007(12)	0.764(15)	0.76(5)	0.77(2)
Archimedes	2	2.76(2)	1.275(6)	1.54(3)	1.50(3)	1.997(10)	1.382(11)	2.001(2)	0.76(2)	0.78(6)	0.76(3)
nc diagonal square	2	2.750(14)	1.273(4)	1.53(2)	1.49(2)	1.992(7)	1.381(8)	2.0005(12)	0.750(14)	0.75(4)	0.76(2)
triangular	2	2.76(2)	1.275(5)	1.51(2)	1.47(3)	2.003(11)	1.388(12)	1.997(2)	0.76(2)	0.71(6)	0.78(3)
Kagomé	2	2.741(13)	1.270(4)	1.53(2)	1.49(2)	1.993(8)	1.381(9)	1.9994(12)	0.741(13)	0.75(5)	0.76(2)
honeycomb	2	2.73(2)	1.268(6)	1.55(4)	1.51(4)	1.990(13)	1.376(14)	2.000(2)	0.73(2)	0.79(8)	0.75(3)
Mitsubishi	2	2.75(2)	1.273(6)	1.54(3)	1.50(4)	1.999(12)	1.387(12)	1.998(2)	0.75(2)	0.77(7)	0.77(3)

From: Huynh, G P, Chew, 2011

The Manna Model has been investigated numerically in great detail.

Imperial College London

Manna on different lattices

Three dimensions

Lattice	$\overline{q}$	$\overline{q^{(v)}}$	$\langle z \rangle$	D	τ	z	α	D_a	τ_a	$\mu_1^{(s)}$	$-\Sigma_s$	$-\Sigma_t$	$-\Sigma_a$
SC	6	1	[0.622325(1)]	3.38(2)	1.408(3)	1.779(7)	1.784(9)	3.04(5)	1.45(4)	2.0057(5)	1.38(2)	1.395(16)	1.36(13)
BCC	8	4	[0.600620(2)]	3.36(2)	1.404(4)	1.777(8)	1.78(1)	2.99(2)	1.444(18)	2.0030(5)	1.36(2)	1.390(19)	1.33(6)
BCCN	14	5	[0.581502(1)]	3.38(3)	1.408(4)	1.776(9)	1.783(11)	3.01(3)	1.44(3)	2.0041(6)	1.38(3)	1.39(2)	1.32(7)
FCC	12	4	[0.589187(3)]	3.35(4)	1.402(8)	1.765(16)	1.78(2)	3.1(2)	1.48(14)	2.0035(11)	1.35(4)	1.37(4)	1.5(5)
FCCN	18	5	[0.566307(3)]	3.38(4)	1.408(7)	1.781(14)	1.787(18)	3.00(4)	1.44(3)	2.0051(8)	1.38(4)	1.40(3)	1.32(9)
Overall				3.370(11)	1.407(2)	1.777(4)	1.783(5)	3.003(14)	1.442(12)	2.0042(3)		1.380(13)	

From: Huynh, G P, 2012

The Manna Model has been investigated numerically in great detail.

Outline

1 SOC: Past and Present

2 Universality Classes
- Early themes
- Relevant fields
- Universality classes

3 Theory of SOC

4 Summary: Any Answers?

Imperial College London

Early themes

- Initially the BTW Model was conceived as the paradigm of SOC and maybe the SOC universality class.
- Zhang and Manna Models were initially suggested to be in that BTW/SOC universality class.
- Starting from the mid-ninties, new universality classes proposed.
- Universality requires (some) robustness.

Dividing lines between models

The following features are generally considered as relevant fields:[1]

- stochastic vs deterministic
- directed vs undirected (isotropy generally)
- Abelian vs non-Abelian (note initial confusion of stochastic=non-Abelian)
- conservative vs non-conservative

Most observations made in variations of BTW and Manna Models.

[1] *e.g.* Ben-Hur and Biham, 1996; Milshtein, Biham, Solomon, 1998; Karmakar, Manna, Stella, 2005

Imperial College London

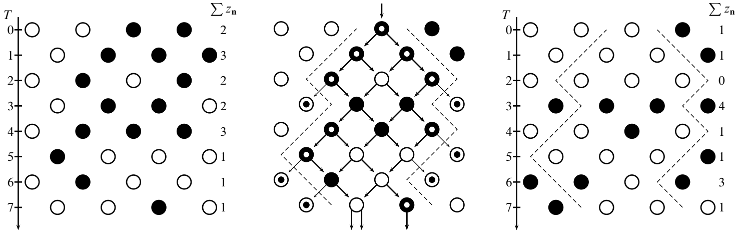
Universality classes

Widely accepted universality classes are:

- **Directed sandpiles (stochastic and deterministic).**
- **Manna universality class in $d = 1, 2, 3, 4$, free above.**
- BTW (multiscaling) in $d = 2, 3, 4$ (free above?), includes possibly the Zhang Model.
- OFC Model (somewhat robust if conservative, class of its own?).
- Forest Fire Model (not robust, class of its own?).
- Bak-Sneppen Model (not robust, class of its own?).

Imperial College London

Directed Models

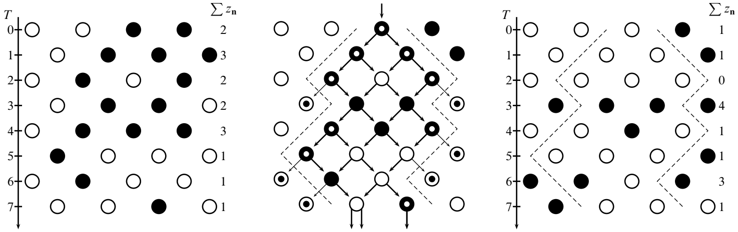


From Pruessner 2012, p.287

- Classic representative: Dhar Ramaswamy-Model (1989).
- Typically solved by mapping to random walker (time is equivalent to one spatial dimension, $d = d_\perp + 1$).
- Exact solutions and controlled approximations.
- $d_\perp = 0, 1, 2$, upper critical dimension is $d_\perp = 2$.

Imperial College London

Directed Models



From Pruessner 2012, p.287

- Plethora of models.
- Two classes: Random distribution to downstream neighbours vs deterministic distribution to downstream neighbours.
- Directedness results in no (or short-ranged or trivial) spatial correlations.
- Fully characterised (Dhar and Ramaswamy, 1989; Paczuski and Bassler, 2000; Bunzarova 2010).

The Manna Universality Class

- The only large universality class in SOC.
- Includes large number of models, which seemingly are very different.
- Spatially isotropic.
- Numerically characterised in $d = 1, 2, 3, 4, 5$ (*e.g.* Luebeck and Heger, 2003).
- Little known analytically, no proper mean field theory.

Imperial College London

Outline

1. SOC: Past and Present
2. Universality Classes
3. Theory of SOC
   - Tools in SOC
   - The Absorbing State Mechanism
   - Field theory for SOC
   - The SOC mechanism
4. Summary: Any Answers?

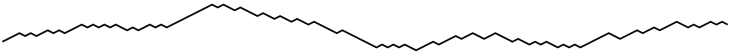
Imperial College London

Tools in SOC

- (Extensive) numerics (BTW, FFM, BS, Manna, Oslo).
- Analytical tools:
    - Exact solutions (so far: directed models only).
    - Mappings to known (understood?) phenomena.
    - Growth processes and field theories.

Link to growth phenomena (generic scale invariance)

Stochastic evolution of sandpile surface.



$$\partial_t \phi(\mathbf{r}, t) = (\nu_{||} \partial_{||}^2 + \nu_\perp \partial_\perp^2) \phi + \eta(\mathbf{r}, t)$$

- *Generic* scale invariance (Hwa and Kardar, 1989, and Grinstein, Lee and Sachdev 1990)
- No mass term $-\epsilon\phi$ on the right $\longrightarrow$ conservative dynamics (finiteness generates ϵ).
- Anisotropy (boundaries?) required in the presence of conserved noise.
- Non-trivial exponents in the presence of non-linearities and non-conserved noise.
- Concept abandoned with the arrival of non-conservative

Imperial College London

Effect of a mass term

Mass term

$$\partial_t \phi = \nu \nabla^2 \phi - \epsilon \phi + \ldots + \eta$$

represents disspation

$$\partial_t \int_V \mathrm{d}^d x \, \phi = \text{surface terms} - \epsilon \int_V \mathrm{d}^d x \, \phi$$

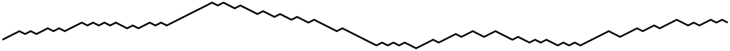
and correlation length

$$\phi = \ldots \mathrm{e}^{-|x|\sqrt{\epsilon/\nu}} \, .$$

But: How can a renormalised $\epsilon = 0$ be maintained without trivialising (no additive renormalisation, $\epsilon = 0$ is the critical point in mean field) the phenomenon?

Field theories for Manna and Oslo

Number of charges interpreted as an interface.



- **Manna model** has a (weird!) Langevin equation.
- **Oslo model** implements **quenched Edwards Wilkinson equation** $\longrightarrow$ interfaces!
- Field theories for both still investigated.
- Mechanism of self-organisation still investigated.
- Link to known universality classes.
- Link to **directed percolation**?

Imperial College London

The Absorbing State Mechanism

Dickman, Vespignani, Zapperi 1998

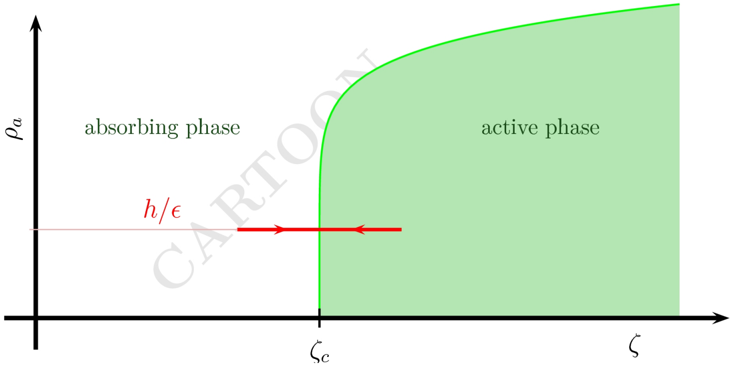
- SOC model: activity ρ_a leads to dissipation
- dissipation reduces particle density ζ
- density is reduced until system is inactive
  $\longrightarrow$ absorbing phase
- external drive increases particle density
  $\longrightarrow$ back to active phase

An SOC model can be seen as an AS model that drives itself into the inactive phase by dissipation ϵ and is pushed back into the active phase by external drive h.

$$\dot{\zeta} = h - \epsilon\rho_a \xrightarrow{\text{stationarity}} \rho_a = h/\epsilon$$

Imperial College London

The Absorbing State Mechanism

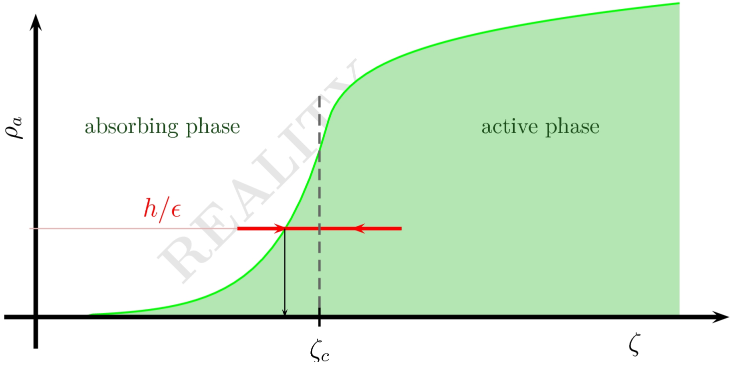


Idea: SOC drives $h/\epsilon = \rho_a$ to 0 as $L \to \infty$

Leading orders: $h(L) = h_0 L^{-\omega}$ and $\epsilon(L) = \epsilon_0 L^{-\kappa}$

Imperial College London

The Absorbing State Mechanism



Problem: SOC exponents would be affected by the way how driving and dissipation are implemented $\longrightarrow$ no universality.

Fey, Levine and Wilson suggest that critical point is not reached.

Outline

1. SOC: Past and Present
2. Universality Classes
3. Theory of SOC
   - Tools in SOC
   - The Absorbing State Mechanism
   - Field theory for SOC
   - The SOC mechanism
4. Summary: Any Answers?

Imperial College London

Field theory for SOC

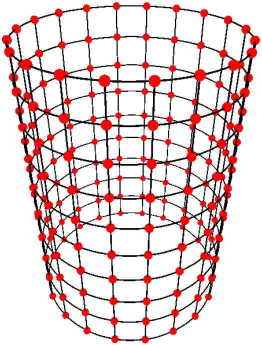
The Manna Model

Field theoretic formulation of the time evolution of the Manna Model.
Note: Before taking any limits, this theory is *exact*.

- Continuum limit
- Simplify. . .
- Diagrams (meaning?, process?, tree level?)
- Renormalisation

Simplification of the field theory

Bare propagators from field theory by inspection.
Simplification by considering periodic boundary conditions in $d-1$ directions. Surface appears in only one dimension.



Imperial College London

Bare propagators

 $= \dfrac{1}{-\imath\omega + D(\mathbf{k}^2 + q_n^2)}$

where $q_n = \frac{\pi}{L} n$ with $n = 1, 2, \ldots$

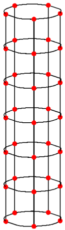
- $d-1$ dimensions can be treated the "usual" way.
- Usually, the gap in the propagator is the mass r_0 in

$$\frac{1}{-\imath\omega + D(\mathbf{k}^2 + r_0)}$$

  found by evaluating the inverse propagator at minimal momentum and frequency magnitude, $\mathbf{k} = 0$ and $\omega = 0$.
- Here, the gap is set by the minimum magnitude of q_n allowed. The effective mass is $q_1^2 = (\pi/L)^2$.

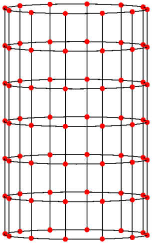
Bare propagators

Consider the system size as the effective mass of the system.
Expect convergence as circumference is increased; critical point controlled by height (L) only.



Bare propagators

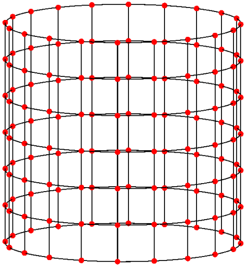
Consider the system size as the effective mass of the system. Expect convergence as circumference is increased; critical point controlled by height (L) only.



Imperial College London

Bare propagators

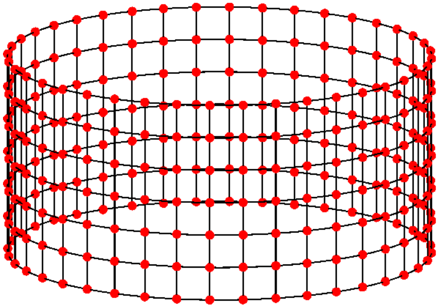
Consider the system size as the effective mass of the system.
Expect convergence as circumference is increased; critical point controlled by height (L) only.



Imperial College London

Bare propagators

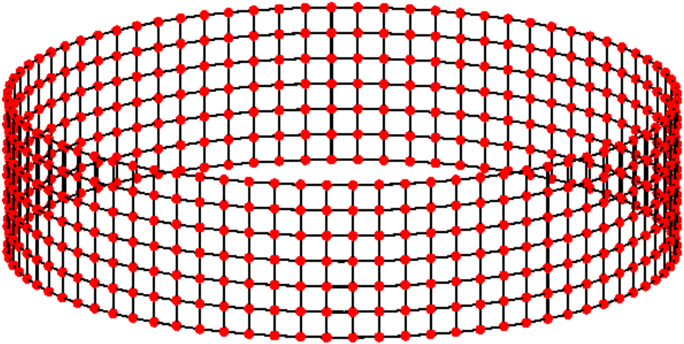
Consider the system size as the effective mass of the system. Expect convergence as circumference is increased; critical point controlled by height (L) only.



Imperial College London

Bare propagators

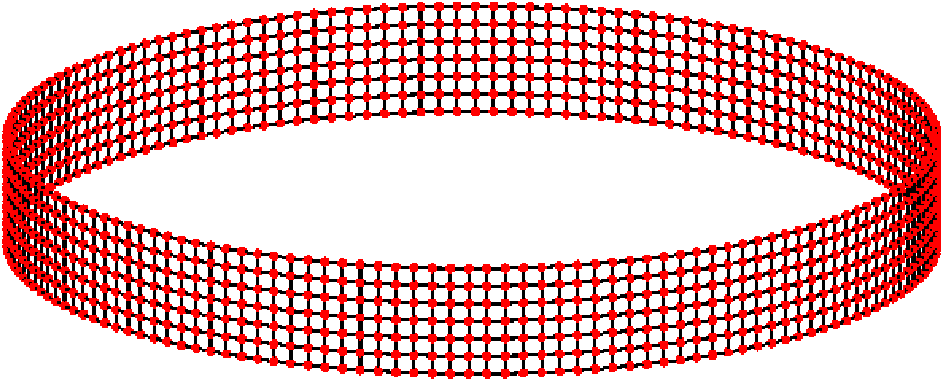
Consider the system size as the effective mass of the system.
Expect convergence as circumference is increased; critical point controlled by height (L) only.



Imperial College London

Bare propagators

Consider the system size as the effective mass of the system.
Expect convergence as circumference is increased; critical point controlled by height (L) only.



Bare propagators

Exact first moments

Circumference does not enter into first moment.
Avalanche size: Total activity (total number of charges).
In one dimension (continuum limit):

$$\langle s \rangle = \frac{1}{6} L^2$$

and $\langle s \rangle = \frac{1}{6}(L+1)(L+2)$ discretely. In higher dimensions:

$$\langle s \rangle = \frac{d}{6} L^2$$

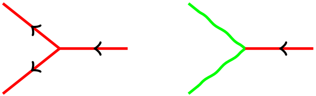
and $\langle s \rangle = \frac{d}{6}(L+1)(L+2)$ discretely.

Imperial College London

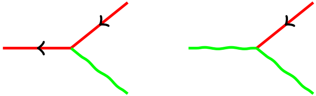
Vertices

The interaction vertices are

- Spontaneous branching and substrate deposition:



- Substrate interaction resulting in attenuation or deposition:

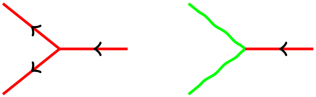


All relevant for $d \leqslant d_c = 4$. Loops occur.

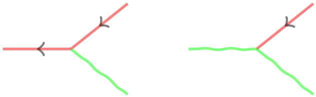
Vertices

The interaction vertices are

- Spontaneous branching and substrate deposition:



- Substrate interaction resulting in attenuation or deposition:

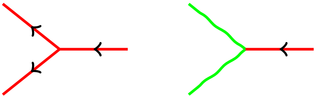


Only the former are relevant for $d > d_c = 4$; as in ϕ^4 the latter enter only for the lowest mode. No loops.

Imperial College London

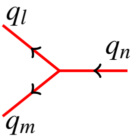
Tree level

Tree level becomes exact above $d_c = 4$. Two vertices are relevant there:



For example:

$$\langle s^2 \rangle = 2 \left(\frac{2}{L}\right)^3 \sum_{\substack{n,m,l \\ \text{odd}}} \frac{4}{q_l q_m}$$



$$\frac{2}{q_n} = \frac{d^3}{140} L^6$$

Higher order moments follow similarly.

Imperial College London

Tree level

Comparison to numerics

Tree level moments can be compared to the numerics of the Manna Model at $d > 4$, here $d = 5$:

Observable	analytical	numerical (leading order)
$\langle s \rangle$	$(d/6)L^2 = 0.833\ldots L^2$	$0.83334(6)L^2$
$\langle s \rangle \langle s^3 \rangle / \langle s^2 \rangle^2$	$3.08754\ldots$	$3.111(11)$
$\langle s^2 \rangle \langle s^4 \rangle / \langle s^3 \rangle^2$	$1.6693\ldots$	$1.70(3)$

Note: Numerical fitting pretty *ad hoc*.

Imperial College London

Tree level: Mean Field Theory

The process corresponding to tree level is the *effective* mean field theory of the Manna Model (random walk, not space-less!).
Parameters are self-organised (see below).

- For that process, avalanche moments can be calculated easily[2] directly (not via the field theory).
- Results coincide with those from field theory and numerics in $d = 5$.

This mean field theory identifies precisely the correlations and fluctuations to be ignored. Not an *ad-hoc* approximantion.
Mean field theories in SOC are usually effective theories of certain observables and do not incorporate space at any level.

Imperial College London

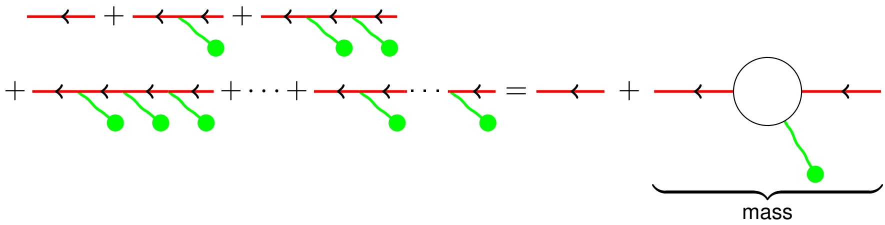
[2] Mathematica takes care of the mess

The SOC mechanism

How does SOC work?

$\longrightarrow$ Organisation to the critical point? Why are the propagators massless?

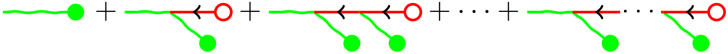
Mass is attenuation (loss of activity). At tree level:



The SOC mechanism

How does SOC work?

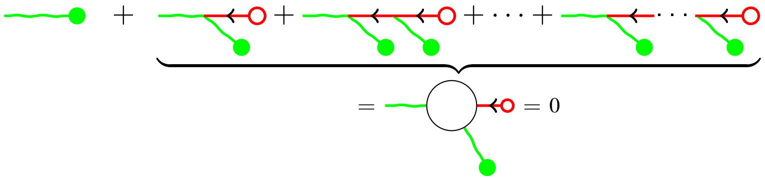
Attenuation leads to deposition by the external drive — diagrams have that symmetry.
Density of particles in the substrate:



The SOC mechanism

How does SOC work?

Attenuation leads to deposition by the external drive — diagrams have that symmetry.
Density of particles in the substrate:



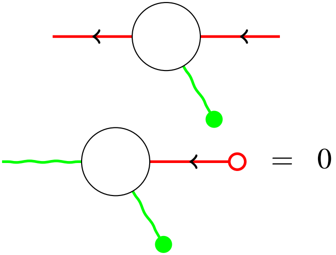
Additional deposition by external drive vanishes at stationarity.

The SOC mechanism

How does SOC work?

Mass:

Additional deposition:



Only difference between the two diagrams: Left most vertex (coupling identical at renormalised and bare level).

Imperial College London

The SOC mechanism

So how does it work then?

- Activity attenuation is mass.
- Conservation links attenuation to (additional) substrate deposition...
- or equivalently, symmetry of vertices equates mass terms of activity and substrate deposition terms.
- Additional substrate deposition vanishes *as we choose to consider stationarity.*
- Terms and conditions apply...

Issue: Deposition without attenuation, by seemingly conservative terms.

Imperial College London

The SOC mechanism

So how does it work then?

- **Stationarity causes criticality.**
  (qualification of Hwa and Kardar: Masslessness by conservation).
- Conservation is secondary to stationarity (links attenuation and deposition, the latter being stationary) — non-conservative SOC is possible!
- (Ward-Takahashi) symmetry of diagrams produces for self-tuning.
- Shift of stationary particle density understood.
- Innocent looking processes (such as “catalytic” diffusion in substrate) destroy critical state.
- Relation to absorbing state mechanism unclear.

Imperial College London

Summary: Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Probably: Superconductors, granular media, earthquakes, precipitation
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Jury is still out.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Imperial College London

Summary: Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Probably: Superconductors, granular media, earthquakes, precipitation
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Jury is still out.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Imperial College London

Summary: Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Probably: Superconductors, granular media, earthquakes, precipitation
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Jury is still out.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Summary: Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Probably: Superconductors, granular media, earthquakes, precipitation
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Jury is still out.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Imperial College London

Summary: Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Probably: Superconductors, granular media, earthquakes, precipitation
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Jury is still out.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Imperial College London

Summary: Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Probably: Superconductors, granular media, earthquakes, precipitation
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Jury is still out.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Imperial College London

Summary: Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Probably: Superconductors, granular media, earthquakes, precipitation
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Jury is still out.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Imperial College London

Summary: Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Probably: Superconductors, granular media, earthquakes, precipitation
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Jury is still out.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Imperial College London

Summary: Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Probably: Superconductors, granular media, earthquakes, precipitation
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Jury is still out.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Summary: Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Probably: Superconductors, granular media, earthquakes, precipitation
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Jury is still out.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!