Shear modulus caused by stress avalanches for jammed granular materials under oscillatory shear

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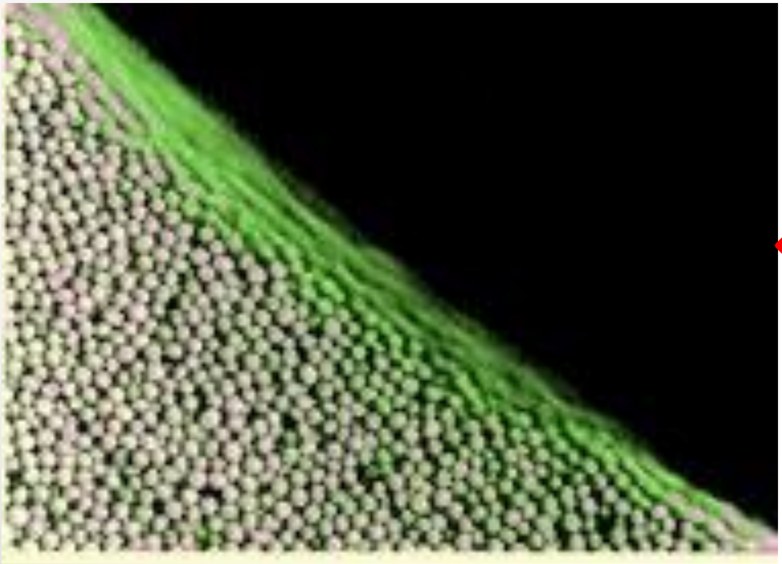
2014/1029 **Avalanches, Intermittency, and Nonlinear Response in Far-From-Equilibrium Solids** at KITP, UCSB,USA

Contents

- Introduction for jamming transition and shear modulus
- Simulation
  - Movie, crossover, scaling plots
- How can we understand exponents?
- Discussion and conclusion

Introduction

- Granular materials behaves as unusual solids and liquids.

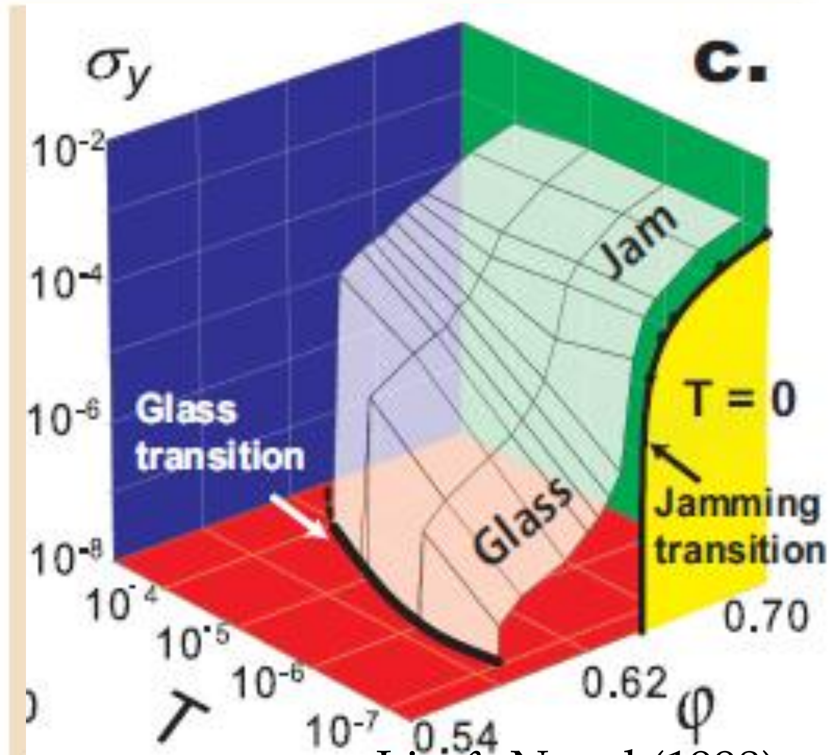


Flow of mustard seeds @Chicago group

Kamigamo shrine (Kyoto!)

Jamming transition

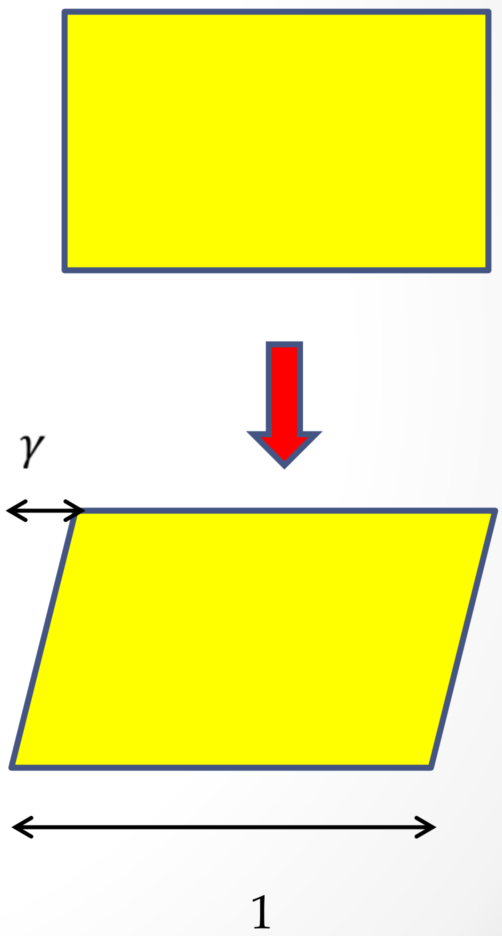
- Granular materials cannot flow above a critical density.
- Above the critical density, the granules has rigidity and behaves as a solid.
- This transition is known as the jamming transition.



Liu & Nagel (1998)
Ikeda-Berthier-Sollich (2012)

Characterization of jamming

- Rigidity of jammed solid is characterized by the shear modulus, $G = S/\gamma$, where S is the shear stress.
- (Storage) modulus becomes nonzero above the jamming transition.



G above the jamming

- The shear modulus is believed to behave as

$$G \sim (\phi - \phi_J)^{1/2}$$

where ϕ and ϕ_J are the volume fraction and the jamming fraction (O'Hern et al. 2002).

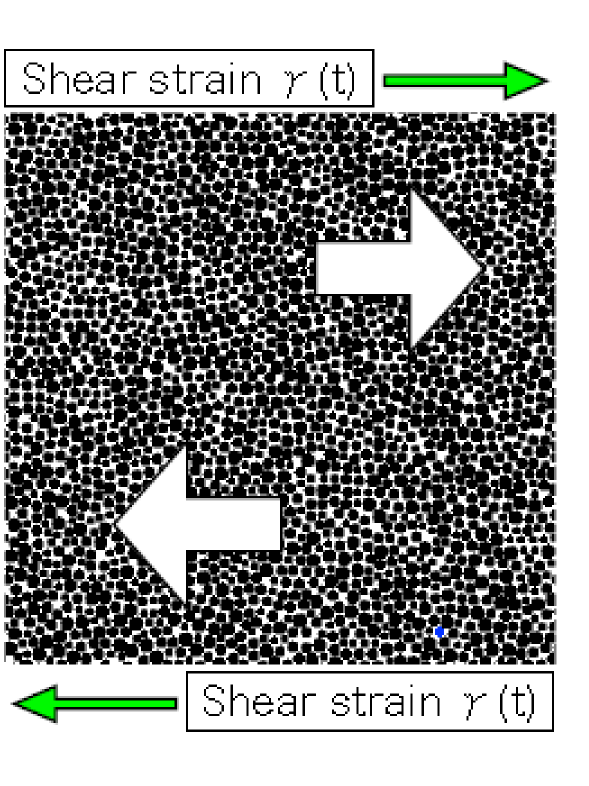
- Some people suggested a different scaling

$$G \sim \gamma_0^{-c}(\phi - \phi_J)$$ c=1/2

(Mason et al., PRE 1996, Okamura & Yoshino 2013, Coulais, Seguin & Dauchot, PRL 2014) .

Purpose

- We would like to clarify the relationship between two different scalings.
- For this purpose, we perform simulation of frictionless granular particles under oscillatory shear.
- See, M. Otsuki & HH, PRE 90, 042202 (2014).



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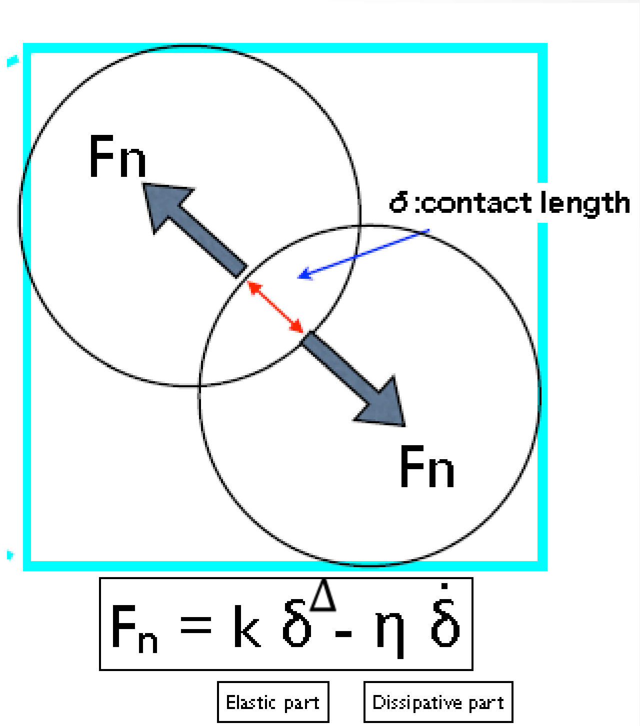
Simulation setup

- Number of grains 16,000.
- Linear or Hertzian spring model
- Shear strain

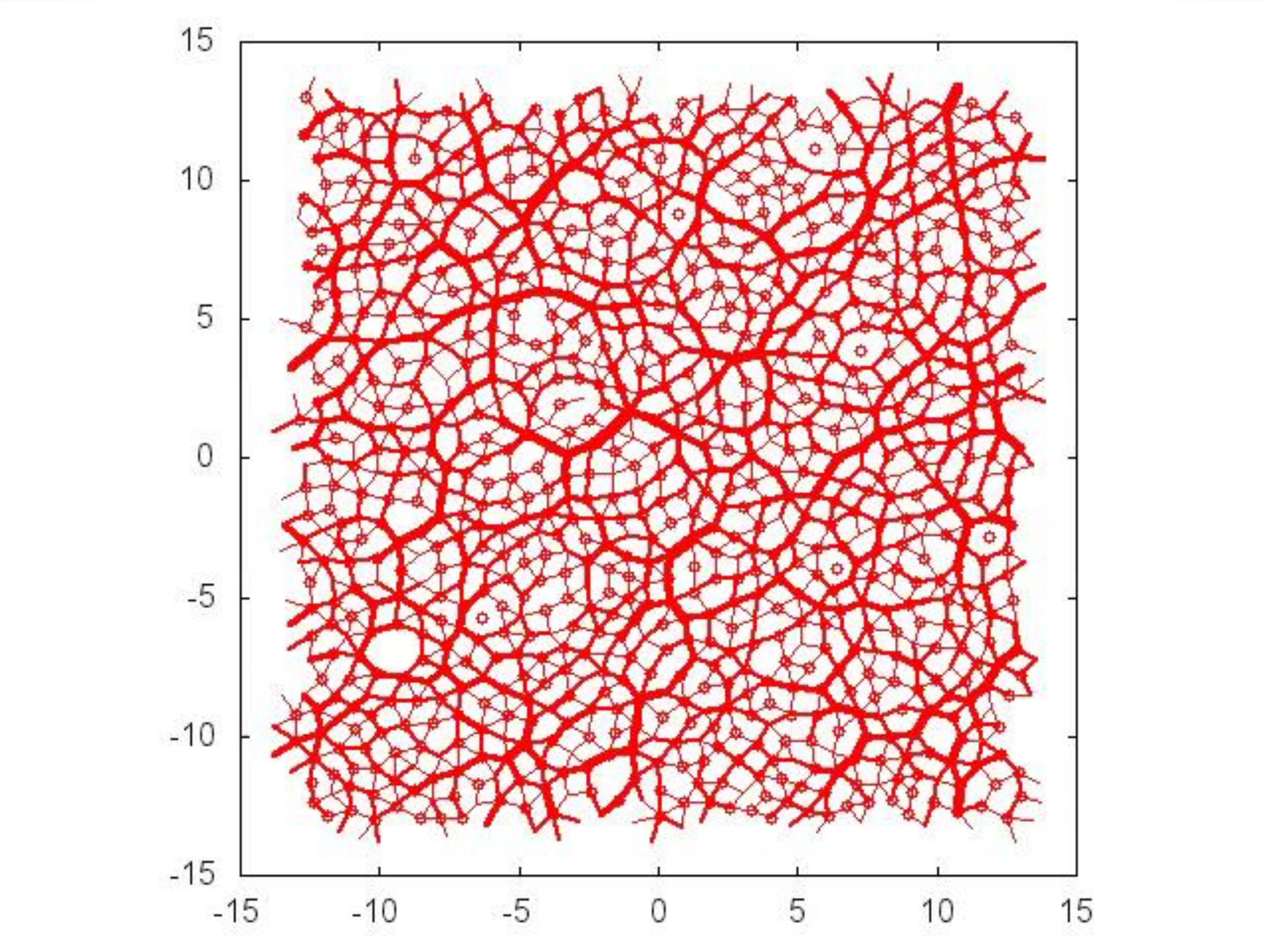
$$\gamma(t) = \gamma_0 \{1 - \cos(\omega t)\}$$



$$\frac{d\boldsymbol{r}_i}{dt} = \frac{\boldsymbol{p}_i}{m} + \dot{\gamma}(t) y_i \boldsymbol{e}_x,$$

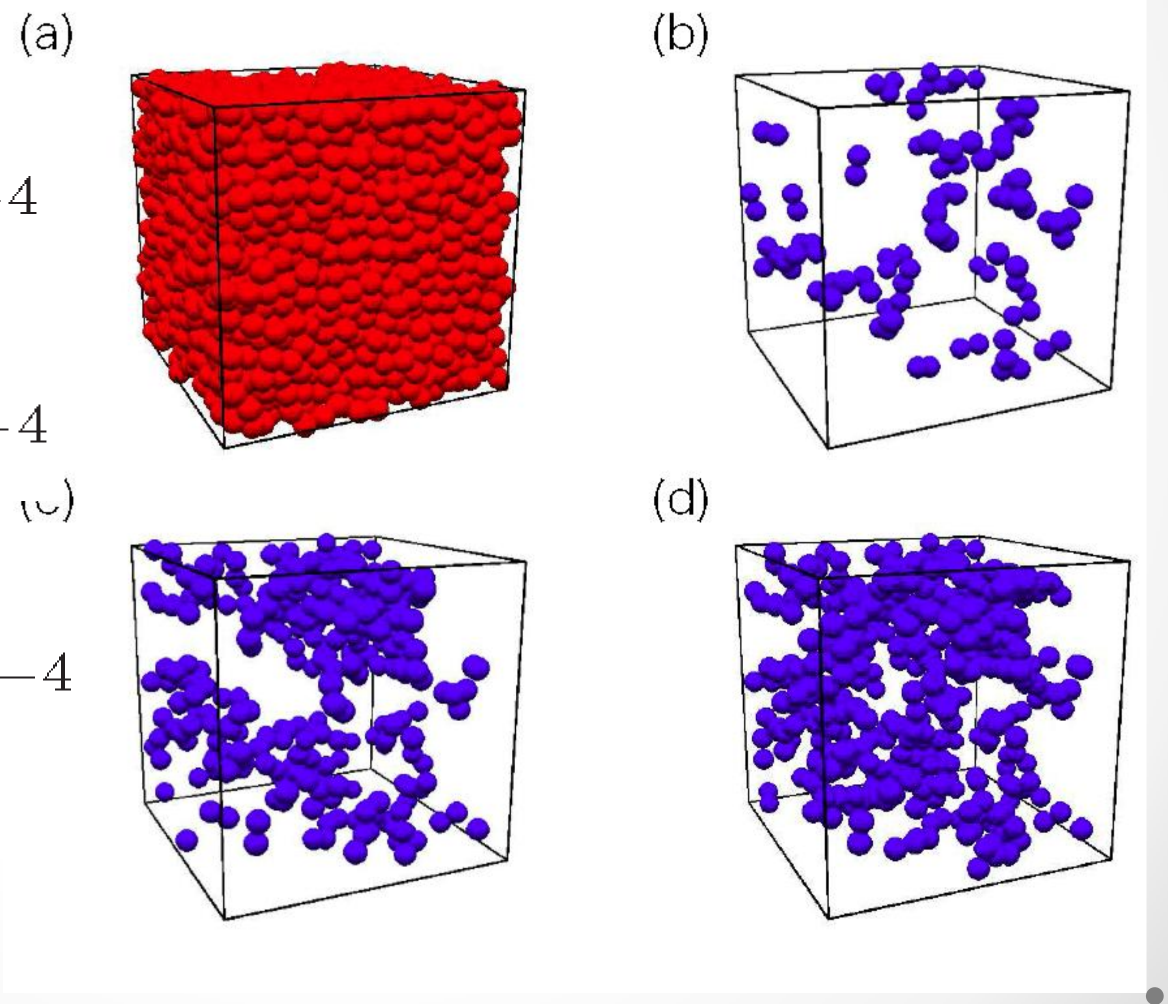
$$\frac{d\boldsymbol{p}_i}{dt} = \sum_{j \neq i} \{\boldsymbol{f}_{ij}^{(\mathrm{el})} + \boldsymbol{f}_{ij}^{(\mathrm{dis})}\} - \dot{\gamma}(t) p_{i,y} \boldsymbol{e}_x,$$

Simulation movie



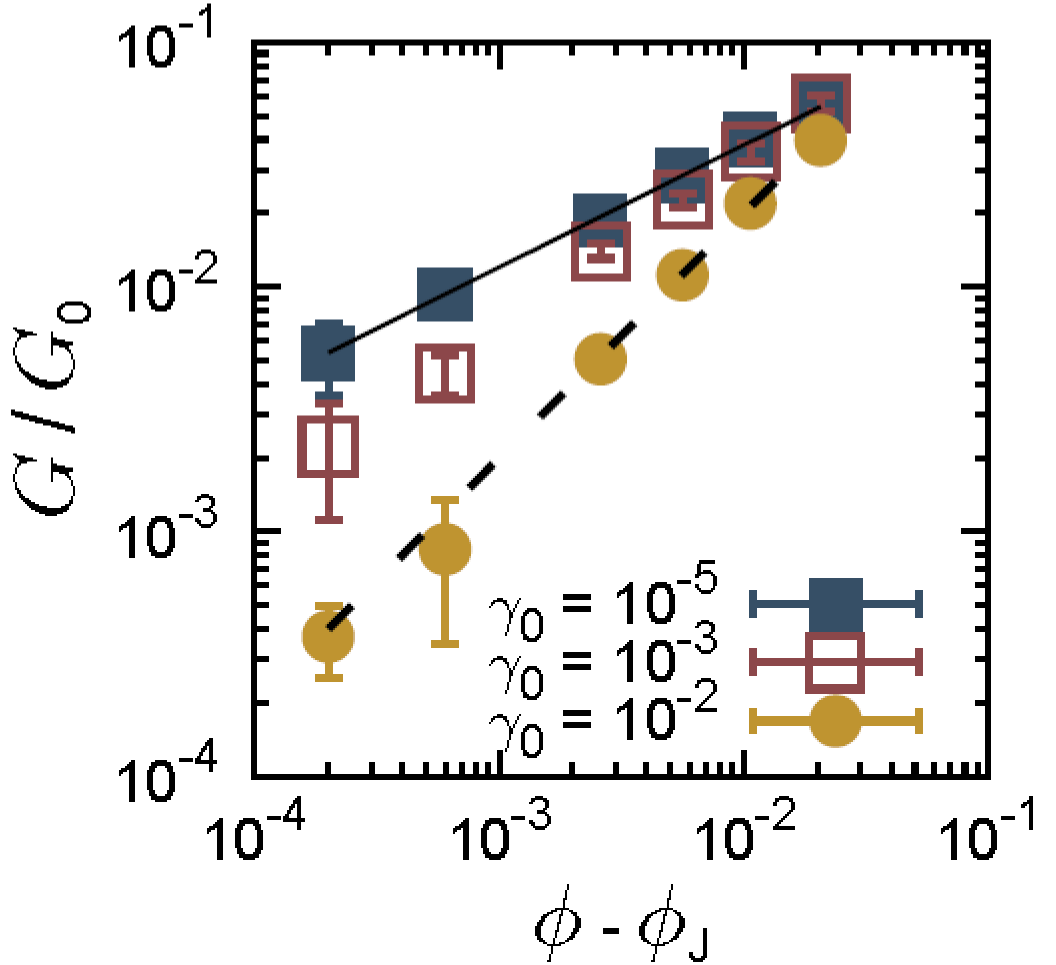
Avalanches in simulation

- (a) $\gamma = 0$
- (b) $\gamma = 1.2 \times 10^{-4}$
- (c) $\gamma = 4.8 \times 10^{-4}$
- (d) $\gamma = 7.5 \times 10^{-4}$



Storage modulus

- Storage modulus strongly depends on the amplitude of oscillatory shear.



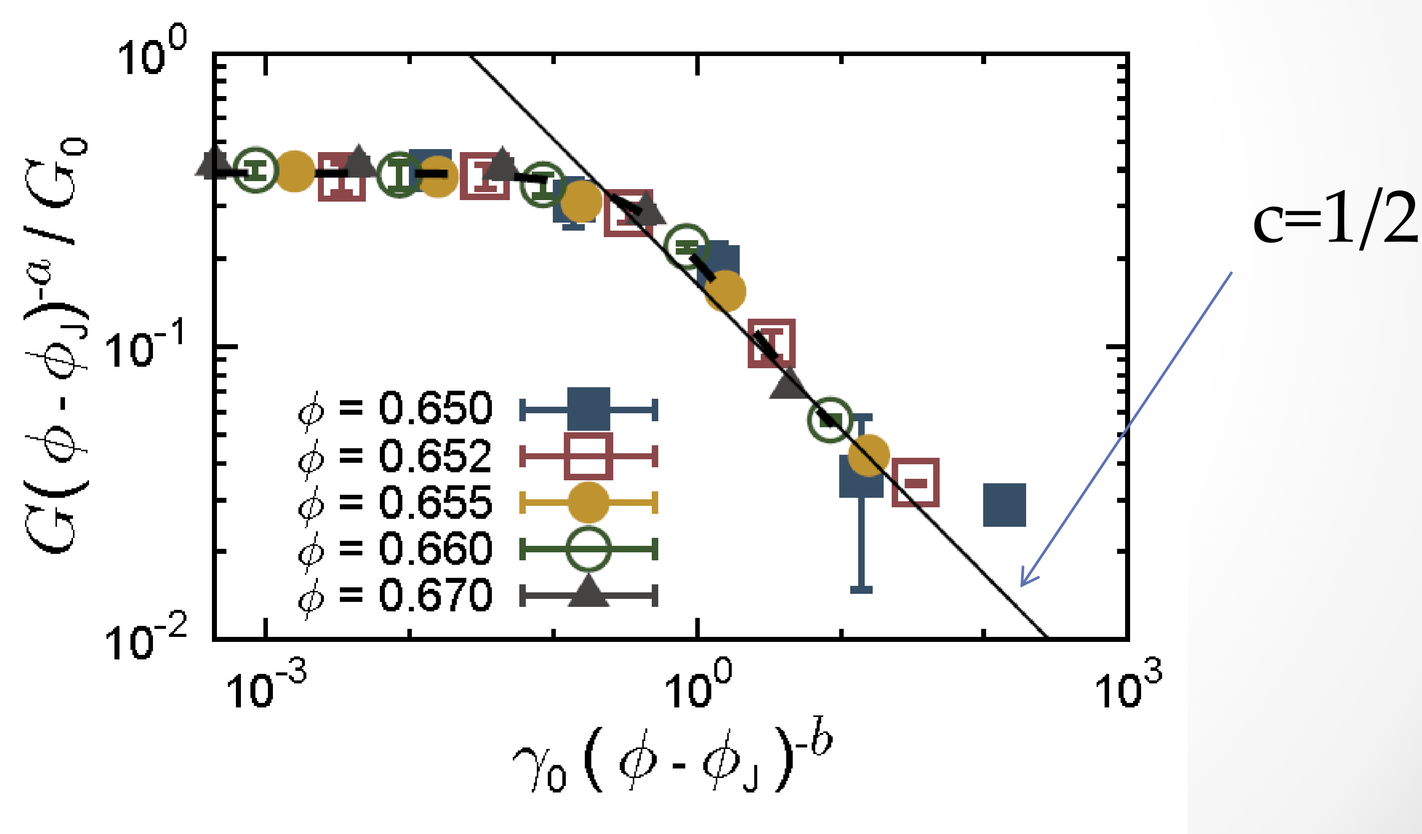
Scaling ansatz

$$G(\phi, \gamma_0) = G_0(\phi - \phi_J)^a \mathcal{G}\left(\gamma_0/(\phi - \phi_J)^b\right)$$

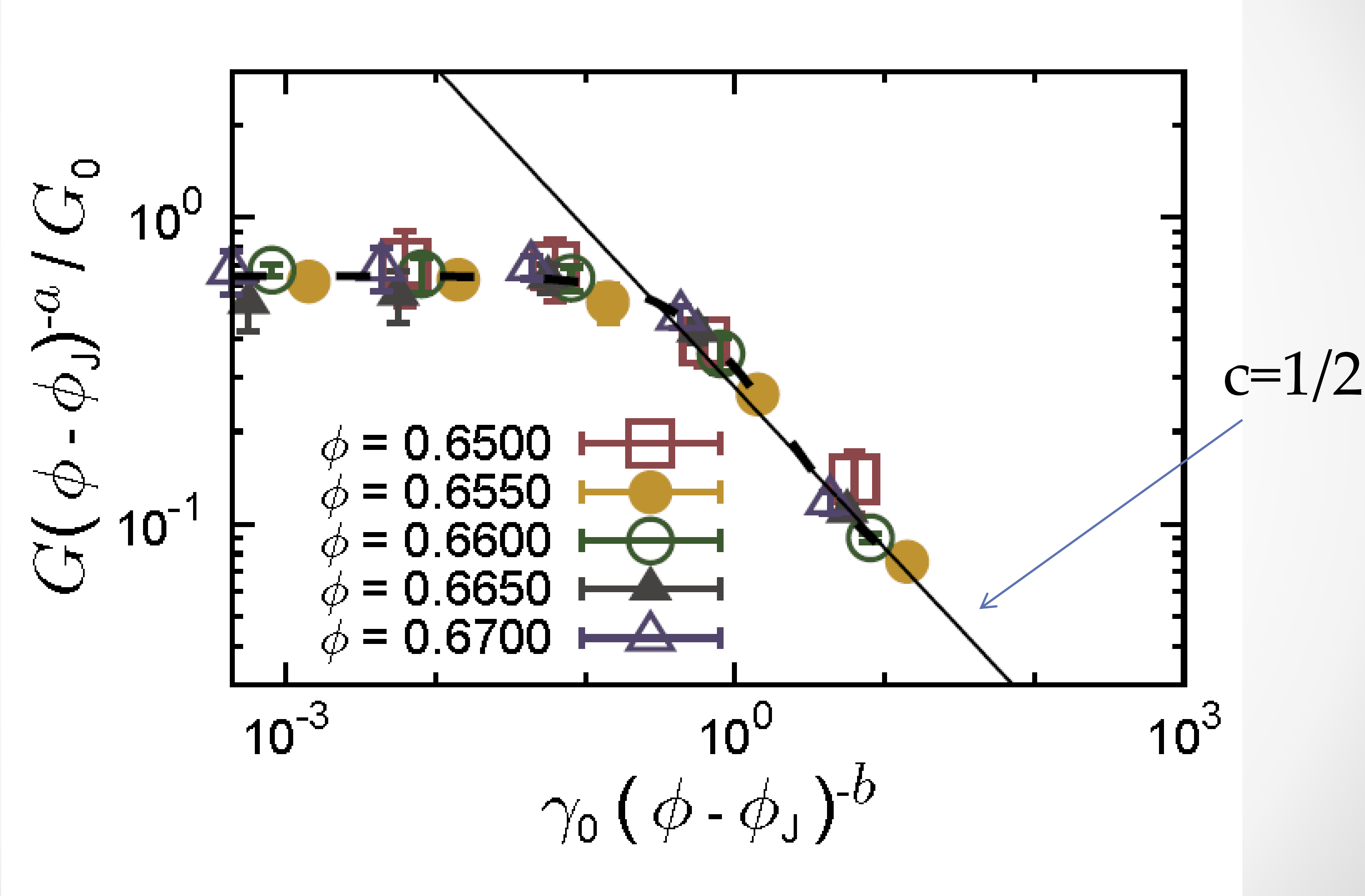
$$\lim_{x \to 0} \mathcal{G}(x) = \text{const.}, \qquad \lim_{x \to \infty} \mathcal{G}(x) = x^{-c}.$$

Scaling for linear spring

$$a = 0.50 \pm 0.02, \quad b = 0.98 \pm 0.02.$$



Scaling for Hertzian model



$$a = 0.99 \pm 0.02, \quad b = 0.98 \pm 0.01$$

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Exponent a

$$f \sim k_{\text{eff}}\delta \sim \delta^{\Delta}$$

$$k_{\text{eff}} \sim \delta^{\Delta-1}$$

$$\delta \sim \phi - \phi_J$$

$$G/k_{eff} \sim \delta z \sim \sqrt{\phi - \phi_J}$$

$$G \sim (\phi - \phi_J)^{\Delta-1/2}$$

$$a = \Delta - 1/2$$

Stress avalanche and exponent c

- Simulation suggests $c = 1/2$

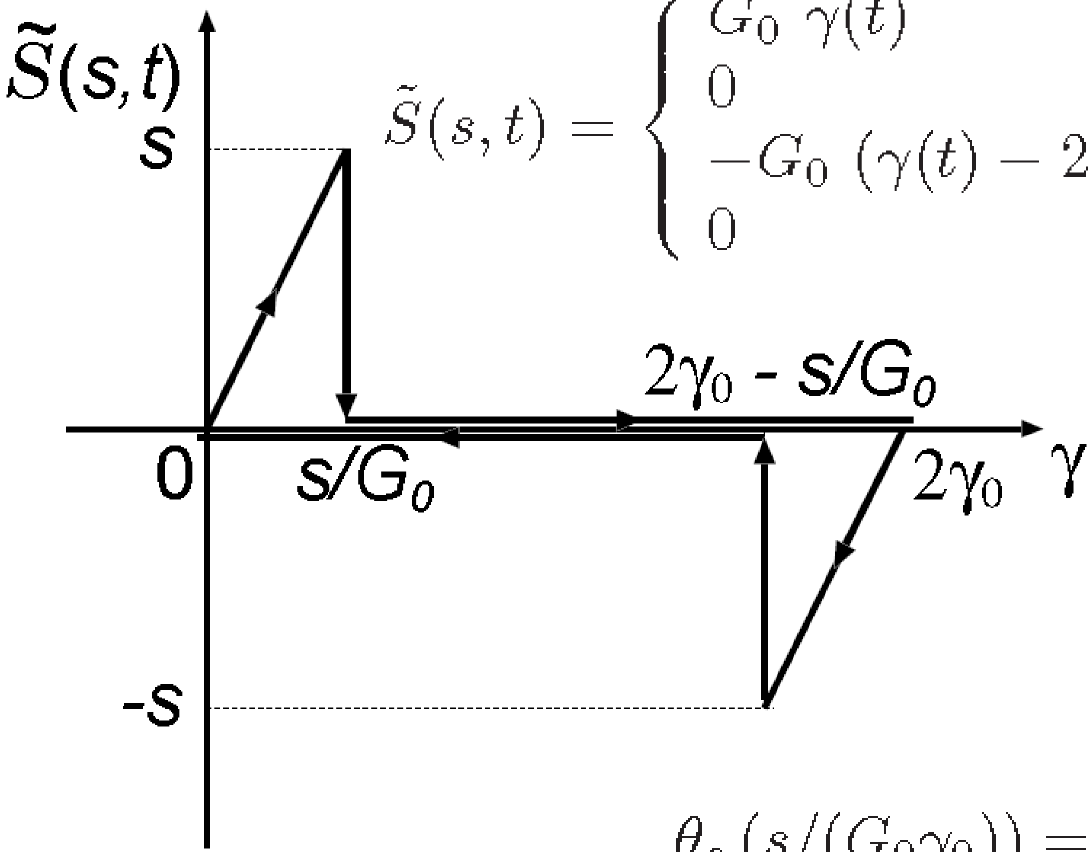
- We may understand this from stress avalanches

$$G = \int_0^\infty ds \tilde{G}(\gamma_0, s)\rho(s)$$

$\tilde{G}(\gamma_0, s)$ Shear modulus of an individual element of stress drop s

$\rho(s)$ Probability density of stress drop s

Stress of indivdual element



$$\tilde{S}(s,t) = \begin{cases} G_0\ \gamma(t) & (0 \le \theta(t) < \theta_c) \\ 0 & (\theta_c \le \theta(t) < \pi) \\ -G_0\ (\gamma(t) - 2\gamma_0) & (\pi \le \theta(t) < \pi + \theta_c) \\ 0 & (\pi + \theta_c \le \theta(t) < 2\pi), \end{cases}$$

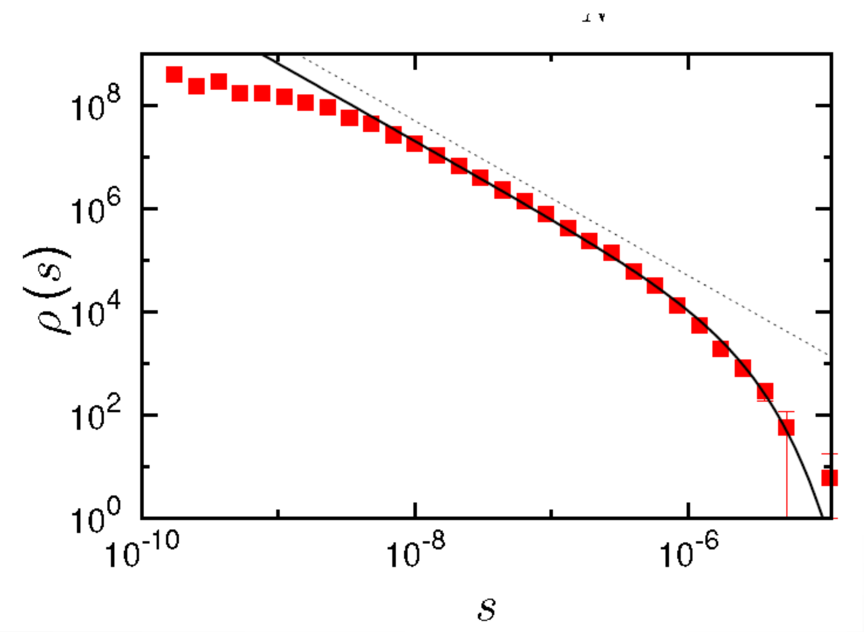
$$\theta(t) = \omega t.$$

$$\theta_c\,(s/(G_0\gamma_0)) = \cos^{-1}\left(1 - \frac{s}{G_0\gamma_0}\right)$$

Stress drop distribution

- The stress drop distribution may obey (Dahmen et al, PRE**58**, 1494 (1998)):

$$\rho(s) \;=\; A(\phi)s^{-3/2}e^{-s/s_c(\phi)}$$



Shear modulus

- From the combination of two contributions we obtain

$$G \simeq A(\phi) G_0^{1/2} \gamma_0^{-1/2} \int_0^\infty dx\, x^{-3/2} F(x)$$

- Then we reach

$$c = 1/2$$

- If size distribution is $\rho(s) \sim s^{-\tau}$ then $c = 1 - \tau$.

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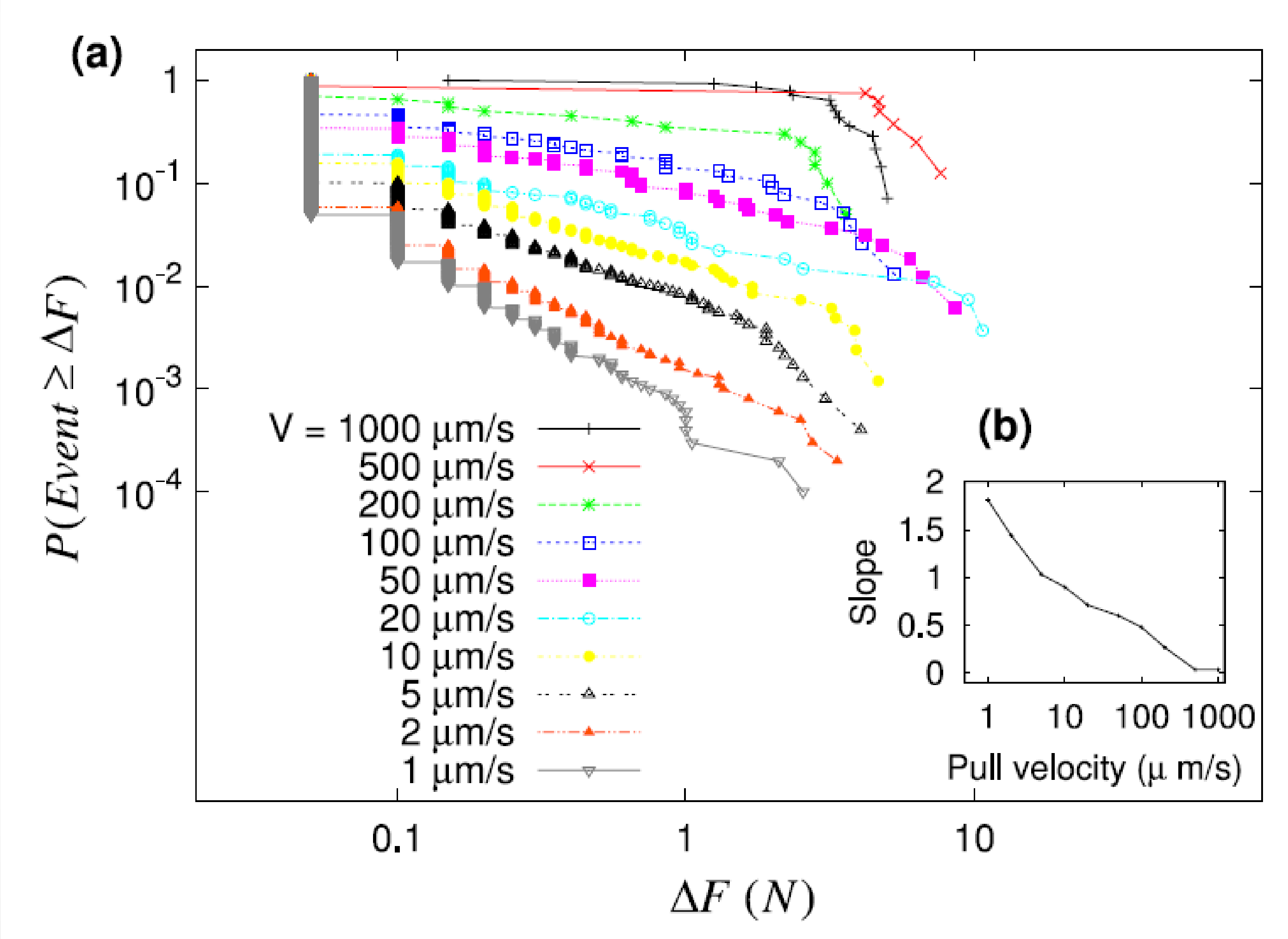
Yamaguchi et al., J. Phys. Condens. **21**, 205105 (2009)



Perspective

- Some experiments such as previous one suggest that the exponent of stress avalanches 3/2 is not universal.
- There are various possibilities to obtain the exponent not equal to 3/2.
  - Renomarilzation method for elastic interfaces
  - Levy process or trapped diffusion or Bessel process etc
- Currently, we do not know what the physical mechanism for non-trivial exponent is.
- Contribution from frictional force=> loss modulus and discontinuous change of stress

Conclusion

- We perform simulations for frictionless grains under oscillator shear.
- We found a crossover from the known exponent for the jamming to the non-trivial behavior.
- Non-trivial exponent can be understood by the mean field theory for stress avalanches.
- See M.Otsuki and HH, PRE90, 042202 (2014) for the details

Appendix

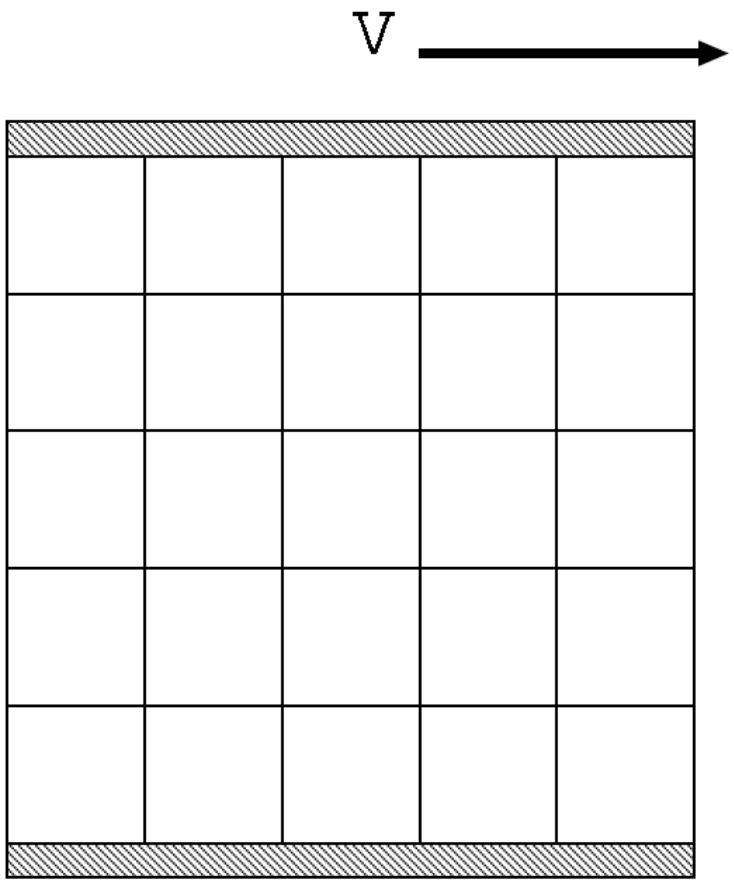
Derivation of stress drop function

- We assume that the mean field theory can be used.

$$\sigma_i = KVt + J\bar{u} - (K+J)u_i,$$

$$\bar{u} = \sum_{j=1}^{N'} u_j / N'$$

$$\sigma = \frac{1}{N'} \sum_{i=1}^{N'} \sigma_i .$$



Derivation (2)

$$\delta u_i = -\frac{\sigma_y - \sigma_a}{K + J},$$

$$s_{self} = -(\sigma_y - \sigma_a),$$

local yield stress σ_y

'arrest stress' σ_a

the stress drop

$$s = (1 - C)(\sigma_y - \sigma_a)n/N'.$$

$$C = \frac{J}{J + K}.$$

$$s_{oth} = C(\sigma_y - \sigma_a)/N'$$

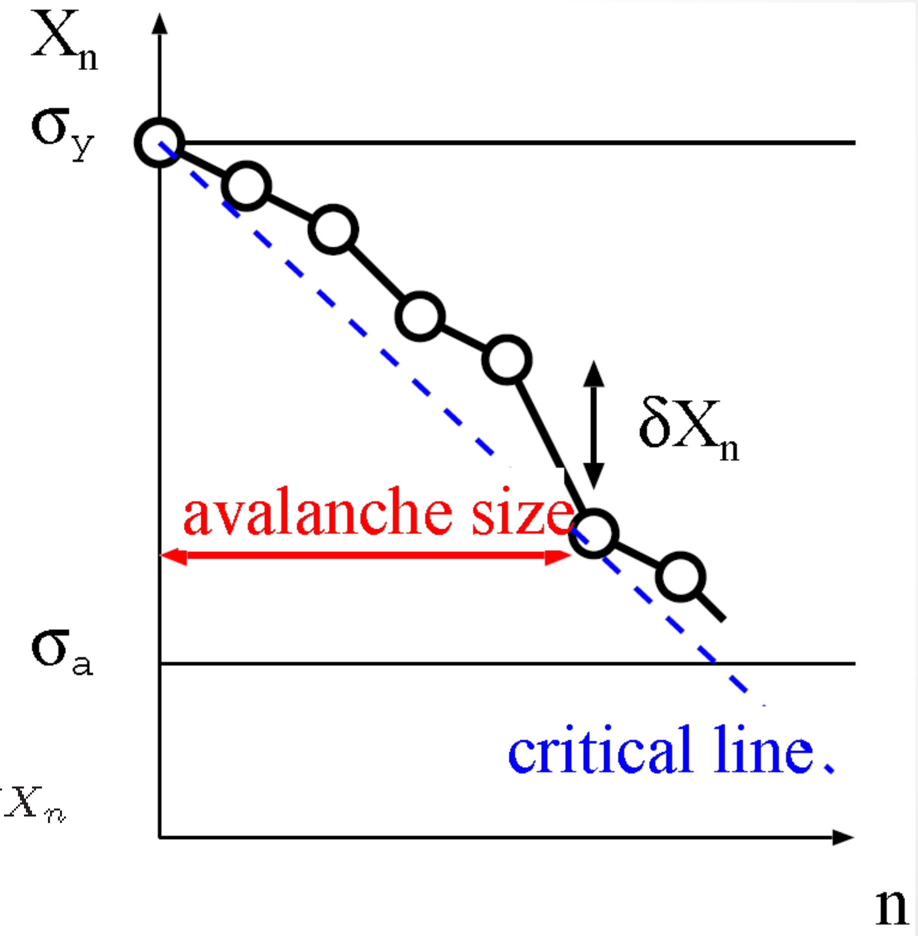
Bernoulli's trial

$$X_n = \sigma_{i(n+1)}$$

$$\delta X_n = X_n - X_{n-1},$$

Poisson distribution

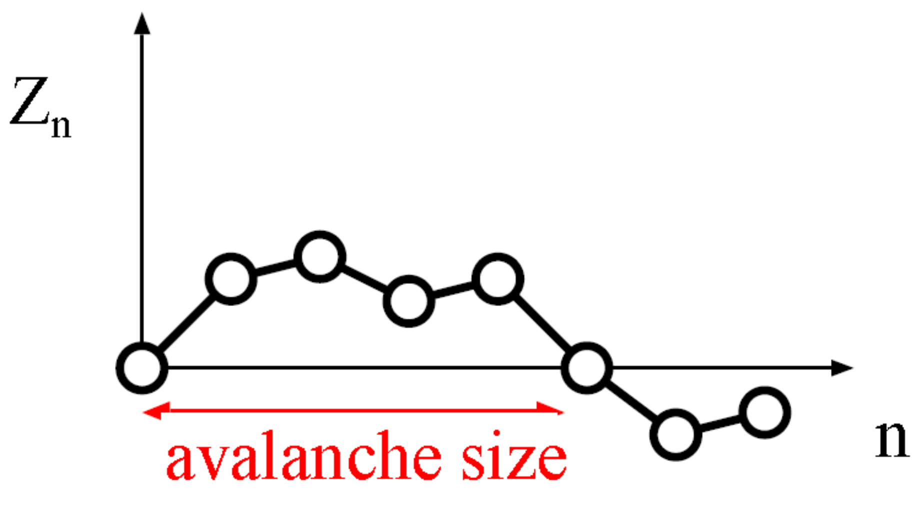
$$\rho_X(\delta X_n) = \frac{N'}{\sigma_y - \sigma_a} e^{-\frac{N'}{\sigma_y - \sigma_a}\delta X_n}$$



$i(n)$ is the index of the site that has the nth largest

Derivation

$$Z_n = X_n - (\sigma_y - n s_{oth}). \quad \delta Z_n = -\delta X_n + s_{oth}$$



$$\delta Z_n = Z_n - Z_{n-1}$$

$$\mu_Z = (2p-1)\Delta x = -(1-C)\frac{\sigma_y - \sigma_a}{N'},$$

$$V_Z = 4\Delta x^2 p(1-p) = \frac{(\sigma_y - \sigma_a)^2}{N'^2}$$

Solution of Bernoulli trial

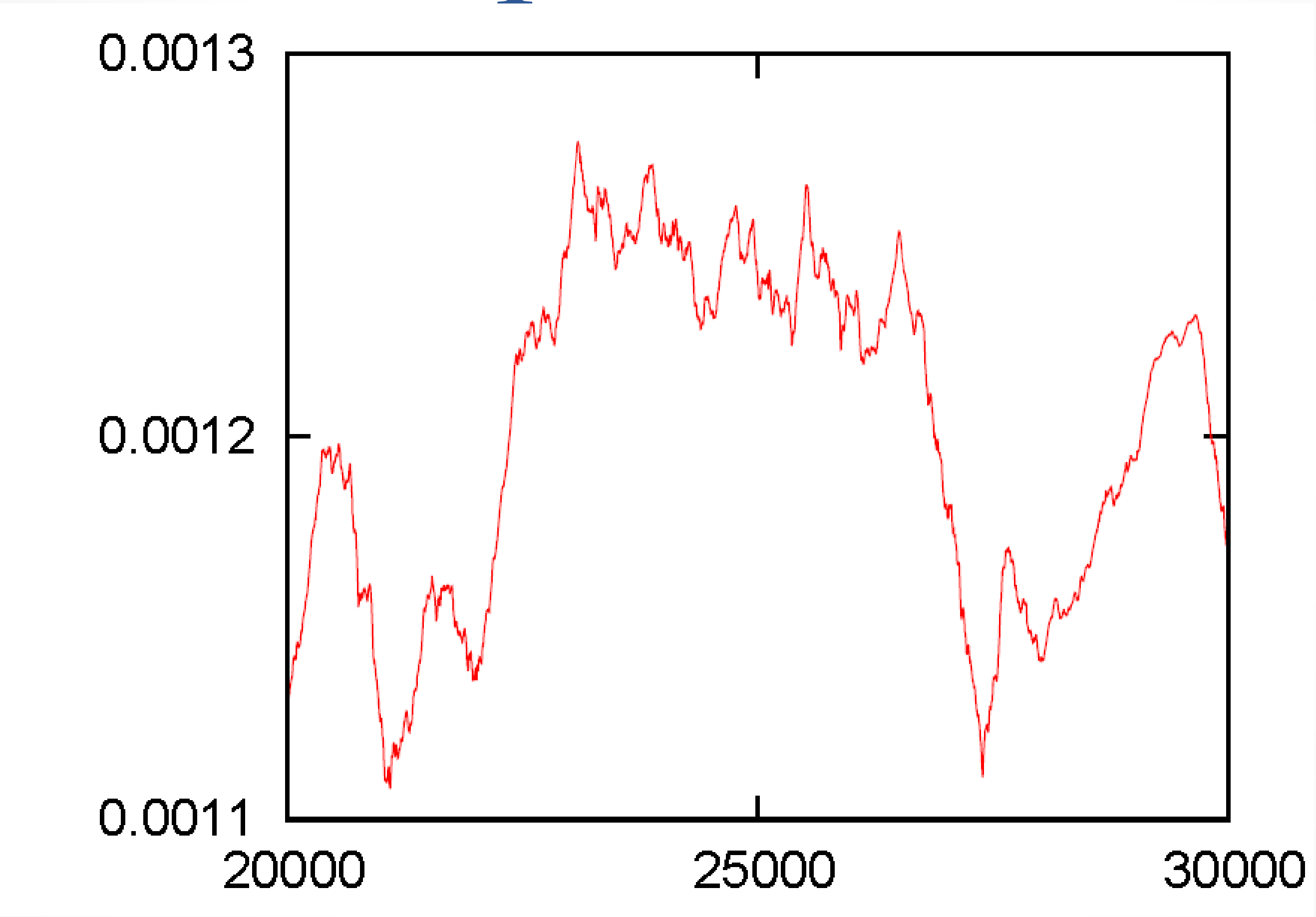
- The solution of Bernoulli's trial is thus given by

$$\begin{aligned} \lambda_{2n-1} &= 0, \\ \lambda_{2n} &= \frac{1}{2p}\begin{pmatrix} 1/2 \\ n \end{pmatrix}(-1)^{(n+1)}\{4p(1-p)\}^n. \end{aligned}$$

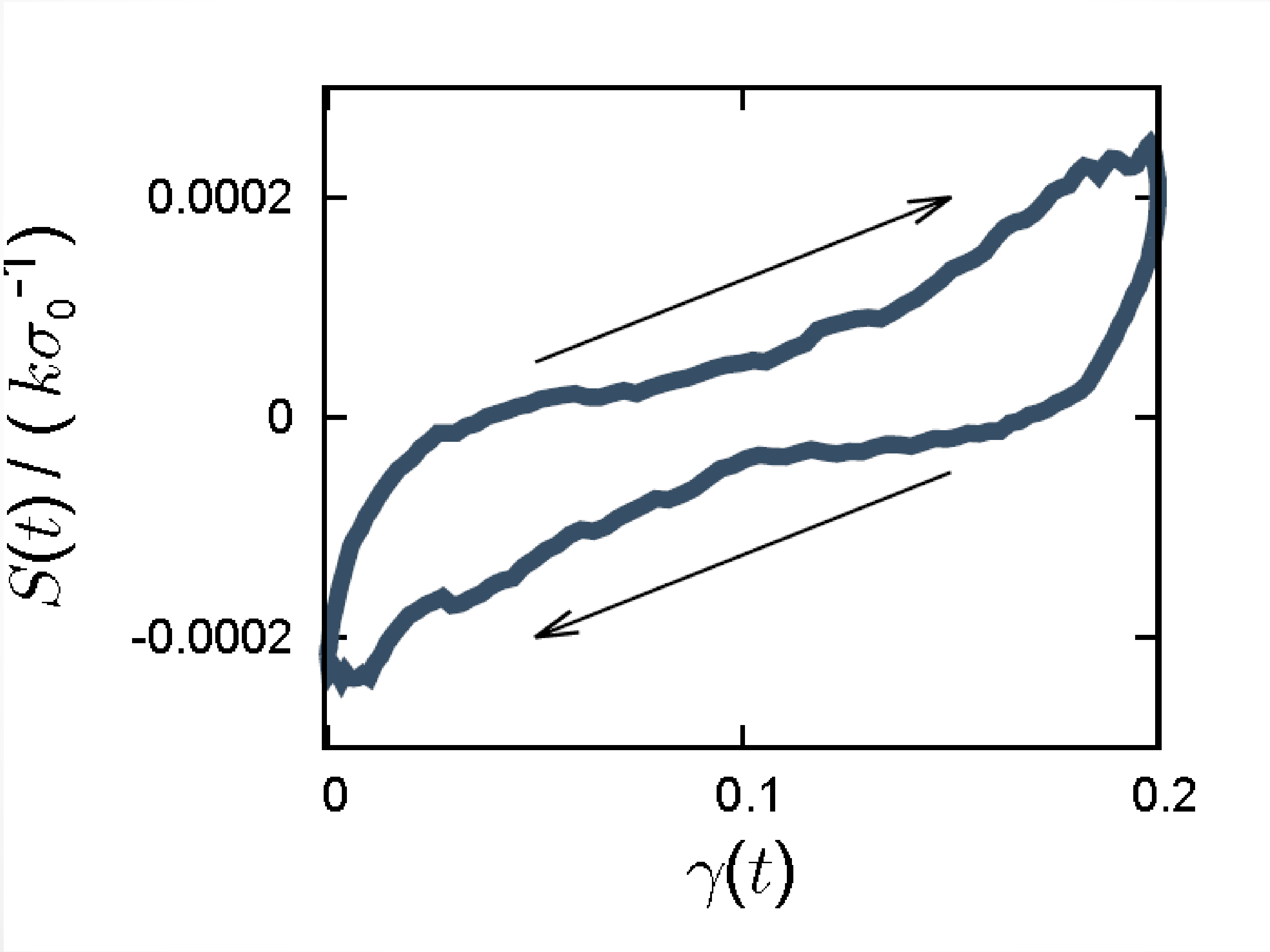
- In the continuum limit, it is reduced to

$$\lambda_{2n-1} = \frac{1}{4\pi^{1/2}p}\frac{1}{n^{3/2}}e^{-n/n_c} \quad n_c = 1/\log(1+(C-1)^2)$$
$$= -1/\log(4p-4p^2)$$

Time sequence of stress



Stress-strain relation



$$\tilde{G}(\gamma_0, s) = -\frac{\omega}{\pi}\int_0^{2\pi/\omega} dt \frac{\tilde{S}(s,t)\cos(\omega t)}{\gamma_0}. \tag{23}$$

Substituting Eq. (18) into Eq. (23), we obtain

$$\tilde{G}(\gamma_0, s) = G_0 F\left(\frac{s}{G_0\gamma_0}\right), \tag{24}$$

where

$$F(x) = \begin{cases} 1 & (x \geqslant 1), \\ T(x)/\pi & (x < 1), \end{cases} \tag{25}$$

with

$$T(x) = \theta_c(x) - 2\sin\theta_c(x) + \frac{\sin 2\theta_c(x)}{2}. \tag{26}$$

Substituting Eqs. (21) and (24) into Eq. (22), we obtain

$$G = A(\phi)G_0 \int_{s_0}^{\infty} ds\, s^{-3/2} e^{-s/s_c(\phi)} F\left(\frac{s}{G_0\gamma_0}\right). \tag{27}$$