ν sterile

WANTED –

DEAD

OR

ALIVE !

Tom Weiler – KITP - Vanderbilt, 03/03

Constraining ν_s:

Sun – CC vs. ES$^{(SK)}$ vs. NC (SNO)
but depends on f_{Boron} [Barger, Marfatia, Whisnant]
& Kamland doesn't help.

Atmosphere –

1. at hi $E \gg \frac{\delta m^2}{A_{NC}} \sim 50\,\text{GeV} \left(\frac{\delta m^2}{3\cdot 10^{-3}\,\text{eV}^2}\right)\left(\frac{1}{4}\right)_{\text{core}}$

$\nu_\mu - \nu_s$ suppressed (as E^{-2}); $\lambda_m \xrightarrow{\text{shrinks}} \sim D_\oplus$
$\nu_\mu - \nu_\tau$ NOT "

[look for $\nu_\mu - \nu_e$ same way]

$\nu_\mu - \nu_\tau$ $\frac{\chi^2}{dof} = \frac{174}{190} \Rightarrow P(\chi^2) = 79\%$

$\nu_\mu - \nu_s$ $= \frac{223}{190} \Rightarrow$ 5%
[Kearns]

$\Rightarrow \nu_s$ fraction $< 0.2\ (0.3)$ @ 68 (99) %

!! But uses Fogli et al. $\{\delta m^2_{atm}, \theta_{atm}, \theta_{\tau s}\}$

2. τ appearance
Expect 85, see $99 \pm 39 \pm 13^{+0}_{-14}$.

mini-BooNE &

"Proton Economics"

TeVatron is limping along,
and
MINOS scheduled to begin Jan '05,
and
6 week shut down this summer,
and
New Shielding going in,
and
⋮

$\exists$ Global fits
with $\{\epsilon_{\mu\mu}, \theta_{\tau s}, \theta_{atm}, \theta_{\odot}, \delta m^2_{LSND}, \delta m^2_{atm}, \delta m^2_{\odot}\}$

[Valle group; Gonzalez-Garcia group]

strongly disfavoring ν_s.

Can $\epsilon_{\mu e}$, ϵ_{ee}, three δ's
invalidate the invalidation?

(PRD)

The Hidden Sterile Neutrino and the (2+2) Sum Rule

Heinrich Päs,[1] * Liguo Song,[2] † and Thomas J. Weiler[2] ‡

[1] *Institut für Theoretische Physik und Astrophysik, Universität Würzburg, Am Hubland, 97074 Würzburg, Germany*

[2] *Department of Physics and Astronomy, Vanderbilt University Nashville, TN 37235, USA*

Abstract

We discuss oscillations of atmospheric and solar neutrinos into sterile neutrinos in the 2+2 scheme. A zeroth order sum rule requires equal probabilities for oscillation into ν_s and ν_τ in the solar+atmospheric data sample. Data does not favor this claim. Here we use scatter plots to assess corrections of the zeroth order sum rule when (i) the 4×4 neutrino mixing matrix assumes its full range of allowed values, and (ii) matter effects are included. We also introduce a related "product rule". We find that the sum rule is significantly relaxed, due to both the inclusion of the small mixing angles (which provide a short-baseline contribution) and to matter effects. The product rule is also dramatically altered. The observed relaxation of the sum rule weakens the case against the 2+2 model and the sterile neutrino. To invalidate the 2+2 model, a global fit to data *with the small mixing angles included* seems to be required.

*E-mail: paes@physik.uni-wuerzburg.de

†E-mail: liguo.song@vanderbilt.edu

‡E-mail: tom.weiler@vanderbilt.edu

Four-flavor mixing is described by six angles and three CP-violating phases. In addition there are three further phases for Majorana neutrinos; since these three phases do not enter into oscillation probabilities, we omit them. The six angles parametrize independent rotations in the six planes of four-dimensional space. For our purposes, it is useful to order these rotations as[2]

$$U = R_{23}(\theta_{\tau s})R_{24}(\epsilon_{\mu\mu})R_{14}(\epsilon_{\mu e})R_{13}(\epsilon_{ee})R_{34}(\theta_{\rm atm})R_{12}(\theta_{\rm sun})\,. \qquad (5)$$

Unitary U transforms from the mass basis (m_4, m_3, m_2, m_1) to the flavor basis $(\nu_\mu, \nu_\tau, \nu_s, \nu_e)$. In suggestive notation, the ϵ's are small angles limited by the short-baseline (SBL) data, $\theta_{\rm sun}$ and $\theta_{\rm atm}$ are the angles dominantly responsible for solar and atmospheric oscillations, respectively, and $\theta_{\tau s}$ is a possibly large angle parametrizing the dominant mixing of the ν_τ & ν_s. Explicitly, the ν_τ-ν_s mixing resulting from $R^T_{23}(\theta_{\tau s})$ is

$$\begin{pmatrix}\nu_+\\ \nu_-\end{pmatrix} = \begin{pmatrix}\cos\theta_{\tau s} & -\sin\theta_{\tau s}\\ \sin\theta_{\tau s} & \cos\theta_{\tau s}\end{pmatrix}\begin{pmatrix}\nu_\tau\\ \nu_s\end{pmatrix}. \qquad (6)$$

As shown in [18], the three phases may be assigned to the double-generation skipping rotations R_{24} and R_{13} and any one of the single-generation skipping rotations, in the following manner: equal and opposite phases are attached to the two non-diagonal elements of the rotation matrix. For the single-generation skipping complex rotation, we choose R_{14}. Since the three angles of R_{24}, R_{13}, and R_{14} are small, this amounts to assigning an arbitrary phase to each of the $\sin(\epsilon_j)\sim\epsilon_j$'s, with ϵ_j to the right of the diagonal and ϵ_j^* to the left.[3] We note that with this parameterization the nonzero phases are associated exclusively with small angles, making the smallness of any observable CP-violation in the 2+2 scheme immediately evident.

When the CP-violating phases are allowed to range from 0 to π, all angles may be restricted to the interval $[0,\pi/2]$. However, when the phases are neglected, the θ's still range over $[0,\pi/2]$, but the ϵ's now range over $[-\pi/2,\pi/2]$ (or over $[0,\pi]$).

$> \delta m^2_{atm}$

δm^2_{LSND}

$> \delta m^2_{\odot}$

The two large-angle mixings on the right in eq. (5) are given by

$$U_\pm = R_{34}(\theta_{\rm atm})R_{12}(\theta_{\rm sun}) = \begin{pmatrix} \cos\theta_{\rm atm} & \sin\theta_{\rm atm} & 0 & 0 \\ -\sin\theta_{\rm atm} & \cos\theta_{\rm atm} & 0 & 0 \\ 0 & 0 & \cos\theta_{\rm sun} & \sin\theta_{\rm sun} \\ 0 & 0 & -\sin\theta_{\rm sun} & \cos\theta_{\rm sun} \end{pmatrix}. \qquad (8)$$

This matrix independently mixes each of the two mass doublets in the 2+2 spectrum. $U_\pm$ approximates the full mixing matrix in the $(\nu_\mu, \nu_+, \nu_-, \nu_e)$ basis. The advantage of locating U_ϵ to the left of $U_\pm$ is that in the full mixing matrix U, the angle $\theta_{\rm atm}$ appears only in the first two columns, and $\theta_{\rm sun}$ appears only in the last two columns. Consequently, atm/LBL amplitudes do not depend on $\theta_{\rm sun}$, and solar amplitudes do not depend on $\theta_{\rm atm}$. Of course, in any parameterization, the SBL amplitudes are insensitive to mixing within either mass doublet, and so depend on neither $\theta_{\rm atm}$ nor on $\theta_{\rm sun}$.

Recent global fits of the 2+2 model to solar and atmospheric data [15–17] have focused on the seven-parameter set $\{\epsilon_{\mu\mu}, \theta_{\tau s}, \theta_{\rm sun}, \theta_{\rm atm}, \delta m^2_{\rm LSND}, \delta m^2_{\rm sun}, \delta m^2_{\rm atm}\}$, neglecting the two small angles $\epsilon_{\mu e}, \epsilon_{ee}$, and the three CP-violating phases.[4] The investigation of the dependence of the sterile neutrino sum rule on the small angles (ϵ's) in this paper suggests that neglect of these small angles, especially $\epsilon_{\mu e}$, may not be warranted.

decoupled except via

$\sqrt{2}\, \nu_\pm = \cos\theta_{\tau s} |\nu_\tau\rangle \pm \sin\theta_{\tau s} |\nu_s\rangle$

III. SUM AND PRODUCT RULES

A. Zeroth order in ϵ's

Let us first discuss the oscillation probabilities in vacuum in the limit where the ϵ's are set to zero. This limit results when $\langle\nu_e|\nu_4\rangle = 0 = \langle\nu_e|\nu_3\rangle$, and $\langle\nu_\mu|\nu_2\rangle = 0 = \langle\nu_\mu|\nu_1\rangle$.

In the limit of vanishing ϵ's, one can use eq. (8) to read off immediately that LBL/atm oscillations of the ν_e are zero and ν_μ oscillates into pure ν_+; at the solar-scale, $\nu_e \to \nu_\mu$ oscillations are zero, and ν_e oscillates into pure ν_-; and there are no SBL oscillations.

Explicitly, the nonzero oscillation amplitudes for ν_μ due to atmospheric-scale oscillations are

$$\boxed{\begin{aligned} A_{\rm atm}(\nu_\mu \to \nu_\tau) &= \sin^2(2\theta_{\rm atm})\,\cos^2\theta_{\tau s} \qquad (12) \\ A_{\rm atm}(\nu_\mu \to \nu_s) &= \sin^2(2\theta_{\rm atm})\,\sin^2\theta_{\tau s} \qquad (13) \\ A_{\rm atm}(\nu_\mu \to \nu_{\not\mu}) &= \sin^2(2\theta_{\rm atm}) \qquad (14) \end{aligned}}$$

while for ν_e due to solar-scale oscillations they are

$$A_{\rm sun}(\nu_e \to \nu_\tau) = \sin^2(2\theta_{\rm sun})\,\sin^2\theta_{\tau s} \qquad (15)$$

$$A_{\rm sun}(\nu_e \to \nu_s) = \sin^2(2\theta_{\rm sun})\,\cos^2\theta_{\tau s} \qquad (16)$$

$$A_{\rm sun}(\nu_e \to \nu_{\not e}) = \sin^2(2\theta_{\rm sun}) \qquad (17)$$

We note that matter effects cannot alter the texture of the off-diagonal elements of the mass-squared matrix in the flavor basis, because the matter potential is diagonal in this basis. Consequently, the block-diagonal structure of the diagonalizing matrix in eq. (8) is maintained in the presence of matter. This further implies that eqs. (12) to (16) are unchanged in form by matter, although the angles (and mass-eigenvalues) assume different values.

The neutrinos arriving at earth from the sun are incoherent mass eigenstates.[5] Thus, the density operator describing the decoherent ensemble of neutrinos arriving at earth from the sun is diagonal in the mass basis. In the adiabatic approximation, it is simply

$$\hat{\rho}^S_{\rm mass} = \sum_j |U^S_{ej}|^2\, |\nu_j><\nu_j|\,. \qquad (20)$$

Here, U^S is the mixing matrix at the center of the sun where the solar neutrinos originate. Let us label the solar neutrino "$\nu_\odot$" to signify its $\nu_e \to \nu_{SUN}$ solar history. The probability to measure neutrino flavor β at earth is then given by

$$P_S(\nu_\odot \to \nu_\beta) = <\nu_\beta|\hat{\rho}^S|\nu_\beta> = \sum_j |U^S_{ej}|^2\, |U^V_{\beta j}|^2\,, \qquad (21)$$

after applying eqs. (1) and (2). The $\nu_\odot \equiv \nu_2$ approximation[6] consists of setting $|U^S_{ej}|^2$ equal to δ_{j2}, yielding

$$P_{\nu_\odot\to\nu_\beta} = |U^V_{\beta 2}|^2 \qquad [\nu_\odot \equiv \nu_2,\ \text{adiabatic}] \qquad (22)$$

In the absence of ϵ's, we have from eqs. (6), (8), and (22), that

$$\nu_2 = \cos\theta_{\rm sun}(\sin\theta_{\tau s}\nu_\tau + \cos\theta_{\tau s}\nu_s) - \sin\theta_{\rm sun}\,\nu_e\,. \qquad (23)$$

¿From this equation, we may read off the values of $U^V_{\beta 2}$ to obtain the oscillation amplitudes for solar neutrinos arriving at the earth in the $\nu_\odot \equiv \nu_2$ approximation:

$$\boxed{\begin{aligned} A_{\rm sun}(\nu_\odot \to \nu_\tau) &= \cos^2(\theta_{\rm sun})\,\sin^2\theta_{\tau s} \qquad (24)\\ A_{\rm sun}(\nu_\odot \to \nu_s) &= \cos^2(\theta_{\rm sun})\,\cos^2\theta_{\tau s} \qquad (25)\\ A_{\rm sun}(\nu_\odot \to \nu_{\not e}) &= \cos^2(\theta_{\rm sun}) \qquad (26)\end{aligned}}$$

In these formulas, the angles are truly vacuum angles.

[6]In our numerical work we will go beyond the $\nu_\odot \equiv \nu_2$ approximation. Nevertheless, $\nu_\odot \equiv \nu_2$ is a good approximation offering simple results. Numerically, we find that $|U^S_{e2}|^2$ = 90%, 98%, and 99% for E_ν = 5, 10, and 15 MeV, respectively, for zero ϵ's, and with little change for nonzero ϵ's.

Thus, unitarity in the context of the 2+2 mass spectrum has led one to a sum rule [9] and a product rule at zeroth order in ϵ:

$$\left[\frac{P(\nu_\odot \to \nu_s)}{P(\nu_\odot \to \nu_{\not e})}\right]_{\rm sun} + \left[\frac{P(\nu_\mu \to \nu_s)}{P(\nu_\mu \to \nu_{\not \mu})}\right]_{\rm atm} = (\cos^2\theta_{\tau s})_V + (\sin^2\theta_{\tau s})_E \to 1 \qquad (27)$$

and

$$\left[\frac{P(\nu_\odot \to \nu_s)}{P(\nu_\odot \to \nu_\tau)}\right]_{\rm sun} \times \left[\frac{P(\nu_\mu \to \nu_s)}{P(\nu_\mu \to \nu_\tau)}\right]_{\rm atm} = (\cot^2\theta_{\tau s})_V \times (\tan^2\theta_{\tau s})_E \to 1 \qquad (28)$$

where the subscripts V and E signify "vacuum" and "earth-matter", and the arrows hold in the limit of negligible earth-matter effects. In the figures to follow and the discussion thereof we will refer to the ratio terms in eqs (27) and (28) as $R_{\rm sun}$ and $R_{\rm atm}$, respectively.

We emphasize that the sum and product rules are not required by any underlying symmetry or principle. Rather, they are accidents of the block-diagonal structure of eq. (8), which results when the small angles are neglected. The values on the right-hand sides of these two rules, eqs. (27) and (28), will differ from unity when the small mixing angles are included, even if matter-effects are absent. Furthermore, although their values are intimately related at zeroth order in the ϵ's, they are not so simply related at higher order in ϵ's.

The purpose of this paper is to evaluate these sum rules, including the small angles neglected in previous work, and including possible earth-matter effects. ~~In the next section,~~

FIGURES

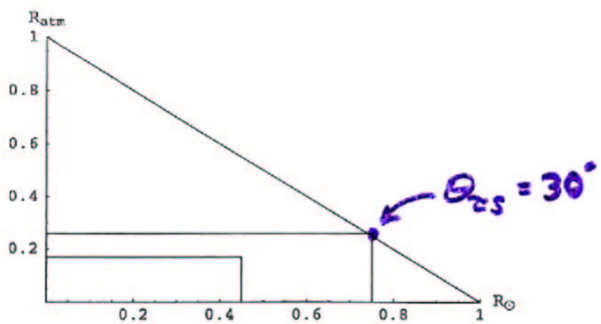

FIG. 1. The zeroth order sum rule compared to the 90% and 99% exclusion boxes obtained from fits to data with small angles set to zero. The vertical axis is R_{atm} and the horizontal axis is R_{sun}.

3 ways to violate Σ rule:

1. Some $\lambda > R_\oplus$

with $\lambda = 2.5 \frac{E}{GeV} \frac{eV^2}{\delta m^2}$ km,

$\lambda_\odot = 50 km \left(\frac{E}{10MeV}\right)\left(\frac{5\cdot 10^{-5} eV^2}{\delta m^2_\odot}\right) \ll R_{\odot core}$

$\Rightarrow$ averaged

$\lambda_{atm} = 8000 km \left(\frac{E}{10GeV}\right)\left(\frac{3\cdot 10^{-3} eV^2}{\delta m^2_{atm}}\right) \gtrsim R_\oplus$

$\Rightarrow$ NOT averaged

$\lambda_{LSND} = 25 km \left(\frac{E}{10GeV}\right)\left(\frac{eV^2}{\delta m^2_{LSND}}\right)$, averaged

2. Matter effects, esp. suppression of $\nu_\mu \to \nu_s$

3. Scale-mixing, ie small angles $\leftrightarrow \delta m^2_{LSND}$ important because of (2.) and (1.).

In our notation, the SBL oscillation amplitudes to order $|\epsilon|^2$ are

$$A_{\rm SBL}(\nu_e \to \nu_{e\!\!/}) = 4\left[|\epsilon_{\mu e}|^2 + |\epsilon_{ee}|^2\right] \tag{30}$$

$$A_{\rm SBL}(\nu_\mu \to \nu_{\mu\!\!/}) = 4\left[|\epsilon_{\mu e}|^2 + |\epsilon_{\mu\mu}|^2\right] \tag{31}$$

$$A_{\rm SBL}(\nu_\mu \to \nu_e) = 4\,|\epsilon_{\mu e}|^2 \tag{32}$$

$$A_{\rm SBL}(\nu_\mu \to \nu_\tau) = 4|\epsilon_{\mu\mu}|^2 \sin^2\theta_{\tau s} \tag{33}$$

$$A_{\rm SBL}(\nu_\mu \to \nu_s) = 4|\epsilon_{\mu\mu}|^2 \cos^2\theta_{\tau s} \quad \sim \cos^2\theta_{\tau s} \tag{34}$$

$$A_{\rm SBL}(\nu_e \to \nu_\tau) = 4|\epsilon_{ee}|^2 \cos^2\theta_{\tau s} \tag{35}$$

$$A_{\rm SBL}(\nu_e \to \nu_s) = 4|\epsilon_{ee}|^2 \sin^2\theta_{\tau s}. \tag{36}$$

These amplitudes, and so the values of the small angles ϵ_{ee}, $\epsilon_{\mu e}$ and $\epsilon_{\mu\mu}$, are bounded from above by the experimental limits on ν_μ and ν_e disappearance in vacuum, and by atmospheric neutrino oscillation data. The positive result of LSND bounds $\epsilon_{\mu e}$ from below, but this constraint is not significant for present purposes. For the allowed LSND region $\delta m^2_{\rm SBL} \sim 1.0$ to 0.2 eV^2, the BUGEY disappearance experiment [21] provides the 90 % C.L. bound[7] $\frac{1}{4}A_{\rm SBL}(\nu_e \to \nu_{e\!\!/}) = |\epsilon_{ee}|^2 + |\epsilon_{\mu e}|^2 \leq 0.01$. The CDHS [22] $\bar\nu_\mu$ (ν_μ) disappearance experiment bounds the amplitude $\frac{1}{4}A_{\rm SBL}(\nu_\mu \to \nu_{\mu\!\!/}) = |\epsilon_{\mu\mu}|^2 + |\epsilon_{\mu e}|^2 \leq 0.2$ for $\delta m^2_{\rm SBL} \gtrsim 0.3\,{\rm eV}^2$. In fact, a more stringent bound on $A_{\rm SBL}(\nu_\mu \to \nu_{\mu\!\!/})$ results from atmospheric neutrino data (a nonzero value for this amplitude is incompatible with maximal ν_μ mixing at the $\delta m^2_{\rm atm}$ scale). A fit to atmospheric data [15] results in $A_{\rm SBL}(\nu_\mu \to \nu_{\mu\!\!/}) < 0.48\,(0.64)$ with 90% (99%) C.L., which translates into $|\epsilon_{\mu\mu}|^2 + |\epsilon_{\mu e}|^2 < 0.12\,(0.16)$. We note that the CHOOZ limit on ν_e-disappearance at the atmosphere scale is of order $(\epsilon_{\mu e}, \epsilon_{ee})^4 \lesssim 10^{-4}$ in the context of the 2+2 model, and so is not of interest.

[7] The KARMEN experiment provides a tighter bound than the BUGEY experiment for $\delta m^2_{\rm LSND} > 0.2{\rm eV}^2$, reaching $\frac{1}{4}A_{\rm SBL}(\nu_\mu \to \nu_e) \sim 0.7 \times 10^{-3}$ at $\delta m^2_{\rm LSND} \sim 1{\rm eV}^2$. However, atmospheric experiments average over the SBL contribution, and so are not sensitive to the value of $\delta m^2_{\rm LSND}$. Accordingly, the appropriate bound to use is the more liberal BUGEY bound, inferred at $\delta m^2_{\rm LSND} \sim 0.2{\rm eV}^2$.

12

$\lesssim$ @ 90% CL, (i) $|\epsilon_{\mu e}|^2 + |\epsilon_{ee}|^2 \leq 0.01$
(ii) $|\epsilon_{\mu\mu}|^2 < 0.12$
$\Rightarrow$ $\epsilon_{\mu e}, \epsilon_{ee} \lesssim 10\%$, $\epsilon_{\mu\mu} \lesssim 35\%$ in vacuum.

Next we turn to the amplitudes for solar neutrino oscillations. We employ again the adiabatic approximation valid for large solar-mixing solutions, and simplify the discussion here with the $\nu_\odot \equiv \nu_2$ approximation. To $\mathcal{O}(\epsilon^2)$, the expansion of ν_2 in flavor states is

$$|\nu_2>= \begin{cases} (\epsilon_{\mu\mu}\cos\theta_{\rm sun} - \epsilon_{\mu e}\sin\theta_{\rm sun})\,|\nu_\mu> \\ +\ (\sin\theta_{\tau s}[\cos\theta_{\rm sun}(1-\frac{1}{2}|\epsilon_{\mu\mu}|^2) + \sin\theta_{\rm sun}\epsilon^*_{\mu\mu}\epsilon_{\mu e}] - \cos\theta_{\tau s}\sin\theta_{\rm sun}\epsilon_{ee})\,|\nu_\tau> \\ +\ (\cos\theta_{\tau s}[\cos\theta_{\rm sun}(1-\frac{1}{2}|\epsilon_{\mu\mu}|^2) + \sin\theta_{\rm sun}\epsilon^*_{\mu\mu}\epsilon_{\mu e}] + \sin\theta_{\tau s}\sin\theta_{\rm sun}\epsilon_{ee})\,|\nu_s> \\ -\ \sin\theta_{\rm sun}(1-\frac{1}{2}|\epsilon_{ee}|^2 - \frac{1}{2}|\epsilon_{\mu e}|^2)\,|\nu_e>. \end{cases} \tag{38}$$

For solar neutrinos arriving at the "day" hemisphere of the earth, all angles assume vacuum values. This result generalizes eqn. (23) to nonzero ϵ's. ¿From this, one can easily calculate the first term in the sum rule (27),

$$\left[\frac{P(\nu_\odot \to \nu_s)}{P(\nu_\odot \to \nu_{e\!\!/})}\right]_{\rm sun} \simeq \frac{|<\nu_s|\nu_2>|^2}{1-|<\nu_e|\nu_2>|^2}, \tag{39}$$

as a function of the large and small angles. Similarly, the first factor in the product rule (28),

$$\left[\frac{P(\nu_\odot \to \nu_s)}{P(\nu_\odot \to \nu_\tau)}\right]_{\rm sun} \simeq \frac{|<\nu_s|\nu_2>|^2}{|<\nu_\tau|\nu_2>|^2}, \tag{40}$$

can be readily calculated.

Finally, we turn to the oscillation amplitudes for the atmospheric neutrinos. The second terms in the sum and product rules, namely,

$$\left[\frac{P(\nu_\mu \to \nu_s)}{P(\nu_\mu \to \nu_\mu)}\right]_{\rm atm} = \frac{P_{\rm LBL}(\nu_\mu \to \nu_s) + P_{\rm SBL}(\nu_\mu \to \nu_s)}{P_{\rm LBL}(\nu_\mu \to \nu_\mu) + P_{\rm SBL}(\nu_\mu \to \nu_\mu)}, \tag{41}$$

and

$$\left[\frac{P(\nu_\mu \to \nu_s)}{P(\nu_\mu \to \nu_\tau)}\right]_{\rm atm} = \frac{P_{\rm LBL}(\nu_\mu \to \nu_s) + P_{\rm SBL}(\nu_\mu \to \nu_s)}{P_{\rm LBL}(\nu_\mu \to \nu_\tau) + P_{\rm SBL}(\nu_\mu \to \nu_\tau)}, \tag{42}$$

respectively, are given by inputting the appropriate amplitudes. In the atmospheric data, the measurement process averages over oscillation scales small relative to the Earth's radius. Hence it is correct and necessary to include contributions from the LBL length-scale and smaller, which here includes the oscillation-averaged short baseline amplitudes. One notes that the SBL amplitudes, given in eqs. (30)-(36), contribute to the sum rules at order ϵ^2 but not order ϵ. The long baseline amplitudes to order ϵ^2 are:

$$\begin{aligned} A_{\rm LBL}(\nu_\mu \to \nu_s) &= \sin^2 2\theta_{\rm atm}\,[\sin^2\theta_{\tau s}(1-|\epsilon_{\mu e}|^2-|\epsilon_{ee}|^2)-|\epsilon_{\mu\mu}|^2] \\ &\quad -\sin 4\theta_{\rm atm}[\tfrac{1}{2}\sin 2\theta_{\tau s}Re(\epsilon_{\mu\mu})+Re(\epsilon_{\mu\mu}\epsilon^*_{\mu e})\sin^2\theta_{\tau s}] \qquad (43)\\ A_{\rm LBL}(\nu_\mu \to \nu_\tau) &= \sin^2 2\theta_{\rm atm}\,[\cos^2\theta_{\tau s}(1-|\epsilon_{\mu e}|^2-|\epsilon_{ee}|^2)-|\epsilon_{\mu\mu}|^2] \\ &\quad +\sin 4\theta_{\rm atm}[\tfrac{1}{2}\sin 2\theta_{\tau s}Re(\epsilon_{\mu\mu})-Re(\epsilon_{\mu\mu}\epsilon^*_{\mu e})\cos^2\theta_{\tau s}] \qquad (44)\\ A_{\rm LBL}(\nu_\mu \to \nu_e) &= -\sin 2\theta_{\rm atm}\,[(|\epsilon_{\mu e}|^2-|\epsilon_{ee}|^2)\sin 2\theta_{\rm atm}+2Re(\epsilon^*_{\mu e}\epsilon_{ee})\cos 2\theta_{\rm atm}] \qquad (45)\\ A_{\rm LBL}(\nu_\mu \to \nu_\mu) &= \sin^2 2\theta_{\rm atm}\,[1-2|\epsilon_{\mu e}|^2-2|\epsilon_{\mu\mu}|^2]-2\sin 4\theta_{\rm atm}Re(\epsilon_{\mu\mu}\epsilon^*_{\mu e}) \qquad (46) \end{aligned}$$

With eqs. (39)-(42) as our guide, it is not difficult to write out the explicit analytic expressions for the sum and product rules. It is also not especially illuminating to do so. One finding is that the order ϵ and ϵ^2 corrections are different for the sum and product rules. Thus these two rules, containing the same information in zero$^{\rm th}$ order, contain different information when the small angles are included. We remind the reader that atmospheric oscillations occur in the earth, and so the mixing angles that enter eqs. (41) and (42) are matter rather than vacuum angles.

Some very interesting sub-block properties of 3-scale (δm^2's) 4x4 mixing matrix in matter!

Coming "Soon" Near You

e.g. Instead of $E \gg \frac{\delta m^2_{atm}}{A_{Ne}}$ SUPPRESSING OSCN,

have $\frac{\delta m^2_{atm}}{A_{Ne}} \ll E \ll \frac{\delta m^2_{LSND}}{A_{Ne}}$

ENHANCING ν_μ AND $\bar{\nu}_\mu$ OSCN

for certain relations among ϵ's!

[Checked that Earth-matter is irrelevant, for Σ rule, in ϵ's $\equiv 0$ approx$^{\underline{n}}$.]

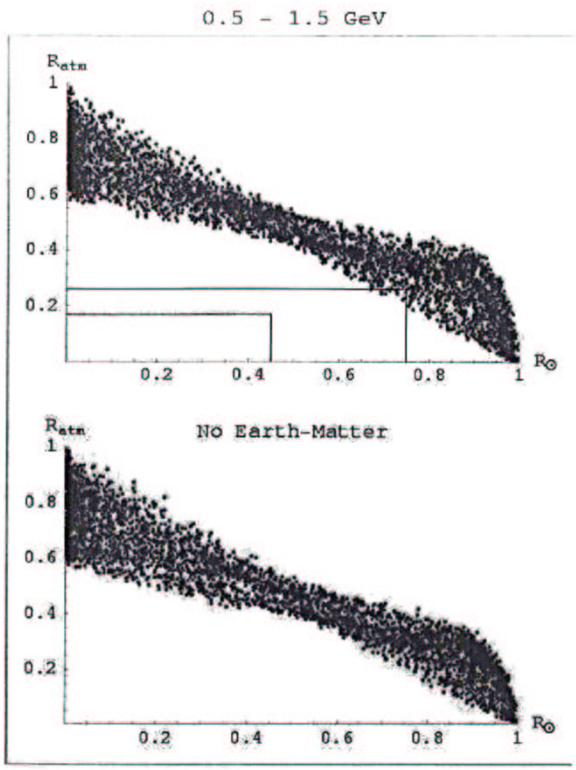

FIG. 2. Top: 4,000 points, each averaged over incident neutrino energies 0.5 GeV $\leq E_\nu \leq$ 1.5 GeV and upcoming angles in $-1.0 \leq \cos\theta_z \leq -0.8$, scattered over $\epsilon_{\mu\mu}$, $\epsilon_{\mu e}$, ϵ_{ee}, and $\theta_{\tau s}$, with matter effects included. The 90% and 99% exclusion boxes obtained with small angles set to zero are shown as a crude reference. Bottom: same as above but with earth-matter omitted. In both plots, the vertical axis is $R_{\rm atm}$ and the horizontal axis is $R_{\rm sun}$.

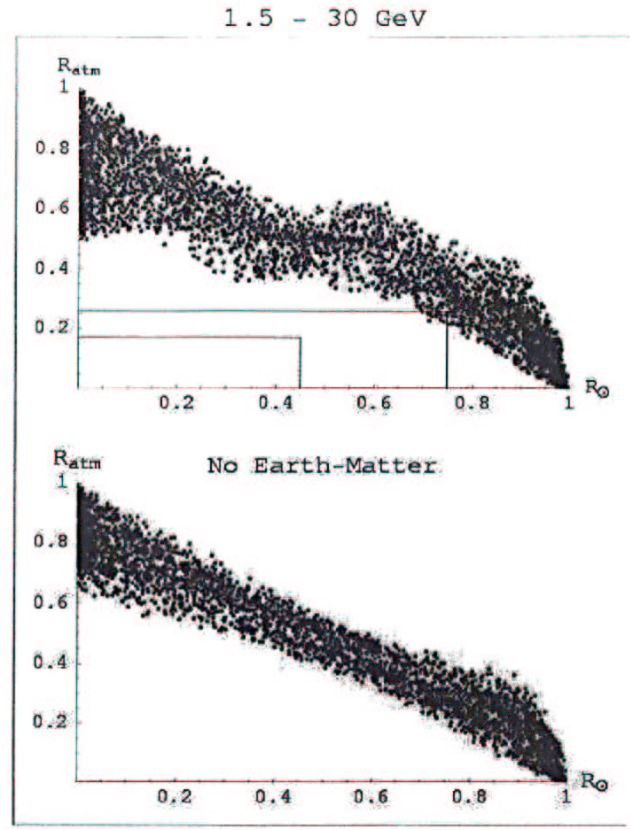

FIG. 3. Same as Fig. 2, but energy-averaged over 1.5 GeV$\leq E_\nu \leq$ 30 GeV.

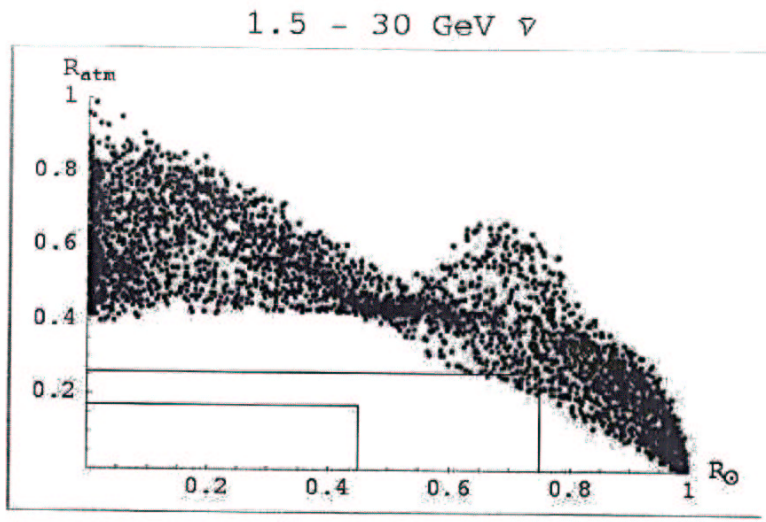

FIG. 11. Sum rule scatter plot for the antineutrino channel, averaged over the energy range 1.5 to 30 GeV.

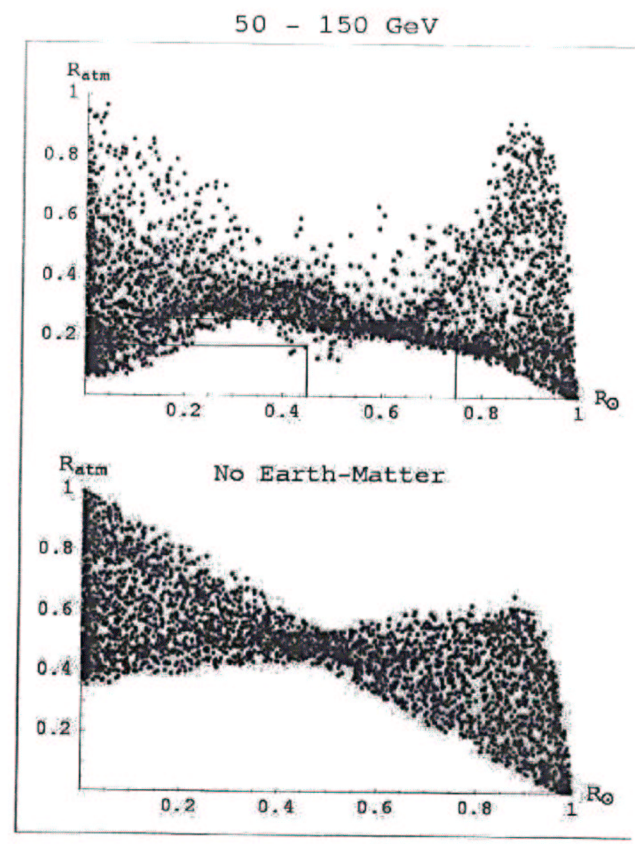

FIG. 5. Same as Fig. 2, but energy-averaged over 50 GeV$\leq E_\nu \leq$ 150 GeV.

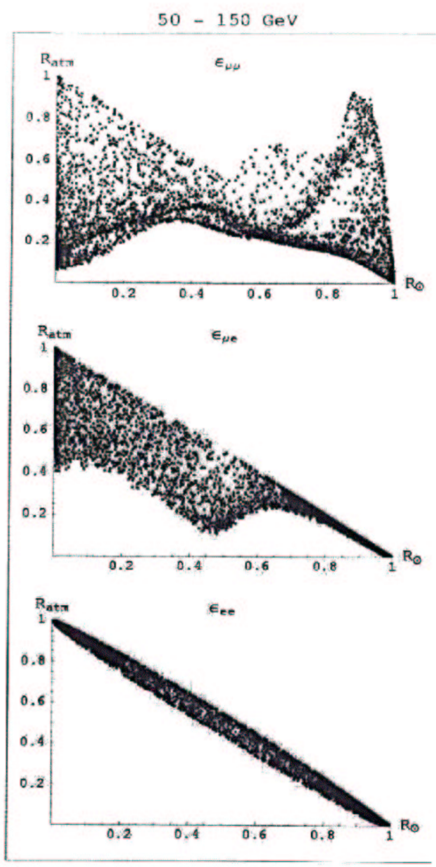

FIG. 9. Same as in Top figure 5, but with only one small-angle (indicated) nonzero.

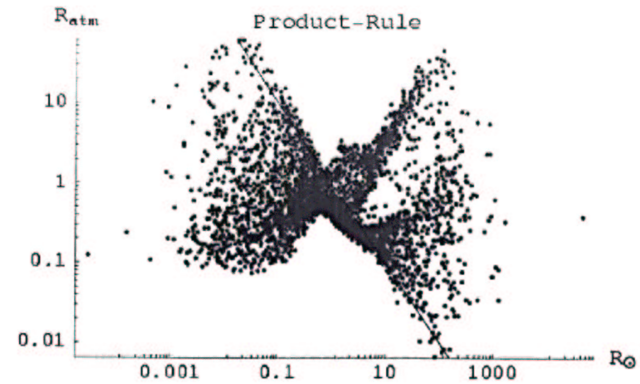

FIG. 10. 4,000 points for the product rule, each averaged over incident neutrino energies 50 GeV $\leq E_\nu \leq$ 150 GeV and upcoming angles in $-1.0 \leq \cos\theta_z \leq -0.8$, scattered over $\epsilon_{\mu\mu}$, $\epsilon_{\mu e}$, ϵ_{ee}, and $\theta_{\tau s}$, with matter effects included. Here we use the same $R_{\rm atm}$ and $R_{\rm sun}$ symbols to denote the ratios of amplitudes appearing in the product rule eq. (28). The diagonal line is the result when small angles are set to zero.