Higgs bundles and D-branes - 3

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- Quadratic duality

- Filtered Koszul duality and Higgs bundles

1. Quadratic duality for graded algebras

To put things in context, let us exhibit the spectral correspondence as a special case of a general class of duality transformations known as *Koszul dualities*.

Let

$$A = \bigoplus_{i \geq 0} A_i, \qquad A_i \cdot A_j \subset A_{i+j}$$

be a non-negatively graded unital complex associative algebra. For simplicity we will also assume that:

• A is connected, i.e. $A_0 = \mathbb{C}$ and the unit 1_A satisfies $1_A = 1 \in \mathbb{C} = A_0 \subset A$.

• A is locally finite dimensional, i.e. $\dim_{\mathbb{C}} A_i < \infty$.

Note that every such A has a natural augmentation with augmentation ideal $A_+ := \bigoplus_{i>0} A_i$.

Definition *A graded algebra A is called* quadratic *if A is generated by A_1 and has relations of degree two.*

This means that the canonical map

$$\mathrm{can} : \boldsymbol{T}A_1 \to A$$

is surjective and that ker(can) is generated by its degree two component

$$R_A := \ker(\mathrm{can}) \cap (A_1 \otimes A_1)$$

as a two-sided ideal in $\boldsymbol{T}A_1$.

Equivalently, a quadratic algebra is uniquely determined by a space of generators $V := A_1$ and an arbitrary subspace $R \subset V \otimes V$ of quadratic relations.

Notation: $A = \{V, R\}$.

Definition *The* quadratic dual *of a quadratic algebra $A = \{V, R\}$ is the quadratic algebra $A^! := \{V^\vee, R^\perp\}$.*

The quadratic duality functor

$$! : (\mathsf{quadAlg}_{\mathbb{C}}) \to (\mathsf{quadAlg}_{\mathbb{C}})^{op}$$

is an involutive equivalence of categories, i.e. $(A^!)^! = A$.

Examples • Let V be a finite dimensional $\mathbb{C}$-vector space. Then $\boldsymbol{T}V = \{V, 0\}$ is quadratic and $(\boldsymbol{T}V)^! = \mathbb{C} \oplus V^\vee \oplus 0 \oplus \ldots$.

• The algebras $S^\bullet V$ and $\wedge^\bullet V$ are obviously quadratic. Moreover $(S^\bullet V)^! = \wedge^\bullet V^\vee$.

We also have quadratic duality for modules over dual quadratic algebras:

Let A be a quadratic algebra and let $M = \bigoplus_{i\geq 0} M_i$ be a graded left module over A. We say that M is *quadratic* if there is a subspace $L \subset A_1 \otimes M_0$ so that

$$M = (A \otimes M_0)/(A \cdot L).$$

If M is a quadratic module, then its *quadratic dual* is a module $M^!$ over the quadratic dual algebra $A^!$ defined by

$$M^! := (A^! \otimes M_0^\vee)/(A^! \cdot L^\perp),$$

where $L^\perp \subset A_1^\vee \otimes M_0^\vee$ is the annihilator of L.

Again the duality functor

$$! : (A - \text{quadmod}) \to (A^! - \text{quadmod})^{op}$$

establishes an involutive equivalence of categories.

Example: ! converts free A-modules into trivial $A^!$-modules and vice versa. In other words we have isomorphisms of $A^!$-modules:

$$\mathbb{C}^! \cong A^! \quad \text{and} \quad A^! \cong \mathbb{C}.$$

The Koszul algebras and modules are particular instances of quadratic algebras and modules. They have the property that the quadratic dual can be described completely in cohomological terms.

Definition *A quadratic algebra A is called Koszul if we have an isomorphism of graded algebras* $\mathrm{ext}_A^\bullet(\mathbb{C},\mathbb{C}) \cong A^!$. *A quadratic module M over a Koszul algebra A is called a Koszul module if we have an isomorphism of $A^!$ modules:* $M^! \cong \mathrm{ext}_A^\bullet(M,\mathbb{C})$.

Here $\mathrm{ext}_A^\bullet$ denotes the Yoneda exts in the category of graded left A-modules. Note that these ext groups are naturally bigraded and so, in particular, if A is Koszul we have that $\mathrm{ext}_A^{ij}(\mathbb{C},\mathbb{C}) = 0$ for all $i \neq j$. It turns out that this is also sufficient for Koszulity. Similarly M is Koszul iff $\mathrm{ext}_A^{ij}(M,\mathbb{C}) = 0$ for $i \neq j$.

Remark: If A is a finite dimensional Koszul agebra, then the quadratic duality functor for modules can be extended to give an equivalence of derived categories of graded modules over A and $A^!$ respectively. More precisely one has the so-called Koszul duality functor

$$D^b(A-\mathrm{mod}^{fg}) \longrightarrow D^b((A^!-\mathrm{mod}^{fg})^{op})$$
$$M \longrightarrow \mathrm{cobar}(A, M).$$

Here cobar(A) is the cobar algebra associated to A, namely cobar(A) is a dg algebra, which is free as a graded algebra, i.e. $\mathrm{cobar}(A) = \boldsymbol{T}(A_+^\vee[-1])$, and has a differential which is uniquely determined by the property that its restriction to $A_+^\vee$ is the dual $A_+^\vee \to A_+^\vee \otimes A_+^\vee$ to the multiplication $A_+ \otimes A_+ \to A_+$.

Similarly $\mathrm{cobar}(A, M) = \mathrm{cobar}(A) \otimes M^\vee$ as a complex of graded modules with a differential extending the action of A on M.

Example: (Bernstein-Gelfand-Gelfand) If V is a finite dimensional vector space, then the derived categories of modules over $S^\bullet V$ and $\wedge^\bullet V^\vee$ are equivalent.

Filtered quadratic duality

In order to make the connection with the Higgs bundles (and their generalizations) we will need a filtered version of quadratic duality which in its most general form is due to Leonid Positselski.

Let

$$F^0A \subset F^1A \subset F^2A \subset \ldots A,$$
$$F^iA \cdot F^jA \subset F^{i+j}A$$

be a filtered unital associative algebra over $\mathbb{C}$. We assume that:

• A is connected, i.e. $F^0A = \mathbb{C}$ and the unit satisfies $1_A = 1 \in \mathbb{C} = F^0A \subset A$;

• A is locally finite dimensional, i.e. $\mathrm{gr}_F(A)$ is a locally finite dimensional graded algebra.

Definition *A filtered algebra A is called a* (filtered) quadratic *algebra if A is generated by $F^1 A$ and has relations in degree $\leq$ two.*

This can be spelled out as follows. Consider the reduced tensor algebra

$$\boldsymbol{T}(1_A \in F^1 A) := \boldsymbol{T}(F^1 A)/\langle 1_{\boldsymbol{T}} - 1_A \rangle$$

generated by the two step filtration

$$[\mathbb{C} \cdot 1_A \subset F^1 A].$$

This is a filtered algebra with a filtration

$$\boldsymbol{T}_0(1_A \in F^1 A) \subset \boldsymbol{T}_1(1_A \in F^1 A) \subset \ldots,$$

given by $\boldsymbol{T}_i(1_A \in F^1 A) := \mathrm{im}(F^1 A)^{\otimes i}$. Now A is a filtered quadratic algebra if the canonical map

$$\mathrm{can} : \boldsymbol{T}(1_A \in F^1 A) \to A$$

is surjective and if ker(can) is generated by its subspace

$$J_A := \ker(\mathrm{can}) \cap \boldsymbol{T}_2(1_A \in F^1 A)$$

as a two-sided ideal in $\boldsymbol{T}(1_A \in F^1 A)$.

Equivalently, a filtered quadratic algebra is uniquely determined by:

• a finite dimensional vector space W (generators);

• a fixed vector $e \in W$ (unit);

• a subspace $J \subset \boldsymbol{T}_2(e \in W)$ (relations).

Indeed, given A we can take $W = F^1 A$, $e = 1_A$ and $J = J_A$. Conversely, given W, e and J we define $A := \boldsymbol{T}(e \in W)/\langle J \rangle$.

Notation: $A = \{e \in W, J\}$.

Note: A filtered quadratic algebra $A = \{e \in W, J\}$ has an associated ordinary quadratic algebra

$$A^{(0)} := \{W/\mathbb{C} \cdot e, J \,\mathrm{mod}\, \boldsymbol{T}_1(e \in W)\}.$$

Moreover $A^{(0)}$ coincides with the quadratic part $q\,\mathrm{gr}_F(A)$ of the associated graded algebra $\mathrm{gr}_F(A)$.

We say that A is a Koszul algebra if $A^{(0)}$ is a Koszul algebra. It turns out that if A is Koszul, then $\mathrm{gr}_F(A)$ is in fact isomorphic to $A^{(0)}$.

Remark: For any graded algebra $A = \oplus_{i \geq 0} A_i$, there is a uniquely defined quadratic algebra qA together with a canonical map $qA \to A$ which is an isomorphism in degree one and a monomorphism in degree two. Explicitly we have

$$qA = \{A_1, \ker(\boldsymbol{T} A_1 \to A) \cap A_1 \otimes A_1)\}.$$

The algebra qA is called *the quadratic part* of A.

For any graded algebra A and any graded A-module M concentrated in non-negative degrees, there is a uniquely defined quadratic module $q_A M$ over the quadratic part qA of A, together with a morphism $q_A M \to M$ of qA-modules which is an isomorphism in degree zero and a monomorphism in degree one. Explicitly

$$q_A M = \{M_0, \ker(qA \otimes M_0 \to M) \cap A_1 \otimes M_0\}_{qA}.$$

The module $q_A M$ is called the *quadratic part* of M.

Hope: In the Koszul case, we may be able to define a quadratic dual of a filtered A by endowing the dual graded algebra $(A^{(0)})^!$ with some extra data remembering the extensions corresponding to the filtration $F^\bullet A$. It turns out that the extra data needed is a differential on the graded algebra $(A^{(0)})^!$.

Definition *A* curved differential graded algebra *is a triple $\boldsymbol{B} = (B, d_B, h_B)$, where $B = \oplus_{i=0}^{\infty} B_i$ is a graded algebra, $d_B : B_i \to B_{i+1}$ is an odd derivation, and $h_B \in B_2$ is an element such that $d_B^2 x = [h_B, x]$ for all x and $d_B h_B = 0$.*

A morphism $\boldsymbol{g} : \boldsymbol{B} \to \boldsymbol{C}$ of curved dg algebras is a pair $\boldsymbol{g} = (g : B \to C, \alpha \in C_1)$, satisfying

$$g(d_B x) = d_C g(x) + [\alpha, g(x)]$$
$$g(h_B) = h_C + d_C \alpha + \alpha^2.$$

A left curved dg module over $\boldsymbol{B}$ is a pair $\boldsymbol{N} = (N, d_N)$ consisting of a graded B-module N and an odd derivation $d_N : N_i \to N_{i+1}$ compatible with d_B and such that $d_N^2 u = h_B u$.

A curved dg algebra is called *quadratic* or *Koszul* if the underlying graded algebra is of the corresponding type. A curved dg algebra witha zero curvature $h_B = 0$ is just an ordinary dg algebra.

Given a filtered quadratic algebra A, choose a subspace $V \subset F^1A$ splitting the map $F^1A \to F^1A/F^0A$. Let R denote the space $J_A \bmod \boldsymbol{T}_1(1_A \in F^1A)$ viewed as a subset in $V \otimes V$. In terms of V and R we have

- $A^{(0)} = \{V, R\}$, $\boldsymbol{T}_i(1_A \in F^1A) = \oplus_{k=0}^{i} V^{\otimes i}$, and
- $J_A \subset \boldsymbol{T}_2(1_A \in F^1A) = \mathbb{C} \oplus V \oplus (V \otimes V)$ is the graph of some linear map

$$\psi = (\varphi, h) : R \to V \oplus \mathbb{C}$$

Moreover the map ψ satisfies

$$(*) \qquad (\psi^{12} - \psi^{23})(V \otimes R \cap R \otimes V) \subset \Gamma_\psi$$

Let $B := (A^{(0)})^!$. By definition $B_2 = R^\vee$ and so φ and h dualize to $\varphi^\vee : B_1 \to B_2$ and $h_B \in B_2$. The condition (*) translates into the fact that φ^* can be extended to an odd derivation d_B of degree 1 and that (B, d_B, h_B) is a curved dg algebra.

Note: The above construction assigns a curved dga $\boldsymbol{B} = (B, d_B, h_B)$ to every filtered quadratic algebra $\{e \in W, J\}$ equipped with a splitting $W/\mathbb{C} \cdot e \xrightarrow{\sim} V \subset W$ of the map $W \twoheadrightarrow W/\mathbb{C} \cdot e$. We will call such algebras *almost-split*.

Choosing a different splitting V' results in an isomorphic curved dga $\boldsymbol{B}' = (B, d'_B, h'_B)$. Indeed, given $V' \subset W$ we can find a linear map $\alpha : V \to \mathbb{C}$ such that

$$V' = \{x - \alpha(x) | x \in V\}.$$

In terms of α the isomorphism $\boldsymbol{f} : \boldsymbol{B} \xrightarrow{\sim} \boldsymbol{B}'$ is given by the pair $\boldsymbol{f} = (\mathrm{id}_B, \alpha^\vee)$.

The assignment $A \mapsto ((A^{(0)})^!, \varphi^\vee, h^\vee)$ gives rise to a filtered quadratic duallity functor

$$! : (\text{filt-quadAlg}_{\mathbb{C}}^{as}) \to (\text{c-dgAlg}_{\mathbb{C}})^{op}$$

which is fully faithful.

Under the Koszulity assumption one can say more:

Theorem [L.Positselski] *The filtered quadratic duality functor establishes an anti-equivalence between the category of almost split filtered Koszul algebras (respectively augmented filtered Koszul algebras) and the category of Koszul curved dg algebras (respectively Koszul dg algebras).*

Similarly one can show:

Theorem [L.Positselski] *For any almost split filtered quadratic algebra A there is a fully faithful functor between the category of quadratic A-modules and the category of curved dg modules over $A^!$. For Koszul algebras this functor induces an anti-equivalence of the respective categories of Koszul modules.*

Remark: The special case of quadratic duality for augmented filtered algebras was worked out originally by S.Priddy. It was rediscovered later by C.Simpson under the name 'duality for split almost polynomial rings of differential operators'.

3. Higgs bundles revisited

We are now ready to recast the spectral construction for Higgs bundles on a smooth complex space S as a filtered Koszul duality for families of algebras over S.

Let as before K be a fixed coefficient bundle and let $X = \text{tot}(K)$ be its total space. Consider the sheaf of algebras $A = S^{\bullet}K^{\vee}$ on S with the natural filtration induced from the grading. The quadratic dual algebra is the trivial dg algebra $(\wedge^{\bullet}K, 0, 0)$:

$$\mathcal{O}_S \xrightarrow{0} K \xrightarrow{0} \wedge^2 K \xrightarrow{0} \ldots \xrightarrow{0} \wedge^n K.$$

Every quasi-coherent sheaf $\mathscr{E}$ on X can be viewed as a filtered module over A and so corresponds by quadratic duality to a dg module $\mathscr{E}^!$ over $(\wedge^\bullet K, 0, 0)$. Explicitly we have

$$\mathscr{E}^! = \left(E \stackrel{\wedge\phi}{\to} E \otimes K \stackrel{\wedge\phi}{\to} \ldots \stackrel{\wedge\phi}{\to} E \otimes \bigwedge^n K \right),$$

where (E, ϕ) is the corresponding Higgs sheaf.

Note that even though the differential in the quadratic dual algebra $A^!$ is trivial, we still can have a non-trivial differential for the module.

The Koszul reinterpretation of the spectral construction is useful because it gives us a way to deform the correspondence. Next we explore various commutative and non-commutative deformations of the spectral construction.

5. Deformations of the spectral construction

Fix a smooth complex variety S and a (coefficient) vector bundle K of rank n on S.

Let $p : X = \mathrm{tot}(K) \to S$ be the total space of K. The spectral correspondence establishes a bijection between the following types of geometric data

(Spectral data) Coherent sheaves $\mathscr{E} \to X$ which are finite over S (B-branes on X).

(K-valued Higgs data) Coherent sheaves $E \to S$ equipped with a Higgs field ϕ, i.e. an $\mathcal{O}_S$-linear K-valued endomorphism

$$\phi : E \to E \otimes K$$

satisfying $\phi \wedge \phi = 0$.

We interpreted the correspondence as a special case of filtered Koszul duality as follows:

- View the spectral sheaf $\mathscr{E} \to X$ as a module over the sheaf of algebras $S^\bullet K^\vee$ over S, i.e. replace $\mathscr{E}$ with the equivalent data

$$(E := p_*\mathscr{E} \to S) + (S^\bullet K^\vee - \text{action on } E).$$

- View the Higgs sheaf (E, ϕ) as a dg module

$$E \stackrel{\wedge\phi}{\to} E \otimes K \stackrel{\wedge\phi}{\to} \ldots \stackrel{\wedge\phi}{\to} E \otimes \wedge^n K$$

over the dga $\mathcal{O}_S \stackrel{0}{\to} K \stackrel{0}{\to} \ldots \stackrel{0}{\to} \wedge^n K$.

- Use filtered Koszul duality to convert modules over the filtered quadratic algebra $S^\bullet K^\vee$ and dg modules over the dg algebra $(\wedge^\bullet K, 0)$.

Remark: • The Koszul reformulation of the spectral correspondence has the adavntage of exhibiting both the Higgs and the spectral data in a manifestly deformable form.

Indeed, by deforming the structures on $S^\bullet K^\vee$ and $(\wedge^\bullet K, 0)$ so that the Koszul duality still holds, we can obtain a new kind of spectral duality between the deformed module structures.

• Note that there are three possible ways in which we can perturb the structure of $S^\bullet K^\vee$ so that the resulting algebra will still be filtered quadratic.

Indeed $S^\bullet K^\vee$ is a filtered quadratic algebra of the most trivial type: it is commutative, augmented and the filtration is completely split.

Thus when we start deforming the product structure on $S^{\bullet}K^{\vee}$ we can perform the deformation so that:

$\Diamond$ the product becomes non-commutative;

$\Diamond$ the augmentation ceases to be an algebra morphism;

$\Diamond$ the filtration is not split anymore.

Similarly we can deform the curved dg algebra structure on $(\wedge^{\bullet}K, 0, 0)$ so that:

$\Diamond$ the product becomes non-commutative;

$\Diamond$ the differential becomes non-zero;

$\Diamond$ the curvature becomes non-zero.

Note: An interesting feature of Koszul duality is that the duality transformation mixes the different types of deformations.