Asymptotic Analysis, Multivalued Morse theory, and a Plan of a proof of Mirror Conjecture

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§0

We assume $M, M^\vee$ are compactifications of dual torus fibrations, and will formulate a series of conjectures which imply

(1) "(GW potential of M)" $=$ Yukawa coupling of $M^\vee$

(2) Lag. submfd + Line bdl on M $(L, \mathcal{L})$
$\Leftrightarrow$ Coherent sheaf on $M^\vee$ $\mathcal{E}(L, \mathcal{L})$

(3) $HF((L_1, \mathcal{L}_1), (L_2, \mathcal{L}_2)) \cong \underline{Ext}(\mathcal{E}(L_1, \mathcal{L}_1), \mathcal{E}(L_2, \mathcal{L}_2))$

((2) + (3) $\Leftrightarrow$ HMS conj. by Kontsevich)

$\mathbb{Z}^n \hookrightarrow \Lambda \to B_0$ $\quad SL(n,\mathbb{Z})$ bdle

$E := \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \to B_0$

$M_0 := E/\Lambda \to B_0$ $\quad T^n \circlearrowright M_0 \to B_0$

$E^* = \mathrm{Hom}(E, \mathbb{R})$

$\Lambda^\vee = \{\alpha \in E^* \mid \alpha(\Lambda) \subseteq \mathbb{Z}\}$

$E^*/\Lambda^* = M_0^\vee \to B_0$

$B \supset B_0 \quad S(B) := B \setminus B_0$ codim ≥ 2

$$\begin{array}{l} M_0 \subset M \searrow \\ \qquad\qquad\qquad B \\ M_0^\vee \subset M^\vee \nearrow \end{array}$$

(2)

We define a multivalued function

$f : B \to \mathbb{R}$.

(M, ω) symplectic

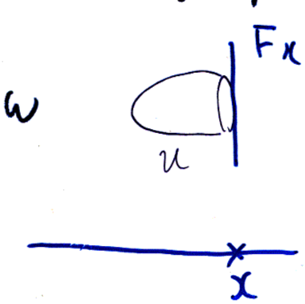
$\begin{cases} \omega|_{F_x} \equiv 0 & (F_x = \pi^{-1}(x) : L \text{ fiber}) \\ \omega|_{0 \text{ section}} \equiv 0 \end{cases}$

Put

$$\tilde{B}_0 := \left\{ (x, [u]) \;\middle|\; \begin{array}{l} x \in B_0 \\ u : (D^2, \partial) \to (M, F_x) \\ [u] \in \pi_2(M, F_x) \end{array} \right\}$$

$\tilde{B}_0 \to B_0 \quad (x, [u]) \to x$ covering space

$f(x, [u]) := \int_{D^2} u^* \omega$



Regard $f : \tilde{B}_0 \to \mathbb{R}$

as a multivalued fcn on B_0

(3)

A model

(M, ω) symplectic

holomorphic map

$\Sigma \longrightarrow M$

B model

$(M^{\vee}, J)$

cpx.

$J = J_0 + b$

T^n equivariant

dim F_x

$\| $

$\varepsilon \to 0$

Morse (homotopy) theory of $f : \tilde{B}_0 \to \mathbb{R}$

(F, Betz-Cohen)

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asym. exp. of b

as $\varepsilon \to 0$

∘ Floer

$HF(L,L) \cong H(L)$

@ F, Oh (Arxiv J. 1)

$M = T^*B_0$

@ Kontsevich-Soibelman

$M = T^{2n}$

$\Sigma = D^2$

$S(B) = \phi$

Witten

Super symmetry and Morse theory

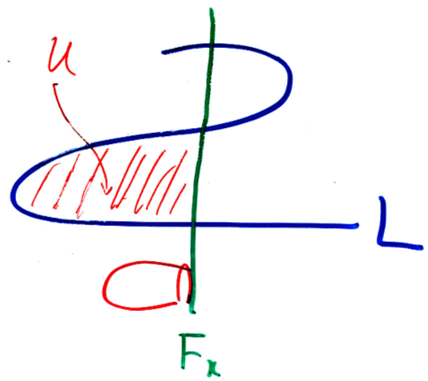
its analogue

• $d \to \bar{\delta}$

• multi valued

• with interaction

Generalization for $L \subseteq M$
$(\omega|_L \equiv 0)$



$$f(x) = \int u^*\omega$$

A model

$(\Sigma, \partial\Sigma) \to (M, L)$
holomorphic

$(D^2, \partial) \to (M, L_0 \cup \dots \cup L_k)$

B model

- g correction to holo. str. of $\mathcal{E}(L)$ (l.d.l.)
- $\bar\partial \subseteq \mathrm{Hom}(\mathcal{E}(L_1), \mathcal{E}(L_2)) \otimes \Lambda^{0,*}$
- Prod on Extensions

⑤

§1 A model

$$T^n \hookrightarrow M_0 \xrightarrow{\pi} B_0^n \quad (M_0, \omega)$$

$$\omega|_{F_x} \equiv 0 \quad (F_x = \pi^{-1}(x) \cong T^n)$$

$x_1, \dots, x_n$ coordinate of B_0
$y_1, \dots, y_n$ " T^n

$$\omega = \sum dx^i \wedge dy^i$$

$\sum g_{ij}\, dx^i dx^j$ metric on B_0
$\downarrow$
J: almost cpx str. on M_0

$$J\left(\frac{\partial}{\partial x^i}\right) = \sum g_{ij} \frac{\partial}{\partial y^j}$$

⑥

∘ Free field

 $\to M$

Let $\begin{cases} \ell : (a,b) \to B_0 & \text{arc.} \\ \gamma \in \mathbb{Z}^n = \pi_1(T^n) & \gamma = (\gamma_1, \cdots, \gamma_n) \end{cases}$

$$\ell^* M_0 \cong (a,b) \times T^n$$

Put

$$u(\tau, t) = (\ell(\tau), t\gamma_1, \cdots, t\gamma_n) \in \ell^* M_0 \subseteq M_0$$

$$u : (a,b) \times S^1 \longrightarrow M_0$$

Prop u is J-holomorphic

$$\Leftrightarrow \frac{\partial \ell}{\partial \tau} = \operatorname{grad} f_\gamma$$

(f_γ: branch of $f = \int u^* \omega \quad u|_\partial = \gamma$)

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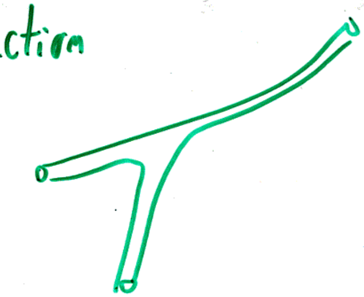
∘ Interaction



$n = 2 \qquad M_0 = \underset{x_1\, x_2}{\mathbb{R}^2} \times \underset{y_1\, y_2}{T^2}$

$$J \frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$$

$\begin{matrix} \gamma = (\gamma_1, \gamma_2) \\ \mu = (\mu_1, \mu_2) \end{matrix} \in \mathbb{Z}^2 \qquad \gamma + \mu \in \mathbb{Z}^2$

$$u_\gamma(\tau, t) \overset{\text{def}}{=} (\tau\gamma_1, \tau\gamma_2, t\gamma_1, t\gamma_2)$$

$$u_\gamma : \mathbb{R} \times S^1 \longrightarrow \mathbb{R}^2 \times T^2$$

u_μ, $u_{\gamma+\mu}$: similar

u_γ, u_μ, $u_{\gamma+\mu}$ are holomorphic

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Prop

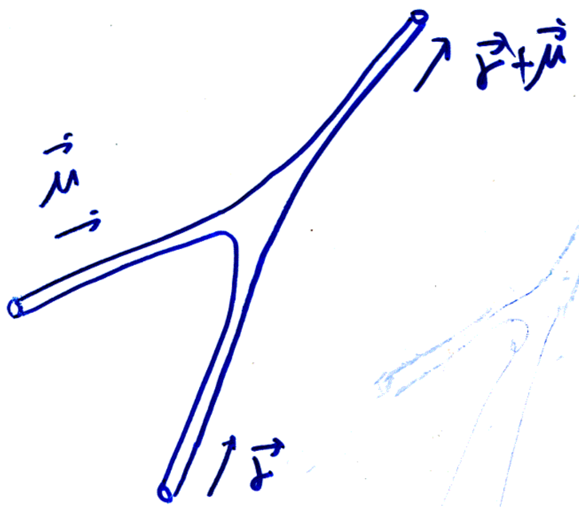
$\exists u : S^2 \setminus \{0, 1, \infty\} \to \mathbb{R}^2 \times T^2$

s.t. $u \sim u_\gamma$ at 0

$u \sim u_\mu$ at 1

$u \sim u_{\gamma+\mu}$ at ∞

multiplicity $= |\gamma_1 \mu_2 - \gamma_2 \mu_1| = |\gamma \cdot \mu|$



⑨

○ Singular fiber

$x_0 \in S(B) = B \setminus B_0$

$U(x_0)$ nbd. $\quad x \in U(x_0) \cap B_0$

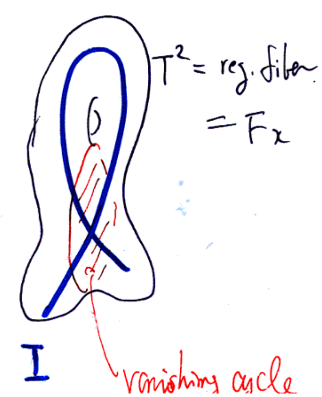
Def.

$[u] \in \pi_2(M, F_x)$ is a vanishing cycle

$\Leftrightarrow$ $\mathrm{Im}\, \pi \circ u \subseteq U(x_0)$

Ex. Type I singular fiber

$S^2 \looparrowright M^4$ (intersect with itself transversally)



⑩

$$C := \left\{ (x, [u]) \;\middle|\; \begin{array}{l} x \in V(x_0) \cap B_0 \\ [u] \in \pi_2(M, F_x) \text{ vanishing cycle} \\ \exists u : (D^2, \partial) \to (M, F_x) \\ \qquad J\text{-holomorphic} \end{array} \right\}$$

$$C' := \left\{ (x, [u]) \;\middle|\; \begin{array}{l} x \in V(x_0) \cap B_0 \\ [u] \text{ vanishing cycle} \\ \lim_{t \to \infty} \exp(-t \operatorname{grad} f_u) x \in S(B) \cap V(x_0) \end{array} \right\}$$

(where f_u : branch of f $\quad f_u = \int u^* \omega$
grad f_u : gradient vect. field
$\exp(-t \operatorname{grad} f_u)$ 1 para gp associated to it)

Conj $C \sim C'$ isotopic

Prop true for type 1 sing. fiber.
([F000] Chap 7)

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° conclusion

$$u : (D^2, \partial) \longrightarrow (M, F_x)$$
$$\Longleftrightarrow (\Gamma, \varphi, \ldots\ldots)$$

similar for S^2, Σ_g^2

Γ : oriented graph, $\pi_1 \Gamma = 1$
$\varphi : \Gamma \longrightarrow B_0$ st.

① vertex of Γ are

only 1

②  $\to$ put its length $\in [0, \infty)$

③ φ of $\downarrow$ $\in S(B)$

④ φ of $\curlyvee$ $= x$

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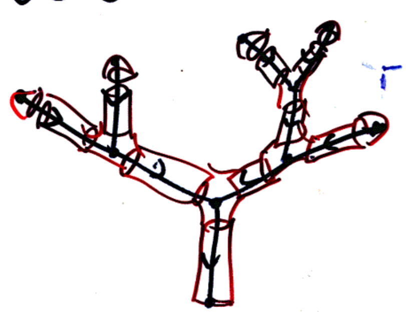
⑤ $\varphi|_{\text{edge}}$ = gradient line of f

⑥ chosen a branch of f for each edge

s.t. (6-1) 

(6-2) 

$\rightsquigarrow$ $u_1+u_2=u_3$

Proof 

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§2 B model

(14)

Semi-flat mirror (review)

$\mathbb{Z}^n \hookrightarrow \Lambda \to B$

$E = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ $\quad \nabla^E$ flat conn.

$T^n \hookrightarrow M_0 = E/\Lambda \longrightarrow B_0$

$\sim$

E^*, $\Lambda^\vee$ dual $\quad \nabla^{E^*}$ flat conn

$T^n \hookrightarrow M_0^\vee = E^*/\Lambda^\vee \to B_0$

- ω (symplectic str. on M_0), $\omega|_{F_x} \equiv 0$, $\omega|_{0\text{ sect}} \equiv 0$

  $\iff$ $\varphi : TB_0 \cong E^*$ st. $\varphi^*\nabla^{E^*}$ is torsion free

- J_0 complex str. on M_0, T^n equivariant

  J_0(Horizontal) $\subseteq$ Virtical, J_0(virtical) $\subseteq$ Horizontal

  $\iff$ $\psi : TB_0 \cong E$ st. $\psi^*\nabla^E$ is torsion free

ω on M_0

$\varphi : TB_0 \cong E^*$

J on M_0

$\psi : TB_0 \cong E$

ω on $M_0^\vee$

$\varphi^\vee : TB_0 \cong (E^*)^* = E$

J on $M_0^\vee$

$\psi^\vee : TB_0 \cong E^*$

ω extends to $M \supseteq M_0$

$M^\vee \supset M_0^\vee$

But J_0 (which is T^n equivariant) does $\boxed{\text{not}}$ extend.

- $\bar\partial \leftrightarrow J$ cpx. str. on $M^\vee$

- $\bar\partial_0 \leftrightarrow J_0$ " $M_0^\vee$, T^n equiv.

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$$\bar\partial = \bar\partial_0 + b \qquad \bar\partial^2 = \bar\partial_0^2 = 0$$

$$b \in \Gamma(M_0^\vee,\ T_{\mathbb{C}} M_0^\vee \otimes \Lambda^{0,1})$$

$$\bar\partial_0 b + \frac{1}{2}[b, b] = 0$$

Put

$$b = \sum_{\substack{\gamma \neq 0 \\ \gamma \in \pi_1(F_x)}} \exp(\sqrt{-1}\,\gamma^*)\,\tilde{b}_\gamma$$

$x \in B_0$ $\qquad F_x = \pi^{-1}(x) \subset M_0$

$F_x^\vee = \pi^{\vee -1}(x) \subset M_0^\vee$ dual

Poin. dual $\pi_1(F_x) \cong \mathrm{Hom}(F_x^\vee, U(1))$

$\gamma \longmapsto \exp(\sqrt{-1}\,\gamma^*)$

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$\gamma^* : \tilde{F}_x^\vee \longrightarrow \mathbb{R}$ grp homo.

$\rightsquigarrow$

$$\Gamma(M_J^\vee, T_J M_J^\vee \otimes \Lambda^{0,1})^{T^n} \leftarrow T^n \text{ equiv. part}$$

$$\cong \Gamma(B_0, TB_0 \otimes \Lambda^1 B_0)$$

$\because$) $x_1, \cdots, x_m$ coordinate of B_0

$y_1, \cdots, y_m$ 〃 T^n

$$J_0\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad z_i = x_i + \sqrt{-1}\, y_i$$

$$\frac{\partial}{\partial z_i} \otimes d\bar{z}_j \Longleftrightarrow \frac{\partial}{\partial x_i} \otimes dx_j$$

$$\tilde{b} \Longleftrightarrow b$$

$\rightsquigarrow$

$$b = \sum_{\substack{\gamma \neq 0 \\ \in \pi_2(M, F_x)}} \exp(\sqrt{-1}\,\gamma^*)\, \tilde{b}_\gamma$$

Fiber wise Fourier expansion

(17)

Conjecture

$$b \underset{\varepsilon \to 0}{\sim} \sum_{\gamma \neq 0} \exp\left(-\frac{f_\gamma}{\varepsilon} + \sqrt{-1}\,\gamma^*\right) \tilde{C}_\gamma$$

- C_γ is described by Morse theory of f
- ε = Diam of fiber

$u : (D^2, \partial) \to (M, F_x)$, $\gamma = u_*$

$\rightsquigarrow$

- Free Field

$$\bar{\partial}_0 b + \cancel{\tfrac{1}{2}[b, b]} = 0$$

Prop. $\begin{cases} b = \sum \exp(\sqrt{-1}\,\gamma^*)\, \tilde{b}_\gamma \\ \bar{\partial}_0 b = 0 \end{cases}$

$\Longleftrightarrow$ $d(\exp(f_\gamma/\varepsilon)\, b_\gamma) = 0$

(18)

<u>Witten</u> Supper symmetry & Morse theory

$f : N \to \mathbb{R}$ Morse f.c.n.

$d_{f,\varepsilon} = e^{-f/\varepsilon} \circ d \circ e^{f/\varepsilon}$

$\Delta_{f,\varepsilon} = - d_{f,\varepsilon}^{*} \circ d_{f,\varepsilon} - d_{f,\varepsilon} \circ d_{f,\varepsilon}^{*}$

$\mathcal{H}^{k}_{f,\varepsilon} = \left\{ u \in \Lambda^k N \,\middle|\, \begin{array}{l} \Delta_{f,\varepsilon} u = \delta u \\ \delta \ll 1 \end{array} \right\}$

$\underset{\text{Witten}}{\cong} \bigoplus_{\substack{p,\ d_p f = 0 \\ \text{Morse index} = k}} \mathbb{R}[p]$

$\langle d S_p, S_q \rangle \sim e^{-(f(q)-f(p))/\varepsilon}$ # grad. lines

$S_p \in \mathcal{H}^{k}_{f,\varepsilon} \longleftrightarrow p$

- Most of the Mass $\|S_p\|$ is in nbd of p
- S_p propagate along grad f.

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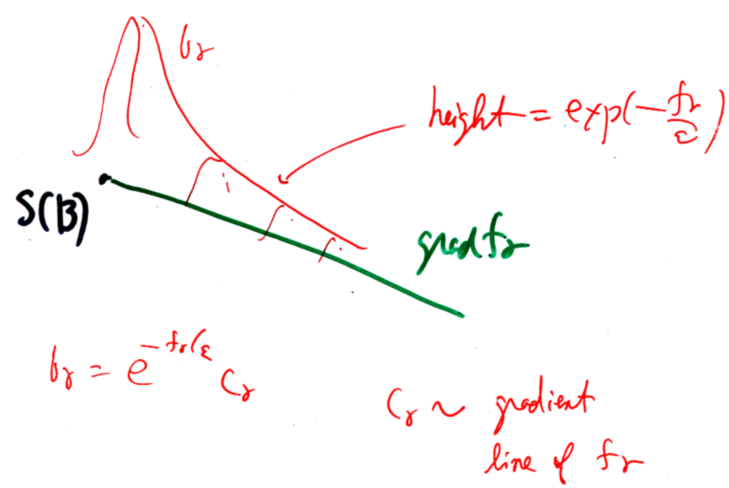
Our case:

$d(e^{f_\gamma/\varepsilon}\, b_\gamma) = 0$

$\leadsto$. Most of Mass $\|b_\gamma\|$ is in the nbd. of singular locus $S(B)$

- It propagate along grad f_γ



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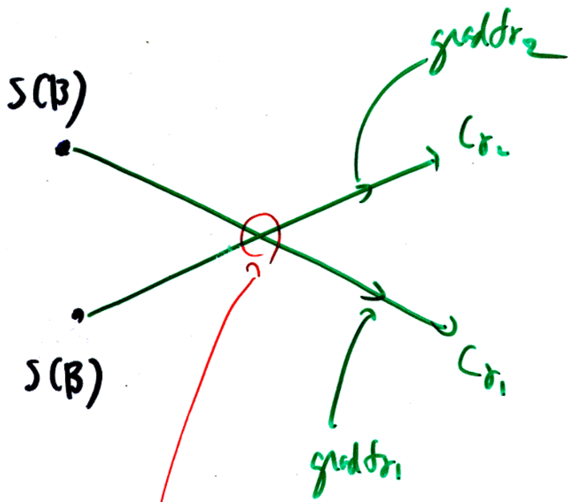
∘ Interaction.

$$\bar{\partial}_0 b + \frac{1}{2}[b,b] = 0$$



$[C_{\gamma_1}, C_{\gamma_2}] = 0$

$$[b_{\gamma_1}, b_{\gamma_2}] = \exp\left(-\frac{\delta_{\gamma_1+\gamma_2}}{\varepsilon}\right)[C_{\gamma_1}, C_{\gamma_2}] \neq 0$$

$$\bar{\partial}_0 b_{\gamma_1+\gamma_2} = -\frac{1}{2}[b_{\gamma_1}, b_{\gamma_2}]$$

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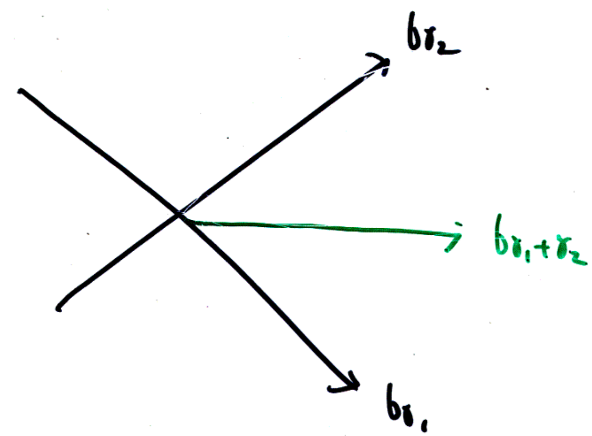
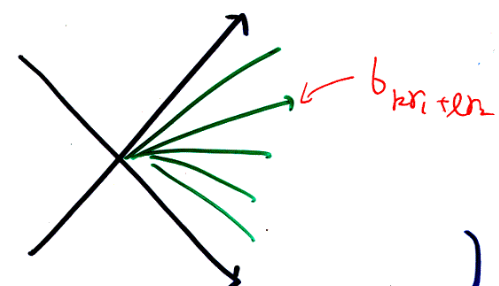


$\sim$

(In fact



)

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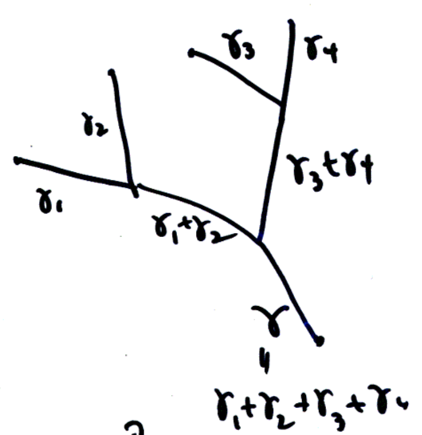
Hence

$$b \sim \overline{\sum} \exp\left(-\frac{f_\gamma}{\varepsilon} + \sqrt{-1}\,\gamma_?\right) \tilde{C}_\gamma$$

C_γ is cal. by



$\subseteq B_0$

the same as A model.

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Conclusion

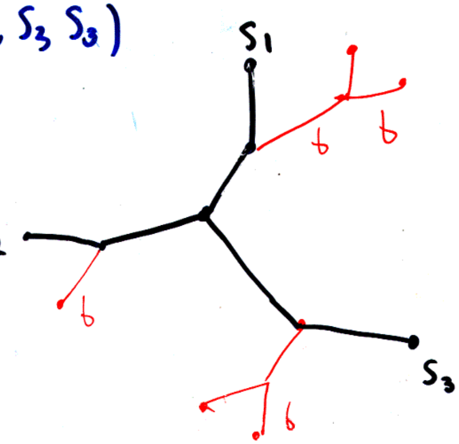
Yukawa coupling

$$s_i \in H^*_{\bar\partial_0 + b}(M^\vee, TM^\vee)$$

$$s_i \sim s_{i,0} + \overline{\sum_\gamma} \exp\left(-\frac{f_\gamma}{\varepsilon} + \sqrt{-1}\,\gamma_*\right) s_{i,\gamma}$$

$$\bar\partial_0 s_i + [b, s_i] = 0$$

$$Y(s_1, s_2, s_3) = \int i_\Omega(s_1 \wedge s_2 \wedge s_3) \wedge \Omega$$



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GW potential

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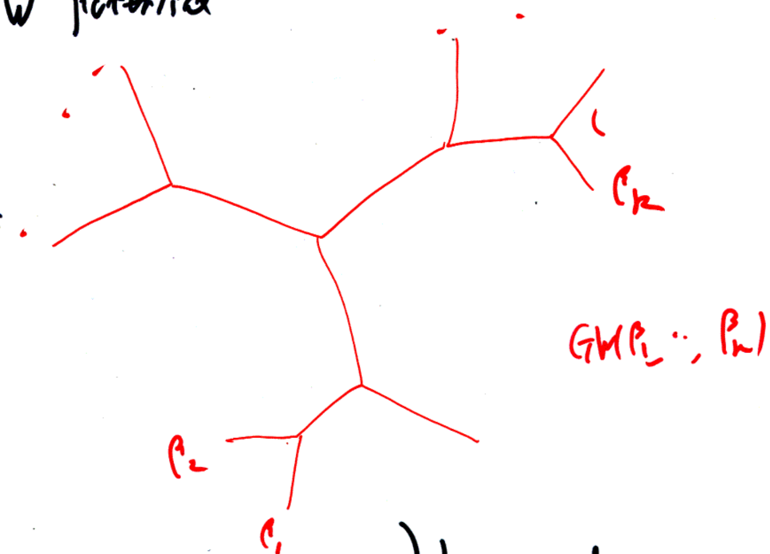

$\Downarrow$ third derivative