CLASSIFICATION OF TOPOLOGICAL INSULATORS AND SUPERCONDUCTORS

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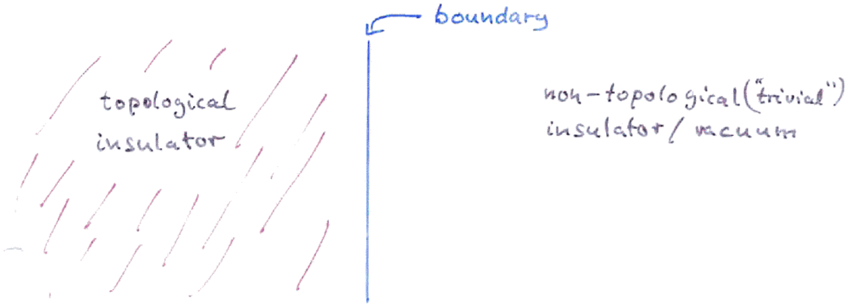
THEME OF THIS TALK:

Classifying topological properties of bulk insulators (or superconductors) by looking at their boundaries



A signature of the topological properties of the bulk is the appearance of gapless boundary degrees of freedom, which are entirely robust to perturbations [respecting the symmetries of the system (← more specific shortly)], including disorder.

THIS TALK:

We will use the appearance of robust gapless degrees of freedom on the sample boundaries, which cannot be Anderson localized by disorder, as a diagnostic of the topological properties of the bulk insulator.

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in other words:

We reduce the problem of classifying topological insulators (superconductors) in d spatial dimension to a problem of Anderson localization in $(d-1)$ dimensions.

We solve this problem of Anderson localization, and thereby solve the classification problem for topological insulators (superconductors).

This talk:

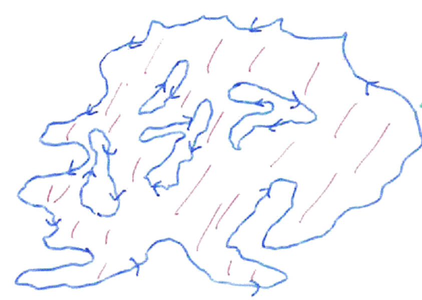
No interactions

(but due to gap: stable to sufficiently weak interactions)

SOME EXAMPLES:

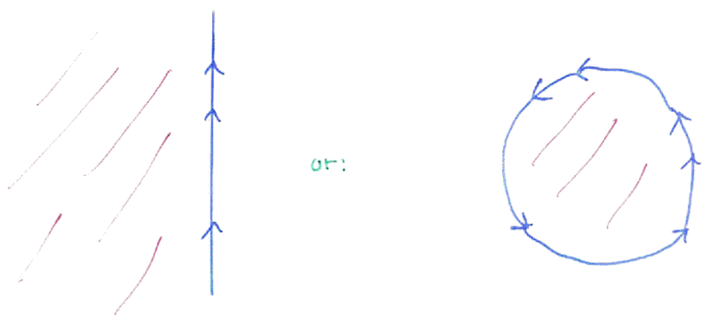
d=2: Integer Quantum Hall insulator (T broken)

Here: because the direction of propagation at the edge is in one direction (due to chirality from T-breaking) the edge state cannot (trivially) be localized by disorder.



{a disordered Quantum Hall insulator}

$d=2$: Chiral $(p_x + i p_y)$ superconductor (T broken)



Similar to Quantum Hall insulator, but no charge transport occurs at the edge, only heat transport [different "symmetries" → more specific shortly].

$d=2$: $\mathbb{Z}_2$ - topological insulator

More recently it was realized that gapped phases supporting topologically protected gapless states appearing at the sample boundaries may also appear in the absence of time reversal symmetry breaking.

These occur in certain band insulators with strong spin-orbit interactions, and are known as $\mathbb{Z}_2$ topological insulators, the "quantum spin Hall state" (QSH).

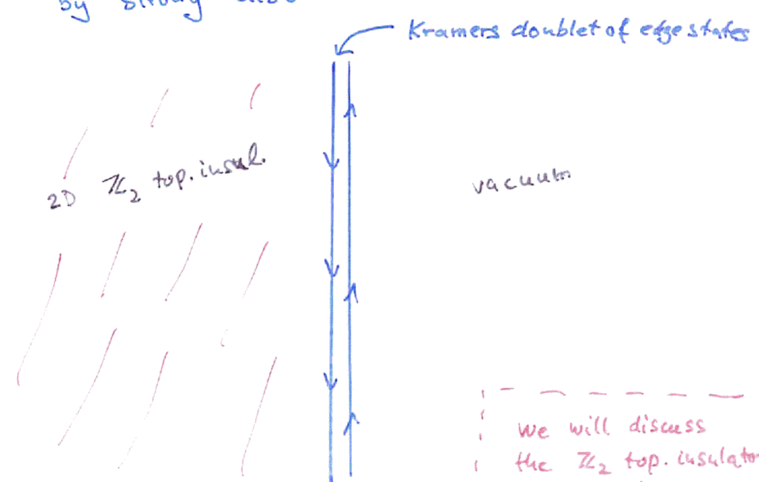
[work by Kane, Mele* and many others].

* e.g.:
PRL 95 (2005) 146802
PRL 95 (2005) 226801

— : experiments: HgTe, BiSb materials

→: A certain non-trivial topological invariant can be associated with these bulk states.

→: In $d=2$, say, $\mathbb{Z}_2$ topological insulators are known to possess gapless edge states consisting of a Kramers doublet. [2 members of doublet related by time-reversal] Even though there is a forward and a backward propagating mode, these cannot be Anderson localized even by strong disorder.



We will discuss the $\mathbb{Z}_2$ top. insulator in $d=3$ below.

This lack of Anderson localization was already observed essentially in 1992 by M. Zirnbauer(*), who studied Anderson localization in (quasi-) 1D wires in the so-called spin-orbit ("symplectic") symmetry class.

For an odd number of Kramers doublets absence of localization in 1D wires with spin-orbit scattering (and T symmetry) was observed [using today's language].

The significance of this was only understood later* when it was noted that an odd number of Kramers doublets cannot appear in 1D wires, except if the 1D system is the boundary of a 2D (topological) bulk.

* P. Brouwer + K. Frahm PRB 53 (1996); Y. Takane (2004)

(*) Zirnbauer, PRL 1992

In this talk we ask the general questions:

- Which systems possess gapped ground states with topologically non-trivial properties?
- How many such systems are there?

- Specifically, we may consider continuously deforming the Hamiltonian giving rise to a gapped phase. We then ask, how many different such phases a system can possess, so that in going from one phase to another a quantum phase transition has to be crossed (gap closes).

$\Rightarrow$: Because of the presence of the bulk gap, the bulk phases are robust against disorder.

Thus one is led to seek a classification of ground states of (in general) random gapped Hamiltonians.

‖ There are only 10 classes of random Hamiltonians. ‖

This underlies the well known classification of random matrices, and universality classes of Anderson localization transitions.

[Zirnbauer (1996), Altland + Zirnbauer (1997), Heinzner, Huckleberry, Zirnbauer (2004);]
also: Bernard + LeClair (2001)

Brief Review: Classification of random Hamiltonians – the "10-fold way" [Zirnbauer (1996), Altland + Zirnbauer (1997)]

In classifying random Hamiltonians one must consider only the most generic symmetries,

time reversal (T)

and

charge-conjugation (particle-hole) (C)

There are only 10 possible behaviors of a Hamiltonian under T and C.
(10 "symmetry classes")

The basic idea is simple to understand:

→: T is antiunitary: $T = \underbrace{\mathcal{U}_T}_{\text{unitary}} \cdot \underbrace{K}_{\text{complex conjugation}}$

$\mathcal{H}$ = 1^{st} quantized Hamiltonian

T: $\quad \mathcal{U}_T \, \mathcal{H}^* \, \mathcal{U}_T^\dagger = \mathcal{H}$

$$T = \begin{cases} 0 & \text{no time reversal invariance} \\ +1 & \text{time reversal invariance and } T^2 = +\mathbb{1} \\ -1 & \text{time reversal invariance and } T^2 = -\mathbb{1} \end{cases}$$

[SU(2) angular momentum = integer]

[SU(2) angular momentum = $\frac{1}{2}$-integer]

→: C is antiunitary: $C = \mathcal{U}_C \cdot K$

C: $\quad - \mathcal{U}_C \, \mathcal{H}^* \, \mathcal{U}_C^\dagger = \mathcal{H}$

$$C = \begin{cases} 0 & \text{no particle-hole symmetry} \\ +1 & \text{particle-hole symmetry and } C^2 = +\mathbb{1} \\ -1 & \text{particle-hole symmetry and } C^2 = -\mathbb{1} \end{cases}$$

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Note:

* There are $3 \times 3 = 9$ choices for $T \times C$

* For 8 of these choices the value of $S = TC$ is uniquely fixed:
  these are all except for "A" and "AIII".

* For "A" and "AIII":

T	C
0	0

=> free to choose $S = 0$ or 1,
yielding "A" and "AIII".

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TABLE – "10 fold way"

['CARTAN Classes']

(CARTAN) Name	T	C	S = CT	Hamiltonian $\mathcal{H}$ element of	(Symmetric space) NLσM Manifold G/H (Fermionic Replicas)	SU(2) SPIN CONSERVED	Examples	
A (unitary)	0	0	0	$U(N)$	$U(2N)/U(N)\times U(N)$	YES/NO	• IQHE • Anderson	WIGNER – DYSON
AI (orthogonal)	+1	0	0	$U(N)/O(N)$	$Sp(4N)/Sp(2N)\times Sp(2N)$	YES	• Anderson	
AII (symplectic)	−1	0	0	$U(2N)/Sp(2N)$	$O(2N)/O(N)\times O(N)$	NO	• Quantum Spin Hall: $\mathbb{Z}_2$-Top. Insulator • Anderson (spin-orbit)	
AIII (chiral unitary)	0	0	1	$U(N+M)/U(N)\times U(M)$	$U(N)$	YES/NO	• Random Flux • Gade	SUBLATTICE (CHIRAL)
BDI (chiral orthogonal)	+1	+1	1	$O(N+M)/O(N)\times O(M)$	$U(2N)/Sp(2N)$	YES	• Bipartite Hopping • Gade • Hatsugai-Wen-Kohmoto	
CII (chiral symplectic)	−1	−1	1	$Sp(2N+2M)/Sp(2N)\times Sp(2M)$	$U(N)/O(N)$	NO	• Bipartite Hopping • Gade	
D	0	+1	0	$O(N)$	$O(2N)/U(N)$	NO	• (p_x+ip_y)-wave 2D • S.C. w/ spin-orbit • TQHE	SUPERCONDUCTORS
C	0	−1	0	$Sp(2N)$	$Sp(2N)/U(N)$	YES	• singlet SC + mag. field $(d+id)$-wave • SQHE	
DIII	−1	+1	1	$SO(2N)/U(N)$	$O(N)$	NO	• $(p+ip)+(p-ip)$ • s.c. with Spin-orbit • He 3B	
CI	+1	−1	1	$Sp(2N)/U(N)$	$Sp(2N)$	YES	• singlet SC	

COMMENT:

SUPERCONDUCTORS:

$\mathcal{H}$ = Bogoliubov – De Gennes Hamiltonian
for quasiparticles
(within MFT of pairing)

(has natural particle-hole symmetry)

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Non linear sigma model (NLσM):

$$S = \frac{1}{g}\int d^d r \; \mathrm{Tr}\left(\partial_\mu Q(r) \; \partial_\mu Q(r) \right)$$

$Q(r)$ = element of the symmetric space G/H

e.g.: $$G/H = \frac{U(2N)}{U(N)\times U(N)}$$

$$Q(r) = U^{+}(r)\, \Lambda \, U(r) \qquad U(r) \in U(2N)$$

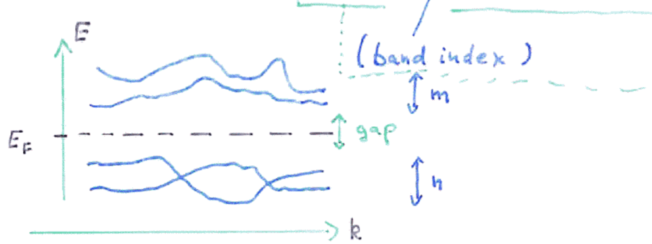
$$\Lambda = \begin{bmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_N \end{bmatrix}$$

(→ N→0
replica limit
→ alternative: "SUSY")

Explicit construction of topological invariants — translationally invariant case

→: Translational invariance ⇒
ground states of non-interacting Fermions (band insulators) are filled Fermi seas in the Brillouin Zone:

k = momentum : $\boxed{\mathcal{H}(k)\,|u_a(k)\rangle = E_a(k)\,|u_a(k)\rangle}$



→: consider Projector onto filled Bloch states:

$$\boxed{P(k) := \sum_{a}^{\text{filled}} |u_a(k)\rangle\langle u_a(k)|}$$ SPECTRAL PROJECTOR

→: Convenient to define: $\boxed{Q(k) := \mathbb{1} - 2P(k)}$

properties: $Q^{+} = Q$, $Q^2 = \mathbb{1}$, $\mathrm{tr}\,Q = m - n$ (empty m, filled n)

(Q = Hamiltonian where $E_a(k) \nearrow -1$ filled, $\searrow +1$ empty)

→: Consider the case of a Hamiltonian without any symmetry conditions (for simplicity): class A

class A: the Hamiltonian $\mathcal{H}$ is a general Hermitean matrix

–: set of eigenvectors = arbitrary unitary matrix ∈

$$\in U(m+n)$$
empty, filled

–: gauge symmetry: $U(m) \times U(n)$ = relabeling filled and empty states

–: ⇒ $Q(k) \in U(m+n)/U(m) \times U(n)$ = "Grassmanian"

$Q:\ BZ \longrightarrow U(m+n)/U(m)\times U(n)$

$\quad k \longrightarrow Q(k)$

The Quantum ground state is a map from the Brillouin Zone into the Grassmanian

→: How many inequivalent (=not deformable into each other) groundstates (= maps) are there?

This is answered by the Homotopy Group:

–: In $d = 2$: $\pi_2 \left[\frac{\mathcal{U}(m+n)}{\mathcal{U}(m) \times \mathcal{U}(n)} \right] = \mathbb{Z} =$

= counts the number of edge states of $d=2$ integer Quantum Hall states

–: In $d = 3$: $\pi_3 \left[\frac{\mathcal{U}(m+n)}{\mathcal{U}(m) \times \mathcal{U}(n)} \right] = \{ \mathbb{1} \}$ (for sufficiently large m, n)

⇒: $\boxed{\text{There are no topological insulators in } d=3 \text{ dimensions in symmetry class } A}$

Note: an element of the "Grassmanian" can be written as:

$$Q = U \cdot \Lambda \cdot U^{+}$$

$$\Lambda = \begin{bmatrix} +\mathbb{1}_m & 0 \\ 0 & -\mathbb{1}_n \end{bmatrix}, \quad U \in \mathcal{U}(m+n)$$

eigenvalues eigenvectors of Q

$\rightarrow$: The spectral projector has the same symmetries as the Hamiltonian.

E.g.: The T-symmetry in the spin-orbit symmetry class AII implies the constraint:

$$\sigma_y \, Q(-k)^* \, \sigma_y = Q(k)$$

Even though there are no different topological sectors for $Q(k)$ in the absence of the constraint, there may be some in the presence of the constraint:

A: no constraint

a, b can be deformed into each other in the absence of the constraint

AII: constraint on $Q(k)$

a, b can no longer be deformed into each other in the presence of the constraint

$\Uparrow$

indeed: AII is $\mathbb{Z}_2$ top. ins. $\Leftrightarrow$

$\Leftrightarrow$ two inequivalent sectors

The sublattice $\underbrace{\text{symmetry}}_{}^{\text{(SLS)}}$ "S" is a source of non-trivial topological sectors: (in 3D)

TABLE: $\Rightarrow$ There are 5 symmetry classes possessing sublattice symmetry "S":

$\Downarrow$

$$\boxed{S = \text{unitary}: \quad S \mathcal{H} S^{+} = -\mathcal{H}, \quad S^2 = \mathbb{1}}$$

$\Rightarrow$ can show: projector $Q(k)$ has block off-diagonal form in some basis

$$Q(k) = \begin{bmatrix} 0 & q(k) \\ q^{+}(k) & 0 \end{bmatrix}, \qquad \text{unitary: } q(k)\, q(k)^{+} = \mathbb{1} \quad (\text{from } Q(k)^2 = \mathbb{1})$$

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sublattice symmetry (more detail):

$\rightarrow$: $S = \text{unitary}$, $S^2 = \mathbb{1}$ ($\Rightarrow S^\dagger = S$ Hermitean)

in a basis: $S = \begin{bmatrix} +\mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix} = \sigma_z$

$\rightarrow$: $\boxed{-S\,\mathcal{H}\,S^\dagger = \mathcal{H}}$ and $\boxed{-S\,Q\,S^\dagger = Q}$

in detail:

$$\underbrace{\begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix}\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}\begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix}}_{\begin{bmatrix} q_{11} & -q_{12} \\ -q_{21} & q_{22} \end{bmatrix}} = -\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \quad \Rightarrow \quad q_{11} = q_{22} = 0$$

$\Rightarrow$: $$\boxed{Q = \begin{bmatrix} 0 & q \\ q^\dagger & 0 \end{bmatrix}}$$

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$\rightarrow$:

{the five symmetry classes with SLS}	S	constraint
AIII	1	none
BDI	1	$q^*(-k) = q(k)$
CII	1	$\sigma_y\, q(-k)^* \sigma_y = -q(k)$
DIII	1	$q(-k)^t = -q(k)$
CI	1	$q(-k)^t = q(k)$

$\rightarrow$: Consider (first, for simplicity) AIII (no constraint) : $q(k) \in \mathcal{U}(m)$ unconstraint

$-$: $q : BZ \longrightarrow \mathcal{U}(m)$
$\quad\; k \longrightarrow q(k)$

$-$: $\pi_3[\mathcal{U}(m)] = \mathbb{Z}$ $\Rightarrow$ $d=3$ topological insulators in symmetry class AIII labeled by integer $\nu(q) \in \mathbb{Z}$

$-$: Explicit form:

$$\boxed{\nu(q) = \frac{1}{24\pi^2}\int_{BZ} d^3k\; \epsilon^{\mu\nu\rho}\, \mathrm{tr}\left[(q^{-1}\partial_\mu q)(q^{-1}\partial_\nu q)(q^{-1}\partial_\rho q)\right]}$$

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→: In remaining four symmetry classes with SLS (i.e.: BDI, CII, DIII, CI) certain integers $\nu(q)$ may not be allowed, due to the constraints.

We will find the answer to this below by counting the number of robust gapless modes appearing at the 2D boundary

Result:

AIII and DIII	:	$\nu \in \mathbb{Z}$ (no change)
CI	:	$\nu \in 2\mathbb{Z}$
CII and BDI	:	$\nu = 0$ {actually: CII is $\mathbb{Z}_2$ top. insulator → below}

SUMMARY: Topological Insulators – so far

(Kane-Mele quantum spin Hall state)

	T	C	S	d=2	d=3
A	0	0	0	$\mathbb{Z}$	–
AI	+1	0	0		
AII	−1	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AIII	0	0	1		$\mathbb{Z}$
BDI	+1	+1	1		
CII	−1	−1	1		$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}$	
C	0	−1	0	$\mathbb{Z}$	
DIII	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}$
CI	+1	−1	1		$\mathbb{Z}$

Gapless degrees of freedom at $d=2$ boundary terminating the $d=3$ bulk insulator

→: Here: want to investigate robustness of the gapless nature against
(i) spatially homogeneous
(ii) random [= not translationally invariant]

perturbations of the Hamiltonian, respecting the symmetries of a given class.

→: Need to look at 10 symmetry classes at the $d=2$ boundary

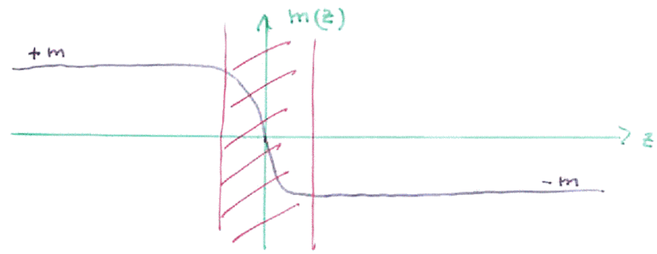
→: Need one extra ingredient:

It is well known* that Dirac Hamiltonians (here in $d=3$ space) possess topological properties:

let the sign of the Dirac mass m change in one spatial direction: $m(z)$

⇒ obtain a $d=2$ Dirac Fermion at the domain wall where $m \equiv 0$

[* Callan + Harvey (1985), Haldane (1988), Ludwig + Fisher + Shankar + Grinstein (1994), ...]



↑ $d=2$ Dirac Fermion localized at boundary

→: Therefore need to allow for $d=2$ Dirac Hamiltonian at the boundary (this is important).

RESULT: CLASSIFICATION OF $d=2$ DIRAC HAMILTONIANS
Bernard + Le Clair (2001) [BL]:

$$\mathcal{H} = \left[\begin{array}{c|c} V_+ + V_- & -i\partial_{\bar z} + A_+ \\ \hline +i\partial_z + A_- & V_+ - V_- \end{array}\right], \qquad \left(\begin{array}{c} \text{matrices:} \\ V_\pm = V_\pm^\dagger \\ A_+^\dagger = A_- \end{array}\right)$$

* There exist 13 symmetry classes, not only 10

* Specifically: In each of AIII, DIII, CI there is an extra, new symmetry class, allowed by the Dirac (block) Structure

→: Dirac description of $d=2$ boundary degrees of freedom is very convenient and important in connection with topological properties of $d=3$ bulk insulator.

- Useful to review the $d=3$ $\mathbb{Z}_2$ top. insulator case (spin-orbit / symplectic symmetry class $A\mathrm{II}$):

* BL tells us that there exists a $d=2$ Dirac Hamiltonian in $A\mathrm{II}$ with $N_f=1$ flavor of gapless (2-component) Dirac Fermions.

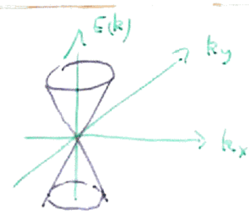
* A single flavor ($N_f=1$) cannot be realized on a $d=2$ lattice ("Fermion doubling"). Thus, this situation must correspond to the boundary of a $d=3$ top. insulator in $A\mathrm{II}$.

* This single flavor ($N_f=1$) Dirac Fermion was constructed explicitly by Fu-Kane-Mele on the lattice.

* BL show that the most general $N_f=1$ flavor $d=2$ Hamiltonian in $A\mathrm{II}$ is:

$$\mathcal{H} = 2\times 2 \text{ matrix} = -i\left[\sigma_x \partial_x + \sigma_y \partial_y\right] + V(x,y)\cdot \mathbb{1}_{2\times 2}$$

(single flavor ($N_f=1$) 2-component Dirac Fermion)



⟸ a single Dirac cone

(• this is in class $A\mathrm{II}$ (spin-orbit) [Ludwig et al. (1994), Ando et al. (1998), ...])

- $V(x,y)$ = homogeneous or random cannot localize the Fermion

*[Ryu et al. (2007), Nomura et al (2007); Ostrovsky et al. (2007), Bardason et al. 2007]

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* $\mathbb{Z}_2$ - top. term in $\frac{O(2N)}{O(N)\times O(N)}$ NLσM prevents appearance of localized phase

• Proof of gaplessness of $d=2$ boundary degrees of freedom in classes $A\mathrm{III}$, $D\mathrm{III}$, $C\mathrm{I}$

* Precisely for $\boxed{A\mathrm{III}, D\mathrm{III}, C\mathrm{I}}$ there are extra symmetry classes for $d=2$ Dirac (BL - as mentioned):

CARTAN CLASS	T	C	S	Bernard/LeClair (BL)	N_f = # of Fermion flavors
$A\mathrm{III}$	0	0	1	$(A\mathrm{III})_a$	$(2n-1)$
				$(A\mathrm{III})_b$	$2n$
$D\mathrm{III}$	-1	+1	1	$(D\mathrm{III})_a$	$(2n-1)$
				$(D\mathrm{III})_b$	$2n$
$C\mathrm{I}$	+1	-1	1	$(C\mathrm{I})_a$	$(2n-1)\cdot 2$
				$(C\mathrm{I})_b$	$2n \cdot 2$

$(n = 1, 2, 3, \ldots)$

* In the extra classes $(A\mathrm{III})_a$, $(D\mathrm{III})_a$, $(C\mathrm{I})_a$ the $d=2$ Hamiltonian must be of the form

$$\boxed{\mathcal{H} = \left[\begin{array}{c|c} 0 & -i\partial_{\bar z} + A_+ \\ \hline -i\partial_z + A_- & 0 \end{array}\right]} \quad \left\{\begin{array}{l}\text{all other} \\ \text{potentials} \\ \text{forbidden} \\ \text{by symmetry}\end{array}\right\}$$

where: $\dim A_\pm = \begin{cases} (2n-1) & \text{for } (A\mathrm{III})_a, (D\mathrm{III})_a \\ (2n-1)\cdot 2 & \text{for } (C\mathrm{I})_a \end{cases}$

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and $A_\pm$ = gauge potentials in the classical groups:

$$\left\{\begin{array}{lll} U(2n-1) & & (A\mathrm{III})_a \\ SO(2n-1) & \text{for} & (D\mathrm{III})_a \\ Sp[2(2n-1)] & & (C\mathrm{I})_a \end{array}\right\}$$

* The gauge potentials, whether homogeneous or random, cannot localize the gapless Dirac Fermions

[Ludwig et al. (1994), Mudry et al. (1996), Tsvelik (1995, 2000), Ludwig (2000)]

* The number of robust gapless 2-component Dirac Fermion flavors N_f is a topological invariant of the $d=3$ gapped bulk insulator.

* "Experimental" signature:

|| universal longitudinal surface conductivity ||

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most realistic case: $n=1$

$U(1)$

—

$Sp(2) \sim SU(2)$

$\rightarrow$: On the other hand, it is very easy to show from the BL classification that in all other classes (except $\boxed{A\,\mathrm{II}}$ discussed above, and $\underline{C\,\mathrm{II}}$ discussed below), the corresponding $d=2$ Dirac Hamiltonian can be made fully gapped while remaining in the respective symmetry class.

$\rightarrow$: Class $\boxed{C\,\mathrm{II}}$: One can show, using BL, that class $C\,\mathrm{II}$ cannot localize iff $N_f = 2 \mod (4)$ [twice an odd integer]

NOTE:

By looking at the UPDATED TABLE of Topological Insulators (Superconductors) we see

(next page)

The symmetry classes for which 3D Topological Insulators (superconductors) we found correspond precisely to those $d=2$ localization problems whose NLσMs allow for special WZW or $\mathbb{Z}_2$-topological terms (those contain no tunable parameter)

UPDATED TABLE: Topological Insulators

	T	C	S	d=2 (top. insulators)	d=3 (top. insulators)
A	0	0	0	$\mathbb{Z}$	–
AI	+1	0	0	–	–
AII	−1	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AIII	0	0	1	–	$\mathbb{Z}$
BDI	+1	+1	1	–	–
CII	−1	−1	1	–	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}$	–
C	0	−1	0	$\mathbb{Z}$	–
DIII	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}$
CI	+1	−1	1	–	$\mathbb{Z}$

NLσM Manifold G/H (Fermionic Replicas)	Top. or WZW term in $d=2$ NLσM
$U(2N)/U(N)\times U(N)$	Pruisken (Θ)
$Sp(4N)/Sp(2N)\times Sp(2N)$	none
$O(2N)/O(N)\times O(N)$	$\mathbb{Z}_2$
$U(N)$	WZW
$U(2N)/Sp(2N)$	none
$U(N)/O(N)$	$\mathbb{Z}_2$
$O(2N)/U(N)$	Pruisken (Θ)
$Sp(2N)/U(N)$	Pruisken (Θ)
$O(N)$	WZW
$Sp(2N)$	WZW

$d=2$ Topological Insulators

→: We can obtain the complete table of $d=2$ Topological Insulators (Superconductors) from the same principle:

Identify all those symmetry classes which completely evade Anderson localization in $d=1$ spatial dimension.

→?: The answer is known from Transfer Matrix studies (many authors):

- : A
- : D
- : C

p = # of right moving modes, q = # of left moving modes — evade localitation when $p \neq q$

- : AII — Zirnbauer 1992 and followup works
- : DIII — Gruzberg + Read + Vishveshwara PRB 2005 [demonstrate lack of localization in 1D]

(other, independent work on $d=2$ Top. Supercond. in DIII: Rahul Roy et al., Qi et al.: equal superposition of (p_x+ip_y) and (p_x-ip_y))

TABLE OF TOPOL. INSULATORS IN 2D

~~[lack of localization at 1D boundary]~~

	T	C	S = CT	Hamiltonian $\mathcal{H}$	Top.
A (unitary)	0	0	0	$U(N)$	$\mathbb{Z}$
AI (orthogonal)	+1	0	0	$U(N)/O(N)$	—
AII (symplectic)	−1	0	0	$U(2N)/Sp(2N)$	$\mathbb{Z}_2$
AIII (chiral unitary)	0	0	1	$\frac{U(N+M)}{U(N)\times U(M)}$	—
BDI (chiral orthogonal)	+1	+1	1	$\frac{O(N+M)}{O(N)\times O(M)}$	—
CII (chiral symplectic)	−1	−1	1	$\frac{Sp(2N+2M)}{Sp(2N)\times Sp(2M)}$	—
D	0	+1	0	$O(N)$	$\mathbb{Z}$
C	0	−1	0	$Sp(2N)$	$\mathbb{Z}$
DIII	−1	+1	1	$SO(2N)/U(N)$	$\mathbb{Z}_2$
CI	+1	−1	1	$Sp(2N)/U(N)$	—

$\boxed{\text{12 classes of transfer matrices}}$

Transfer matrix

element of

$U(p,q)/U(p)\times U(q)$

$Sp(2n,\mathbb{R})/U(2n)$

$\frac{SO^*(4n)}{U(2n)}$ (even), $\frac{SO^*(4n+2)}{U(2n+1)}$ (odd)

$\frac{Gl(n,\mathbb{C})}{U(n)}$

$\frac{Gl(n,\mathbb{R})}{O(n)}$

$\frac{U^*(2n)}{Sp(2n)}$

$\frac{SO(p,q)}{SO(p)\times SO(q)}$

$\frac{Sp(2p,2q)}{Sp(2p)\times Sp(2q)}$

$\frac{SO(2n,\mathbb{C})}{SO(2n)}$ (even), $\frac{SO(2n+1,\mathbb{C})}{SO(2n+1)}$ (odd)

$\frac{Sp(2n,\mathbb{C})}{Sp(2n)}$

Classification of topological insulators in $d=1$ spatial dimensions

Same principle:

A diagnostic of a $d=1$ topological insulator is the appearance of "gapless degrees of freedom" at its boundaries.

In $d=1$ the boundaries are points.

A gapless degree of freedom appears if there are zero modes = states at zero energy at the boundary points.

$\Rightarrow$: Need to study the Hamiltonian $\mathcal{H}$ = a random matrix

Dmitri Ivanov (2001) (zero modes in random matrix theory)

Results:

$\mathrm{D\underline{III}}$ [– odd] : $\mathcal{H} \in SO(4N+2)/U(2N+1)$
D [– odd ("B")] : $\mathcal{H} \in SO(2N+1)$

"Majorana" zero mode

$\mathrm{A\underline{III}}$: $\mathcal{H} \in U(p+q)/U(p)\times U(q)$
BDI : $\mathcal{H} \in SO(p+q)/SO(p)\times SO(q)$
$\mathrm{C\underline{II}}$: $\mathcal{H} \in Sp[2(p+q)]/Sp(2p)\times Sp(2q)$

$(q-p)$ zero modes – Verbaarschot (1994)

sublattice symmetry: appear for one dimerization:

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TABLE OF TOPOL. INSULATORS IN 1D

Name	Hamiltonian $\mathcal{H}$ element of	d=1 top. ins.
A	$U(N)$	
AI	$U(N)/O(N)$	
AII	$U(2N)/Sp(2N)$	
AIII	$U(p+q)/U(p)\times U(q)$	$\mathbb{Z}$
BDI	$SO(p+q)/SO(p)\times SO(q)$	$\mathbb{Z}$
CII	$Sp(2p+2q)/Sp(2p)\times Sp(2q)$	$\mathbb{Z}$
D(even)	$SO(2N)$	
D(odd) = "B"	$SO(2N+1)$	$\mathbb{Z}_2$
C	$Sp(2N)$	
DIII (even)	$SO(2N)/U(N)$	
DIII (odd)	$SO(4N+2)/U(2N+1)$	$\mathbb{Z}_2$
CI	$Sp(2N)/U(N)$	

RESULT:

FULLY UPDATED TABLE: Topological Insulators in $d=1,2,3$

	T	C	S	d=1	d=2	d=3
A	0	0	0	–	$\mathbb{Z}$	–
AI	+1	0	0	–	–	–
AII	–1	0	0	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AIII	0	0	1	$\mathbb{Z}$	–	$\mathbb{Z}$
BDI	+1	+1	1	$\mathbb{Z}$	–	–
CII	–1	–1	1	$\mathbb{Z}$	–	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	–
C	0	–1	0	–	$\mathbb{Z}$	–
DIII	–1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI	+1	–1	1	–	–	$\mathbb{Z}$

RESULT: in all dimensions ($d=1,2,3$):
* three $\mathbb{Z}$ top. insulators
* two $\mathbb{Z}_2$ " " insulators

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{ LOOK AT SIMPLY REORDERED TABLE } "SHIFT"

see very recently all dimensions "d": Alexei Kitaev ("Bott periodicity")

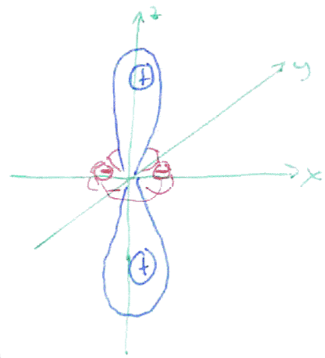
					d=1	d=2	d=3
A					–	$\mathbb{Z}$	–
AIII					$\mathbb{Z}$	–	$\mathbb{Z}$
AI					–	–	–
BDI					$\mathbb{Z}$	–	–
D					$\mathbb{Z}_2$	$\mathbb{Z}$	–
DIII					$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII					–	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII					$\mathbb{Z}$	–	$\mathbb{Z}_2$
C					–	$\mathbb{Z}$	–
CI					–	–	$\mathbb{Z}$

Lattice model for topological gapped superconductor in symmetry class CI
[Singlet S.C. with T-symmetry]

→: Consider a diamond lattice
(two Face-Centered-Cubic sublattices shifted by $\frac{1}{4}$ along the body diagonal)

→: First:

1.) $d_{3z^2-r^2}$ -like pairing on the same sublattice only



2.) nearest neighbor hopping

1.) + 2.) : gapless Dirac

→: Then:

3.) Add a next nearest hopping which generates a mass.

(can then add additional pairing terms between the two sublattices, at least if small enough)

By explicit computation find non-vanishing "winding number" $\nu = \pm 2$

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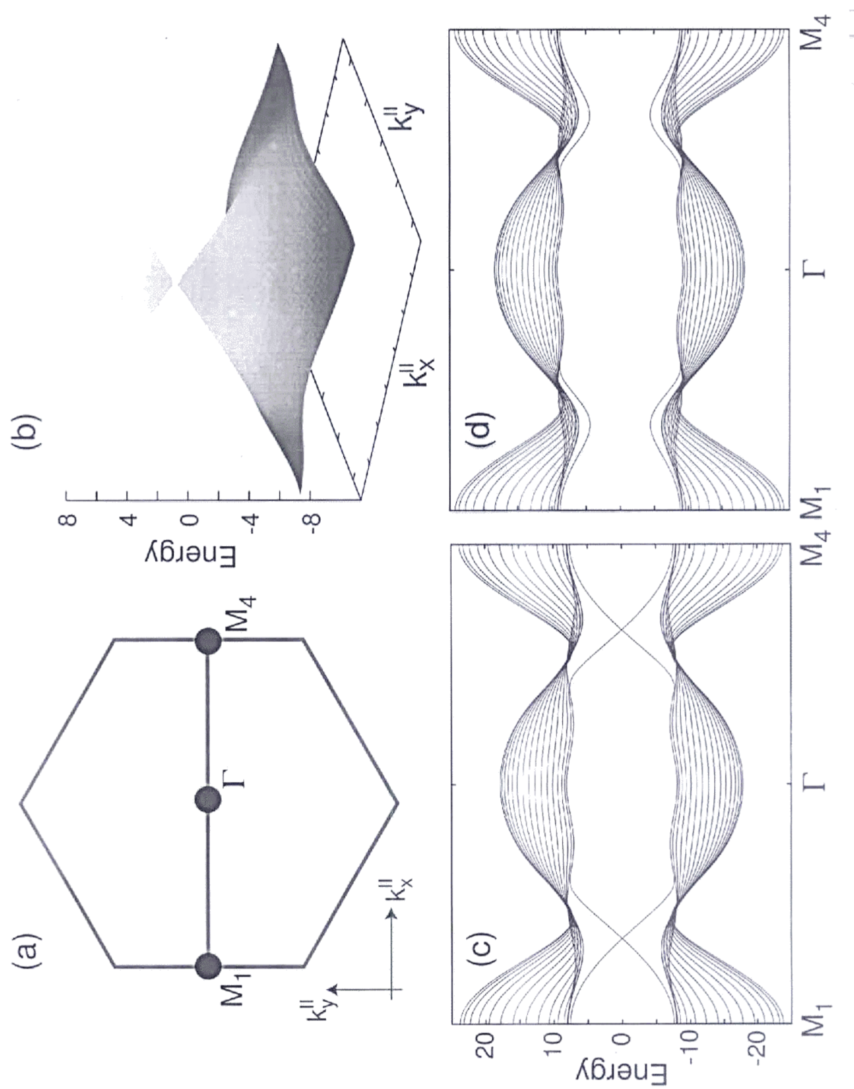
surface states at 111 - surface terminating 3D bulk



The surface Dirac fermion modes cannot be gapped or localized by any deformation of the Hamiltonian respecting the symmetries (TRS, PHS):

the only allowed perturbation is an SU(2) gauge potential

$$\mathcal{H} = (k_x + \vec{a}_x \cdot \vec{\sigma})\,\tau_x + (k_y + \vec{a}_y \cdot \vec{\sigma})\,\tau_y$$

(a^a_μ = real ; $\mu = x, y$, $a = 1, 2, 3$)

Shifts location of Dirac nodes

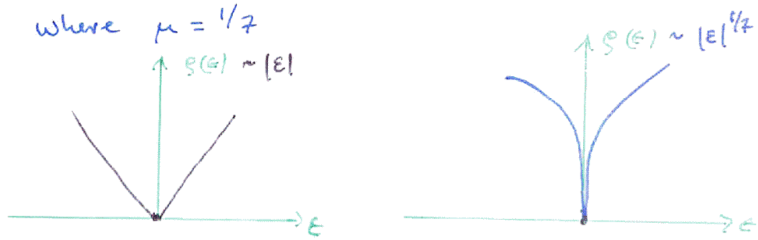
Random SU(2)-gauge field:

Randomness has dramatic effects – cannot be studied perturbatively, but has exact non-perturbative solution

(Tsvelik, Caux, Ludwig, Leclair) R.G. flow to WZW fixed pt.

* surface conductivity (spin or thermal) is unchanged as compared to free Dirac

* However, local density of surface states (accessible by tunneling experiments) change from linear behavior to $\rho(\varepsilon) \sim |\varepsilon|^{\mu}$ where $\mu = 1/7$



no disorder: free Dirac

with disorder: "WZW"

Example:
- In $d=3$ the superconductor class D_{III} has a "strong-" and a "weak-" pairing phase (=the topological gapped phase).
- This describes the Balian Werthamer pairing realized in the B-phase of liquid He 3
- This is the exact $d=3$ analog of the $d=2$ $(p_x + i p_y)$ weak pairing phase of the $d=2$ (p_x+ip_y) superconductor.

Specifically:

strong pairing | weak pairing $\to \mu$

$\nu = 0$ | 0 | $\nu = 1$ (topological phase)

$$\mathcal{H} = \begin{bmatrix} \xi & \Delta \\ -\Delta^{+} & -\xi^{t} \end{bmatrix}; \quad \xi_k = \frac{k^2}{2m} - \mu, \quad \Delta_k = (\vec{k}\cdot\vec{\sigma})\, i\sigma_y$$

$\mu < 0$ (strong pairing)	$\mu > 0$ (weak pairing)
$g_k = (-1)\dfrac{(\vec{k}\cdot\vec{\sigma})\, i\sigma_y}{\lvert\mu\rvert}$,	$g_k = (-2)\lvert\mu\rvert\dfrac{(\vec{k}\cdot\vec{\sigma})\, i\sigma_y}{k^2}$
$\boxed{g(r) = \dfrac{\sigma_\mu\, i\sigma_y}{\lvert\mu\rvert}\, i\partial_\mu \delta^{(3)}(r)}$,	$\boxed{g(r) = (-2)\lvert\mu\rvert \dfrac{i(\vec{\sigma}\cdot\vec{r})\, i\sigma_y}{4\pi\, r^3}}$

Long-range pair wave fct.

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→: BCS ground state wave fct. at fixed electron number

$$\Psi(r_1\alpha_1; r_2\alpha_2; \dots; r_N\alpha_N) = Pf\left[g(r_i\alpha_i; r_j\alpha_j)\right] =$$

$$= Pf\left[\frac{i\,\vec{\sigma}_{\alpha_i\alpha_j}\cdot(\vec{r}_i-\vec{r}_j)}{|\vec{r}_i-\vec{r}_j|^3}\right]$$

→: compare 2D "Moore-Read" Pfaffian:

$$\Psi(z_1,\dots,z_N) = Pf\left[\frac{1}{z_i-z_j}\right]$$