Aspects of gauge/gravity duality

Original AdS/CFT correspondence:

between \( N=4 \) SYM

and type IIB string theory on \( AdS_5 \times S^5 \)

\[
ds^2 = \frac{R^2}{z^2} \left( dx^2 + dz^2 \right) + R^2 d\Omega_5^2
\]

\( \frac{1}{AdS_5} \)

\( \frac{1}{S_5} \)

(this is a solution to Einstein's equation

\[
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T
\]

\( F_{\mu \nu} \),

\( F_{\mu \nu, \nu} \),

\( F_{\mu \nu, \nu}, \ldots \)

Large 't Hooft limit \( \leftrightarrow \) small curvature

\[
g^2 N_c \to \infty \quad \leftrightarrow \quad \sqrt{\alpha_s G} \to 0
\]

\( \to \) strong theory \( \to \) supergravity

correlation functions are computable
The dictionary of gauge/gravity duality:

<table>
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<tr>
<th>Gauge Theory</th>
<th>Gravity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Operator $\phi$</td>
<td>Field $\phi$</td>
</tr>
<tr>
<td>$T_{uv}$</td>
<td>Graviton $h_{uv}$</td>
</tr>
<tr>
<td>Dimension of operator</td>
<td>Mass of field</td>
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<tr>
<td>Global symmetry</td>
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<tr>
<td>Conserved current</td>
<td>Gauge field</td>
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<tr>
<td>Anomaly</td>
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</table>

Many more examples of gauge/gravity duality have been discovered

including theories with confinement and chiral symmetry breaking

To be able to compute using string theory:

strong gauge coupling

\[
g^2 N \rightarrow \frac{1}{\Lambda_{\text{QCD}}} 
\]

consequence: 2 different scales

\[
\Lambda_{\text{QCD}} \ll \frac{1}{\sqrt{a}}
\]

- Masses of lowest hadrons: Spin $\leq 2$
- String excitations: Arbitrary spin
Hadronic physics

4D
hadron

5D
normalizable mode \( \psi(z) \)

\( \lim_{z \to 0} \frac{\psi'(z)}{z} \)
eigenvalues of 5D operator

hadron mass
decay constant \( \langle 0 | J_\mu | p \rangle \)

meson coupling

overlap integral \( \int \psi(z) \bar{\psi}(x) \eta(x) \)

Finite-temperature AdS/CFT correspondence:
black 3-brane solution:

\[
ds^2 = \frac{r^2}{R^2} \left( -f(r) dt^2 + dx^2 \right) + \frac{R^2}{r^2 f(r)} dr^2 + R^2 d\Omega_5^2
\]

\( f(r) = 1 - \frac{r_o^6}{r^6} \)
corresponds to

\( N = 4 \) SYM at temperature
\( T = T_H = \frac{r_o}{\pi R^2} \)

Entropy

\[
A = \int dx dy dz \sqrt{g_{xx} g_{yy} g_{zz}} \times \pi^3 R^5
\]

\( A \sim \frac{\pi^3}{20} r_o^3 R^2 \sim T^3 \)

\( G \sim \frac{\pi^5}{N^2} \)

\( \Rightarrow S_H = \# N^2 T^3 \)

\( \text{coefficient} = \frac{3}{4} \times (\text{free gas}) \)
Compare with QCD thermodynamics

\[ \frac{\varepsilon}{\varepsilon_{SB}} \left( T - 3T_c \right) \approx 0.8 \]

pure coincidence?

Next:
- Viscosity in thermal field theories from AdS/CFT

- The viscosity/entropy density ratio

Kovtun, Son, Starinets 2004
Polchrostro, Son, Starinets 2001
Herzog, Buchel, Liu
Viscosity: introduced by C. L. M. H. Navier (1822)

\[ p \frac{\partial \mathbf{v}}{\partial t} + p (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P + \eta \nabla^2 \mathbf{v} \]

\[ \mathbf{v} \cdot \mathbf{v} = 0 \]

Viscosity

From real-world experience: ANY finite-temperature interacting system can be described, at largest time and length scales, by hydrodynamic equations:

- local
- few variables

Finite-T QFT also behave hydrodynamically

Idea: use gauge/gravity duality to investigate the hydrodynamic regime of field theory

finite-T QFT $\leftrightarrow$ black hole with translationally invariant horizon "black brane"

Example:

\[ ds^2 = H^{-\frac{1}{2}} \left( -dt^2 + dx^2 + dy^2 + dz^2 \right) + \frac{1}{H^{1/2}} \left( \frac{dr^2}{r^4} + r^2 d\Omega_5^2 \right) \]

\[ H = 1 + \frac{r_0^4}{r^4} \quad f = 1 - \frac{r_0^4}{r^4} \quad r_0 \ll R \]

Hawking temperature $T = \frac{r_0}{\pi R^2}$

\[ \text{AdS} \quad \text{flat} \]

Dual to $\mathcal{N} = 4$ SYM
Dynamics of flat horizons:

\[ T \sim r_0 = r_0(x) \]

Generalizing black hole thermodynamics \( M, Q \) ... to black brane hydrodynamics

\[ T = T_N(x), \quad \mu = \mu(x) \quad \ldots \]

Event horizons behave as viscous fluids

\[ S = \frac{\text{Area of horizon}}{4G} \]

Bekenstein Hawking

\[ W=4 \text{ SYM: } S(Q^N \to \infty) = \frac{3}{4} S(Q^N \to 0) \]

What is viscosity from the point of view of gravity?
Viscosity: Kubo's formula

\[ \eta = \lim_{\omega \to 0} \frac{1}{2 \omega} \int \frac{d\omega}{d \Phi} \left< \left[ T_{xy}(0, \Phi), T_{xy}(0, 0) \right] \right> \]

\[ = \lim_{\omega \to 0} \lim_{\Delta \to 0} \frac{1}{\omega} \text{Im} G_{xy, xy}^R(\omega, \Delta) \]

\[ \uparrow \text{retarded Green's function of } T_{xy} \]

Similar relations exist for other kinetic coefficients (diffusion constants, conductivities...)

Gravity counterpart of Kubo's formula:

\[ \text{AdS/CFT \text{ "dictionary"}} \]

Coupling:

\[ h_{\mu \nu} T_{\mu \nu} \]

Bulk graviton

Boundary stress energy

Stack of N D3-branes

1997 Klebanov: absorption of a graviton falling at right angle to the black brane

\[ \sigma_{\text{abs}} = -\frac{2k^2}{\omega} \text{Im} G^R(\omega) \]

\[ = \frac{k^2}{\omega} \int d^4x e^{i\omega t} \left< \left[ T_{xy}(x), T_{xy}(0) \right] \right> \]

Viscosity = absorption cross section of low-energy gravitons

\[ \eta = \frac{\sigma_{\text{abs}}(0)}{2k^2} = \frac{\sigma_{\text{abs}}(0)}{16\pi G} \]
Absorption cross section can be found classically.

\[ \Box h_{xy} = 0 \]

\[ h_{xy}'' + \frac{5r^4 - r^2 + r^4}{r(r^2 - r_0^2)} h_{xy}' + \frac{r^4(r^4 + r_0^4)}{(r^2 - r_0^2)^2} h_{xy} = 0 \]

The computation of \( \sigma_{\text{abs}} \) is made easy by 2 theorems, valid for a wide class of backgrounds:

- Equation for \( h_{xy} \) is the same as of a minimally coupled scalar.
- For a minimally coupled scalar,

\[ \lim_{\omega \to 0} \sigma_{\text{abs}}(0) = \text{Area of event horizon} \]

Das, Gibbons, Mathur

Consequences of 2 theorems:

\[ \eta = \frac{\sigma_{\text{abs}}(\omega \to 0)}{16\pi G} = \frac{A}{16\pi G} \]

\[ S = \frac{A}{4G} \]

\[ \Rightarrow \frac{\eta}{S} = \frac{1}{4\pi} \]
Restoring $\frac{\hbar}{s}$ and $c$:

$$\frac{\eta}{s} = \frac{\hbar}{4\pi}$$

in theories with gravity dual

This is a very small value of $\eta/s$

weakly coupled theories $\frac{\eta}{s} \gg \hbar$

Water (1 bar, 25°C) $\frac{\eta}{s} \approx 380 \frac{\hbar}{4\pi}$

Liquid helium (including superfluid) $(\frac{\eta}{s})_{\text{min}} \approx 9 \frac{\hbar}{4\pi}$

A viscosity bound conjecture

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}$$

does not contain $c, G$

c.f. Entropy bound $S \leq \frac{c^3}{\hbar G} \frac{A}{4}$

Bekenstein bound $S \leq \frac{c}{\hbar} 2\pi R M$

Applications:
- Quark-Gluon Plasma
- Trapped atomic gases
Finding transport coefficient on the lattice is hard

- analytic continuation from a discrete set of frequencies \( i \omega_n = i \frac{2\pi n T}{2\pi} \) to real frequencies

or: restoring the spectral density \( \rho(\omega) \) from

\[
G_\omega(\omega_n) = \int \frac{d\omega'}{2\pi} \frac{\rho(\omega')}{\omega' - i\omega_n}
\]

need apriori information on \( \rho(\omega) \) (Ansatz)

Transport coefficients \( \sim \lim_{\omega \to 0} \frac{\rho(\omega)}{\omega} = \rho'(0) \)

sensitive on the Ansatz

We now argue that \( \rho'(0) \) is easier to find in strongly coupled theories than in weakly coupled theories

Difficulties at weak coupling:

- viscosity \( \sim \) mean free path diverges at weak coupling

In gauge theories

\[
\ell_{\text{mfp}} \sim \frac{1}{T} \left( \frac{1}{g^4 \ln \frac{1}{g}} \right)
\]

\( \Rightarrow \) the slope of \( \rho(\omega) \) sets at very low values of \( \omega \)

![Graph showing \(\rho(\omega)\) vs \(\omega\)]

\( w_0 \sim \frac{1}{\ell_{\text{mfp}}} \ll T \)

extremely difficult to find
on the other hand, at strong coupling,

\[ l_{\text{mfp}} \sim \frac{1}{T} \]

so \( \rho'(\omega) \) should not behave too violently
as \( \omega \) changes from 0 to \( 2\pi T \).

\( \Rightarrow \) one can hope to be able to find transport coefficients with some reasonable accuracy

AdS/CFT calculations can suggest an Ansatz for \( \rho(\omega) \)
Conclusion:

- Gauge/gravity duality allows solution of QCD-like theories at strong coupling.

- Can address wide range of problems within these theories:
  - High energy scattering \( \text{Polchinski} \), \( \text{Strassler} \)
  - Vector meson dominance
  ...

- Unexpected suggestion of a lower bound on \( \frac{n}{s} \)