Lattice Field Theory at Non-zero Chemical Potential

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- The Sign Problem and the Silver Blaze Problem
- Progress at small $\mu/T$
- Taylor Expansion of the Free Energy
- NJL model: Fermi Surface and Superfluidity
- Speculations

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The QCD Phase Diagram

- Hadronic fluid
- Nuclear matter
- Color superconductor
- Quark–gluon plasma
- Compact stars
- Critical endpoint
- Crossover
- GSI?
- $T_c$ (MeV)

$T_c$ (RHIC/ALICE)
The Sign Problem for $\mu \neq 0$

In Euclidean metric the QCD Lagrangian reads

$$\mathcal{L}_{QCD} = \bar{\psi}(M + m)\psi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

with $M(\mu) = \mathcal{D}[A] + \mu \gamma_0$

Straightforward to show $\gamma_5 M(\mu) \gamma_5 \equiv M^\dagger(-\mu) \Rightarrow$

$$\det M(\mu) = (\det M(-\mu))^*$$

ie. Path integral measure is not positive definite for $\mu \neq 0$

Fundamental reason is explicit breaking of time reversal symmetry

Monte Carlo importance sampling, the mainstay of lattice QCD, is ineffective
A formal solution to the Sign Problem is *reweighting* ie. to include the phase of the determinant in the observable:

\[
\langle \mathcal{O} \rangle \equiv \frac{\langle [\mathcal{O} \arg(\det M)] \rangle}{\langle \arg(\det M) \rangle}
\]

with \(\langle \ldots \rangle\) defined with a positive measure \(|\det M|e^{-S_{boson}}\)

Unfortunately both denominator and numerator are exponentially suppressed:

\[
\langle \arg(\det M) \rangle = \frac{\langle 1 \rangle}{\langle \langle 1 \rangle \rangle} = \frac{Z_{true}}{Z_{fake}} = \exp(-\Delta F) \sim \exp(-\#V)
\]

Expect signal to be overwhelmed by noise in thermodynamic limit \(V \to \infty\)
What goes wrong with the usual positive HMC measure?

\[ \det M^\dagger M \begin{cases} M & \text{describes quarks } q \in 3 \\ M^\dagger & \text{describes conjugate quarks } q^c \in \bar{3} \end{cases} \]

In general \( \exists \, qq^c \) gauge singlet bound states with \( B > 0 \)

In QCD some \( qq^c \) states degenerate with the pion
\Rightarrow unphysical onset of “nuclear matter” at \( \mu_0 \approx \frac{1}{2} m_\pi \).

bug for QCD, feature for Two Color QCD...

Calculations with the true complex measure \( \det^2 M \) nullify effects of \( qq^c \) states for the vacuum with \( T = 0 \),
\( \frac{1}{2} m_\pi < \mu \lesssim \frac{1}{3} m_N \) by cancellations among configurations with different signs/phases

The Silver Blaze Problem...
This has been numerically verified, eg. in TSMB simulations of Two Color QCD with $N = 1$ adjoint staggered quarks.

SJH, Montvay, Scorzato, Skullerud, EurPJ C22 (2001) 451

The fake transition to a superfluid phase, forbidden by the Pauli Principle, at $\mu_o a \simeq 0.35$ disappears once configurations with $\det M < 0$ are included with the correct weight.
Analytic solution for Random Matrix model in the mesoscopic limit $V \to \infty$ with $m^2 \pi f^2 \pi V$ fixed

Akemann, Osborn, Splittorff & Verbaarschot, hep-th/0411030

$\langle \bar{\psi} \psi \rangle = \lim_{m \to 0} \lim_{V \to \infty} V^{-1} \int dx dy \frac{\rho(x, y, m; \mu)}{x + iy + m}$

For $\mu = 0$ or $N_f = 0$ $\rho$ is real, but in general it is a complex-valued spectral density.

In region $x > m$ $\rho$ develops oscillatory structure with wavelength $\sim V^{-1}$, amplitude $\sim e^V$ \Rightarrow here is where the Silver Blaze cancellations take place.
Two Routes into the Plane

(I) Analytic continuation in $\mu/T$ by either
Taylor expansion @ $\mu = 0$
Gavai & Gupta; QCDTARO

Simulation with imaginary
$\tilde{\mu} = i\mu$
de Forcrand & Philipsen;
d'Elia & Lombardo

effective for $\frac{\mu}{T} < \min\left(\frac{\mu E}{T E}, \frac{\pi}{3}\right)$

(II) Reweighting along transition line $T_c(\mu)$
Overlap between $(\mu, T)$ and $(\mu + \Delta\mu, T + \Delta T)$ remains large, so multi-parameter reweighting unusually effective

Fodor & Katz
The Bielefeld/Swansea group used a hybrid approach; ie. reweight using a Taylor expansion of the weight:

\[ \ln \left( \frac{\det M(\mu)}{\det M(0)} \right) = \sum_n \frac{\mu^n}{n!} \frac{\partial^n \ln \det M}{\partial \mu^n} \bigg|_{\mu=0} \]

This is relatively cheap and enables the use of large spatial volumes \((16^3 \times 4\) using \(N_f = 2\) flavors of p4-improved staggered fermion). Note with \(L_t = 4\) the lattice is coarse: \(a^{-1}(T_c) \simeq 700\text{MeV}\)
The (Pseudo)-Critical Line

Remarkable consensus on the curvature...

Same curvature also seen in direct HMC simulations with

$$\mu_I = \frac{1}{2}(\mu_u - \mu_d) \neq 0$$


J. Kogut & D. Sinclair, PRD70:094501,2004
The pseudocritical line found lies well above the \((\mu_B, T)\) trajectory marking chemical freezeout in RHIC collisions.

\[ \Rightarrow \text{is there a region of the phase diagram where } \text{hadrons interact very strongly (ie. inelastically)?)} \]
In our most recent work we develop the Taylor expansion of the free energy to \( O((\mu_q/T)^6) \) (recall \( c_6^{SB} = 0 \)):

\[
\frac{p}{T^4} = \sum_{n=0}^{\infty} c_n(T) \left( \frac{\mu_q}{T} \right)^n \quad \text{with} \quad c_n(T) = \frac{1}{n!} \frac{\partial^n (p/T^4)}{\partial (\mu_q/T)^n} \Big|_{\mu_q=0}
\]

Similarly we define expansion coefficients

\[
c_n^I(T) = \frac{1}{n!} \frac{\partial^n (p/T^4)}{\partial (\mu_I/T)^2 \partial (\mu_q/T)^{n-2}} \Big|_{\mu_q=0, \mu_I=0}
\]
Equation of State  

Allton et al PRD68(2003)014507, hep-lat/0501030

Pressure change \( \Delta p / T^4 \)

Quark density \( n_q / T^3 \)

\[
\Delta \frac{p(\mu, T)}{T^4} = \frac{p(\mu, T) - p(0, T)}{T^4} = \sum_{n=1}^{n_{\text{max}}} c_n(T) \left( \frac{\mu}{T} \right)^n ; \quad n_q = \frac{\partial p}{\partial \mu}
\]
Growth of Baryonic Fluctuations

Quark number susceptibility $\chi_q = \frac{\partial^2 \ln Z}{\partial \mu_q^2}$ appears singular near $\mu_q/T \sim 1$; isospin susceptibility $\chi I = \frac{\partial^2 \ln Z}{\partial \mu_T^2}$ does not.

Massless field at critical point a combination of the Galilean scalar isoscalars $\bar{\psi}\psi$ and $\bar{\psi}\gamma_0\psi$?
The Critical Endpoint \( \mu_E/T_E \)

Reweighting estimate via Lee-Yang zeroes
\[ \mu_E/T_E = 2.2(2) \]

Taylor expansion estimate from apparent radius of convergence
\[ \mu_E/T_E \gtrsim |c_4/c_6| \sim 3.3(6) \]
Allton et al PRD68(2003)014507
\[ \mu_E/T_E \gtrsim 1.1(2) \]
Gavai & Gupta hep-lat/0412035

Analytic estimate via Binder cumulant \( \langle (\delta O)^4 \rangle / \langle (\delta O)^2 \rangle^2 \)
evaluated at imaginary \( \mu \Rightarrow \mu_E/T_E \sim O(20)! \)
P. de Forcrand & O. Philipsen NPB673(2003)170
The QCD Phase Diagram

- **Hadronic fluid**
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- **Quark-gluon plasma**
- **Color superconductor**
- **Compact stars**

- **Critical endpoint**
- **Crossover**
- **RHIC/ALICE**
- **GSI?**

- **\( T(\text{MeV}) \)**
- **\( \mu(\text{MeV}) \)**

- **\( T_c \)**
- **\( (\mu_E, T_D) \)**
- **\( \mu_{\text{onset}} \)**
$\chi_{SB}$ vs. Cooper Pairing

Diagram: Two plots showing different energy ($E$) vs. wave vector ($k$) behaviors. The left plot illustrates pairing instability in a Dirac Sea, while the right plot shows 2$\Sigma$ and 2$\Delta$ in a Fermi Sea. The diagrams highlight the relationship between these two concepts in the context of superconductivity.
Color Superconductivity

In the asymptotic limit $\mu \to \infty$, $g(\mu) \to 0$, the ground state of QCD is the color-flavor locked (CFL) state characterised by a BCS instability, [D. Bailin and A. Love, Phys.Rep. 107(1984)325] ie. diquark pairs at the Fermi surface condense via

$$\langle q_i^\alpha (p) C \gamma_5 q_j^\beta (-p) \rangle \sim \varepsilon^{A\alpha\beta} \varepsilon_{Aij} \times \text{const.}$$

breaking $\text{SU}(3)_c \otimes \text{SU}(3)_L \otimes \text{SU}(3)_R \otimes \text{U}(1)_B \otimes \text{U}(1)_Q \longrightarrow \text{SU}(3)_\Delta \otimes \text{U}(1)_{\tilde{Q}}$

The ground state is simultaneously superconducting (8 gapped gluons, ie. get mass $O(\Delta)$),

superfluid (1 Goldstone),

and transparent (all quasiparticles with $\tilde{Q} \neq 0$ gapped).

At smaller densities such that $\mu/3 \sim k_F \lesssim m_s$, expect pairing between $u$ and $d$ only $\Rightarrow$ “2SC” phase

$$\langle q_i^\alpha (p) C \gamma_5 q_j^\beta (-p) \rangle \sim \varepsilon^{\alpha\beta3} \varepsilon_{ij} \times \text{const.}$$

$SU(3)_c \rightarrow SU(2)_c \Rightarrow 5/8$ gluons get gapped

Global $SU(2)_L \otimes SU(2)_R \otimes U(1)_B$ unbroken

In the electrically-neutral matter expected in compact stars, $k_F^d - k_F^u = \mu_e = -2\mu_I \sim 100\text{MeV} \Rightarrow \langle qq \rangle$ condensate can have $\vec{k} \neq 0$ breaking translational invariance $\Rightarrow$ crystallisation

Other ideas: a 2SC/normal mixed phase (plates? rods?)
or a gapless 2SC where $\langle qq \rangle \neq 0$ but $\Delta = 0$?

The most urgent issue of all – whether quark matter exists in our universe – requires quantitative knowledge of the EOS $p(\mu), \epsilon(\mu)$ for all $\mu > \mu_o$
Four Fermi Models with $\mu \neq 0$

Effective description of soft pions interacting with constituent quarks

$$\mathcal{L}_{NJL} = \bar{\psi} (\not{\partial} + m + \mu \gamma_0) \psi - \frac{g^2}{2} [(\bar{\psi} \psi)^2 - (\bar{\psi} \gamma_5 \vec{\tau} \psi)^2]$$

$$\sim \bar{\psi} (\not{\partial} + m + \mu \gamma_0 + \sigma + i \gamma_5 \vec{\pi} \cdot \vec{\tau}) \psi + \frac{2}{g^2} (\sigma^2 + \vec{\pi} \cdot \vec{\tau})$$

Full global symmetry is $SU(2)_L \otimes SU(2)_R \otimes U(1)_B$

Dynamical $\chi$SB for $g^2 > g_c^2 \Rightarrow$

isotriplet Goldstone $\vec{\pi}$, constituent quark mass $\Sigma \gg m$

Scalar isoscalar diquark $\psi^{tr} C \gamma_5 \otimes \tau_2 \otimes A^{color} \psi$ breaks $U(1)_B$

$\Rightarrow$ diquark condensation signals superfluidity
Lattice four-fermi models:

Preserve QCD-like symmetries and have either an interacting continuum limit $(2+1d)$ or can be considered a cutoff effective theory (eg. $a^{-1} \approx 700 \text{MeV}$ for NJL in $3+1d$)

Exhibit spontaneous $\chi$SB at low $\mu$, but no confinement physics, and no sign of any “nuclear matter” phase where both $\langle \bar{\psi}\psi \rangle > 0$ and $n_B > 0$

$\Rightarrow$ for $\mu > \mu_c \sim \Sigma$ describe “relativistic quark matter”

Simulable with $\mu > 0$ because the Goldstone channel dominated by

\[ \gamma_5 \times \gamma_5 \times \gamma_5 \times \gamma_5 \times \gamma_5 \]

which are only available to $\bar{q}q$, and not $qq^c$

*Can break $U(1)_B$ with a real measure because excluded from Vafa-Witten theorem*
The fermion dispersion relation is fitted with

\[ E(|\vec{k}|) = -E_0 + D \sinh^{-1}(\sin |\vec{k}|) \]

yielding the Fermi liquid parameters

\[ K_F = \frac{E_0}{D}; \quad \beta_F = D \frac{\cosh E_0}{\cosh K_F} \]
Meson Correlation Functions (2+1d)

\[ \sum_{\tilde{x}} \bar{\psi}(\psi(0) \bar{\psi}(x) \exp(i\tilde{k}.\tilde{x}) \]

For \( \tilde{k} \neq 0 \) can always excite a particle-hole pair with almost zero energy \( \Rightarrow \) algebraic decay of correlation functions

\[
\begin{align*}
|\tilde{k}| &< \mu & \Rightarrow C \sim \frac{1}{x_0^2} \\
|\tilde{k}| &= 2\mu & \Rightarrow C \sim \frac{1}{x_0^{3/2}} \\
|\tilde{k}| &> 2\mu & \Rightarrow C \sim \frac{e^{-(|\tilde{k}|-2\mu)x_0}}{x_0^{3/2}}
\end{align*}
\]
eg. in the spin-1 channel at $\mu a = 0.6$, $C_{\gamma \perp}$ (left) looks algebraic as predicted by free field theory, but $C_{\gamma \parallel}$ (right) decays exponentially.

The interpolating operator for $C_{\gamma \parallel}$ in terms of continuum fermions is $\bar{q}(\gamma_0 \otimes \tau_2)q$

ie. with same quantum numbers as baryon charge density
Dispersion relation $\omega(|\vec{k}|)$ extracted from meson channel interpolated by an operator $\bar{\psi}(\gamma_0 \otimes \tau_2)\psi$

A massless vector excitation?

Sounds Unfamiliar?

Light vector states in medium are of great interest: Brown-Rho scaling, vector condensation... In the Fermi liquid framework a possible explanation is a collective excitation thought to become important as $T \to 0$: Zero Sound

Ordinary FIRST sound is a breathing mode of the Fermi surface: velocity $\beta_1 \approx \frac{1}{\sqrt{2}} \frac{k_F}{\mu}$

ZERO sound is a propagating distortion of the Fermi surface: velocity $\beta_0 \sim \beta_F$ must be determined self-consistently
Diquark Condensation (3+1d)

\[ \langle \chi \chi \rangle \]
\[ n_B \]

\[ \propto \text{Volume within the Fermi Surface} \]
Diquark Condensation (3+1d)

Add source

\[ j [\psi^{tr} \Gamma_A \psi + \bar{\psi} \Gamma_A \bar{\psi}^{tr}] \]

Diquark condensate estimated by taking \( j \rightarrow 0 \)

Our fits exclude \( j \leq 0.2 \)
The Superfluid Gap

Quasiparticle propagator:

\[ \langle \psi_u(0) \bar{\psi}_u(t) \rangle = Ae^{-Et} + Be^{-E(L_t-t)} \]
\[ \langle \psi_u(0) \psi_d(t) \rangle = C(e^{-Et} - e^{-E(L_t-t)}) \]

Results from \(96 \times 12^2 \times L_t, \mu a = 0.8\) extrapolated to \(L_t \rightarrow \infty\) (ie. \(T \rightarrow 0\)) then \(j \rightarrow 0\)

The gap at the Fermi surface signals superfluidity

• $\Delta/\Sigma_0 \simeq 0.15 \Rightarrow \Delta \simeq 60\text{MeV}$
in agreement with self-consistent approaches

• Similar formalism to study non-relativistic model for
  EITHER nuclear matter (with or without pions)
  $\Rightarrow$ calculation of $E/A$
  D. Lee & T. Schäfer nucl-th/0412002
  OR Cold atoms with tunable scattering length
  $\Rightarrow$ study of BEC/BCS crossover
  M. Wingate cond-mat/0502372

In either case non-perturbative due to large dimensionless parameter $k_F|a| \gg 1$, with $a$ the $s$-wave scattering length.

N.B. $\mu_I \neq 0$ or $m_\pi < \infty$ reintroduces sign problem!
Cold Quarks, Hot Glue...

NJL models permit study of Fermi surface – Zero Sound, perhaps the most interesting effect, is expected in systems with short-ranged interactions, but not in gauge theories.

Conjugate quarks supposedly invalidate the quenched approximation (Gocksch, Stephanov) – arguments assume tightly bound $q q^c$ states resulting from confinement.

What if we could generate cold, non-confining gluon configurations?

We have tried generating deconfining 3$d$ configurations using the Dimensional Reduction approach to hot QCD, then “reconstructing” the timelike direction to permit quenched inversions of $(M(\mu) + m)$

Resulting model contains only static modes, so NOT a systematic effective description of high densities.
The DR model is $3d$ SU(2) gauge-Higgs with parameters $\beta, \kappa, \lambda$

Large $\kappa$ yields results similar to Two Color QCD
Small $\kappa$ qualitatively different – no variation of $\langle qq_+ \rangle$ with $\mu$

Gauge-fixed quark propagator $G(\vec{k}, t)$ indicates restored chiral symmetry (sawtooth) and exponential decay ($\Delta > 0$)

Intriguingly, $G$ has no significant $\vec{k}$-dependence $\Rightarrow$ no means of identifying $k_F$ or the Fermi surface

What is the gauge invariant signal for a Fermi surface?

Hart & Philipsen, NPB 572 (2000) 243
Are we using the right basis?

Large cancellations between diagrams/configurations hint at low calculation efficiency. Maybe gauge covariant quarks and gluons not natural degrees of freedom at high density?

Intriguing $3d$ example: approximate duality between scalar QED and complex scalar field theory


$$\mathcal{L}_{SQED} = \frac{1}{4} F^2 + |D\phi|^2 + m^2 |\phi|^2 + \lambda |\phi|^4 - \frac{1}{2} \varepsilon_{ijk} H_i F_{jk}$$

$H$ is a real source term coupled to a real $B$-field

$$\mathcal{L}_{SFT} = [(\partial - \tilde{e}H)_k \tilde{\phi}^*][(\partial + \tilde{e}H)_k \tilde{\phi}] + \tilde{m}^2 |\tilde{\phi}|^2 + \tilde{\lambda} |\tilde{\phi}|^4 + \cdots$$

$\tilde{e} H_3$ with $\tilde{e} = 2\pi/e$ is a real chemical potential for the conserved charge density $2\text{Im}(\tilde{\phi}^* \partial_3 \tilde{\phi})$

*Duality exact at Coulomb/Higgs $\Leftrightarrow$ broken/symmetric phase transition*
What is the physical origin of the sign problem? eg:

- Two Color QCD with $N = 1$ adjoint staggered quarks – superconductor at large $\mu$?
- Repulsive Hubbard model away from half-filling – model of cuprate superconductivity?
- Technicolor – chiral fermions in complex representations
- “$\tau_3$-QED” describing planar superconductivity by giving the photon a mass via a mixed Chern-Simons term

\[ \text{det} M \neq \text{det} M^* \text{ since } \{\gamma_5, \slashed{D}\} \neq 0 \]

Dorey & Mavromatos NPB 386 (1992) 614

- QCD itself?

Conjecture: any system exhibiting spontaneous breaking of a local symmetry by a pairing mechanism has a sign problem when formulated in terms of local gauge covariant degrees of freedom
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Summary

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And finally...