QCD at non-zero Chemical Potential: Insights from the Large Nc Limit?

TDC, Phys.Rev. D70 (2004) 116009
TDC, PRL 91 (2003) 222001
TDC, PRL 91 (2003) 032002

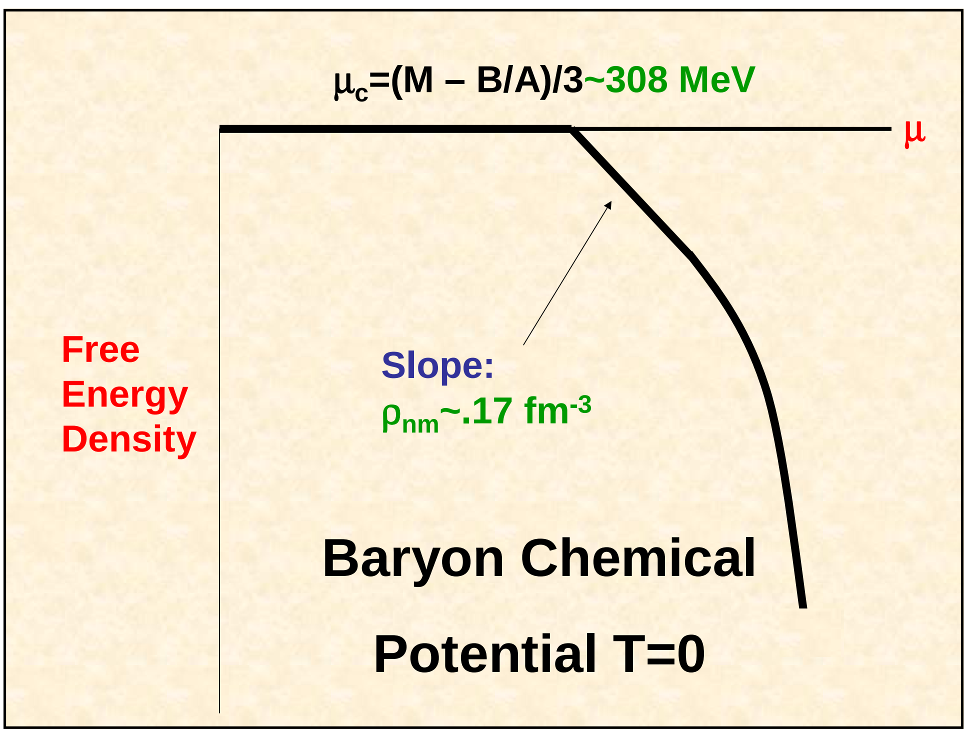
Outline

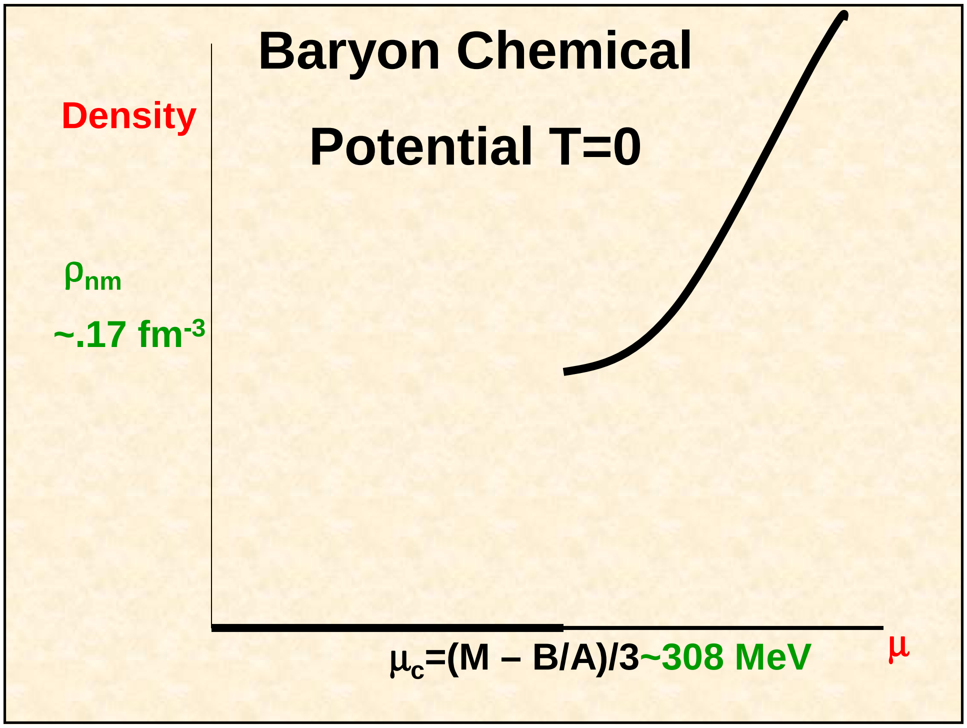
- Introduction
  - Cold QCD at low chemical potential: the “The Silver Blaze Problem”
  - Isospin μ vs Baryon μ
- A Large Nc Puzzle: DiagramaticallyQCD at nonzero baryon chemical potential differs from QCD at nonzero isospin chemical potental only at $\mathcal{O}(N_c^{-1})$; phenomenologically they differ radically at *T=0*
  - Some clues & some red herrings
  - Open Questions

Introduction

- Nuclear matter is *the* fundamental problem in low-energy nuclear physics
  - The interior of all nuclei are similar
  - Neglect coulomb effects and one can have arbitrarily large nuclei.
  - Infinite nuclear matter properties (at T=0) extracted from finite nuclei:
    - Density: .17 fm^{-3}
    - Binding energy per nucleon: 16 MeV
    - First order transition in chemical potential at T=0
      - Critical chemical potential (per quark):

μc=(M – B/A)/Nc~308 MeV



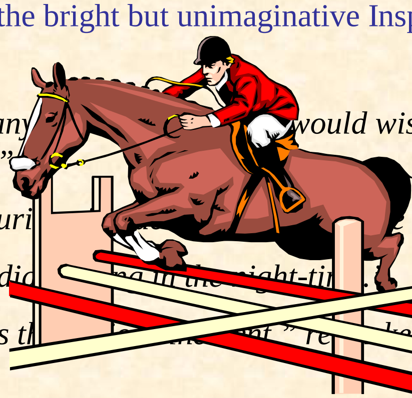


- Can we understand cold Nuclear Matter from QCD?
  - Lattice is useless at present: fermion sign problem
  - There has been significant recent progre ng hot QCD with a finite chemical potential example Philippe de Forcrand's talk)
  - These methods fail for cold matter. The d ent of such method is a major open problem fo field theory.
  - It is critical to get a new handle on the problem. To do so it may be useful to view the problem from some new perspectives:
    - Cold QCD at very low chemical potential (below the critical value): "The Silver Blaze Problem"
    - The contrast between QCD with a baryon μ and an isopsin μ.
    - Large Nc QCD

A Literary Interlude

From “Silver Blaze’’ in ***The Memoirs of Sherlock Holmes*** by Arthur Conan Doyle

The story involves the theft a race horse and the apparent murder of its trainer. The following exchange takes place between Sherlock Holmes and the bright but unimaginative Inspector Gregory:



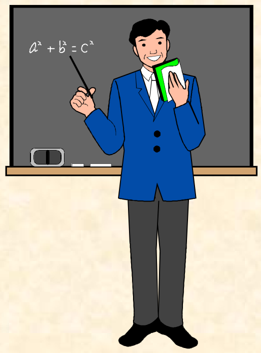
“Is there any ... would wish to draw my attention?”

“To the curi ... night-time.”

“The dog di ...”

“That was th ... ked Sherlock Holmes.

And now back to our previously scheduled scientific talk



Cold QCD at low μ: A QCD Dog Not Barking in the Night

- Consider QCD at zero temperature but with a nonzero baryon chemical potential
- There will be a critical chemical potential below which the system remains in the vacuum. If the chemical potential is less than the energy of the state with smallest nonzero energy per unit baryon number, nothing happens.
- The reason is clear from phenomenological considerations based on energetics. However, it is quite obscure from the perspective of a Euclidean Space Functional Integral.

Euclidean Space Functional Integrals

$$Z_B(\mu_B) = \int d[A] \, \text{Det}\left(\not{D} + m - \mu_B \gamma_0\right)^2 e^{-S_{\text{YM}}}$$

The *only* way that the functional integral knows about μ is in Det. The Det is simply the product of eigenvalues.

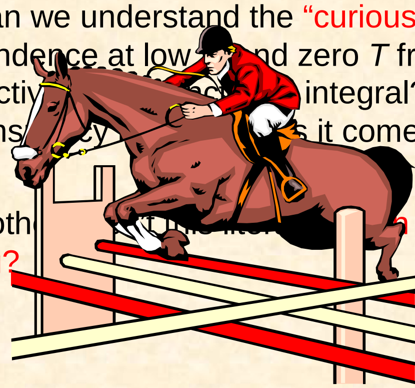
$$\text{Det}\left(\not{D} + m - \mu_B \gamma_0\right) = \prod_j \lambda_j$$

$$\text{with} \quad \left(\not{D} + m - \mu_B \gamma_0\right)\left|\psi_j\right\rangle = i\lambda_j \left|\psi_j\right\rangle$$

The Silver Blaze Problem

- How can we understand the "curious incident" of μ independence at low ... nd zero T from the perspecti... integral? Clearly there is a cons...cy ... s it come about?

- Why both... ado about nothing?



In order to understand what happens above the critical chemical potential one needs to understand why nothing happens below this point.

Possible Resolutions of the "Silver Blaze Problem"

- For $\mu<\mu_{crit}$, there is a conspiracy such that the relevant set of functional determinants are all equal to their $\mu=0$ value.

---or---

- For $\mu<\mu_{crit}$, there is a conspiracy in which the phases of the determinant oscillate in such a way as lead to complete cancellations of the shifts in the magnitudes.

- In fact,
  – for $\mu<m_\pi/2$ there is a conspiracy such that the relevant set of functional determinants are all equal to their $\mu=0$ value.

  – For $m_\pi/2<\mu<\mu_{crit}$, there is a conspiracy in which the phases of the determinant oscillate in such a way as lead to complete cancellations of the shifts in the magnitudes.
- This can be seen by comparing QCD with an isospin chemical potential to QCD with a baryon chemical potential

Compare functional integrals

- Euclidean functional integral perspective (work at finite T=1/β, two flavor QCD) free energy with baryon chemical potential

$$G_B(\mu,T) = -\frac{Log\left(Z_B\right)}{\beta}$$ Free Energy

$$Z_B(\mu) = \int d[A]\,\mathrm{Det}\left(\not{D} + m - \mu\gamma_0\right)^2 e^{-S_{\mathrm{YM}}}$$

Standard boundary conditions

Periodic in time: for gluons, antiperiodic for quarks

Determinant is squared due to two flavors (assuming isospin is a good symmetry)

Isospin Chemical Potential

$$G_I(\mu,T) = -\frac{Log\left(Z_I\right)}{\beta}; \qquad Z_I(\mu,T) =$$

$$\int d[A]\,e^{-S_{\mathrm{YM}}}\mathrm{Det}\left(\not{D} + m - \mu\gamma_0\right)\mathrm{Det}\left(\not{D} + m + \mu\gamma_0\right)$$

Two determinant with opposite signs for μ; this reflects the isospin of the up and down quarks.

Neat fact: the two determinants are related by complex conjugation (easy proof)

Compare

$$Z_B(\mu) = \int d[A]\ \mathrm{Det}\left(\not{D} + m - \mu\gamma_0\right)^2 e^{-S_{YM}}$$

$$Z_I(\mu) = \int d[A] \left|\mathrm{Det}\left(\not{D} + m - \mu\gamma_0\right)\right|^2 e^{-S_{YM}}$$

•The only difference is due to the phase of the determinant.

•This phase is the origin of all phenomenological differences between isospin matter (pion condenates) and nuclear matter

Note: $\left|\mathrm{Det}\ \left(D + m - \mu\gamma_0\right)\right|^2 \geq \mathrm{Re}\left(\mathrm{Det}\ \left(D + m - \mu\gamma_0\right)^2\right)$

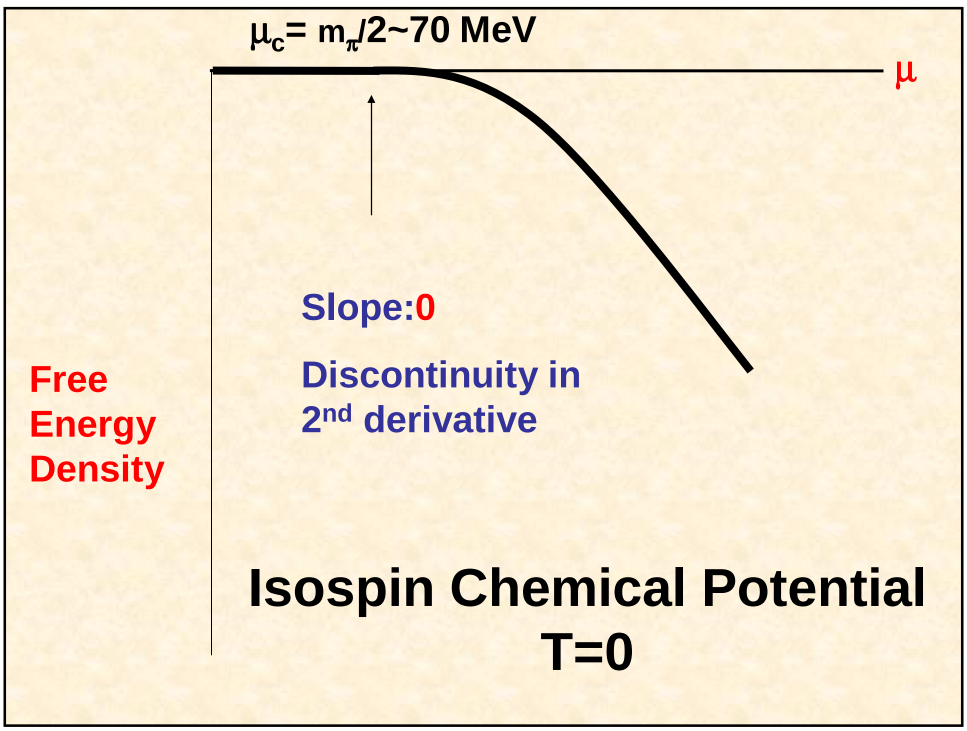
Thus

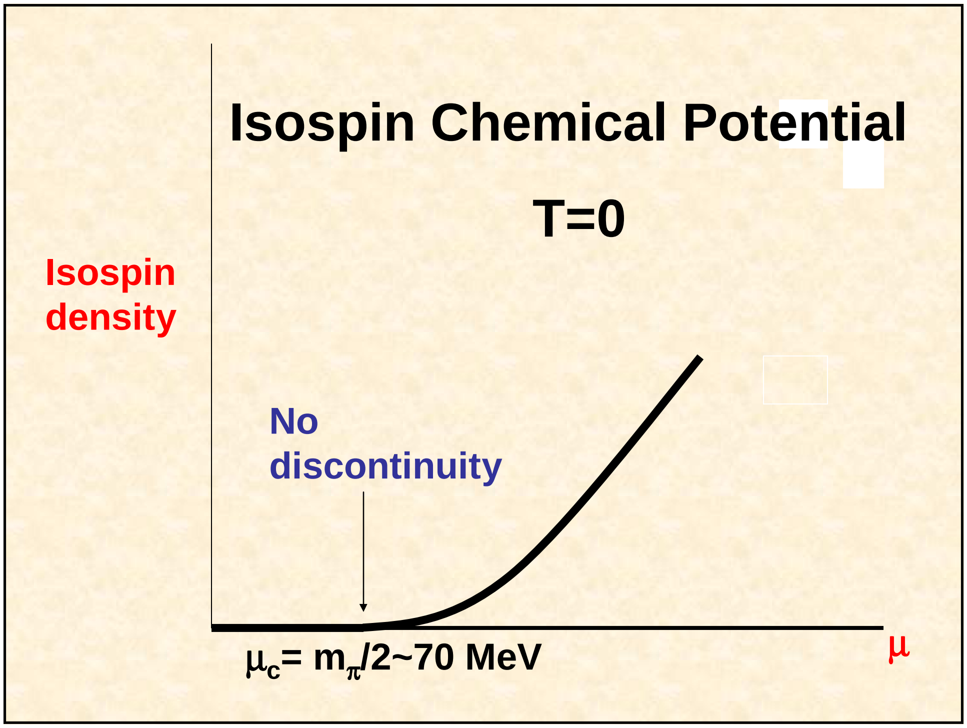
$$Z_I(\mu) \geq Z_B(\mu) \quad \text{or} \quad G_I(\mu) \leq G_B(\mu)$$

- Free energy for isospin matter is always less than for nuclear matter at same chemical potential
- Possible useful constraint in modeling dense matter
- Strong implications for “Silver Blaze”

Phenomenology is radically different in two cases

- Isospin chemical potential
  - Second order transition in μ (at T=0)
  - Critical chemical potential is $m_\pi/2$ **~70 MeV**
    - Critical value fixed by physics of chiral symmetry breaking
- Baryon chemical potential
  - First order transition in μ (at T=0)
  - Critical μ is **μ_c=(M – B/A)/Nc~308 MeV**
    - Critical value is not fixed by physics of chiral symmetry breaking





Some Key Facts

- For case of isospin chemical potential the fermion determinant is real.
- Only solution to the isospin “Silver Blaze” problem for $\mu<m_\pi/2$ is for all relevant configurations to have same functional determinants as at $\mu=0$.
- For $\mu<m_\pi/2$ and $T=0$ $G_I(\mu)=G_B(\mu)=G_I(0)$.
- $G_I(\mu)=G_B(\mu)$ only if the phases of all relevant configurations are the same.

- Ergo, for the baryon case with $\mu<m_\pi/2$ all relevant configurations have the same the fermion determinant as for $\mu=0$. *It is not due to cancellations due to the phase.*
- Conversely, for $m_\pi/2< \mu< \mu_{crit}$ the $G_I(\mu)<G_B(\mu)= G_B(0)$. This can only be due to the phases.
- The conspiracy needed for the $\mu<m_\pi/2$ regime is relatively easy to understand.
- The conspiracy needed for the $m_\pi/2<\mu<\mu_{crit}$ regime remains mysterious

The $\mu<m_\pi/2$ regime

- Focus on eigenspectrm of $\gamma_0\left(\not{D} + m - \frac{\mu_I \gamma_0}{2}\right)$ rather than usual Dirac operator.
- Exploit the identity

$$\mathrm{Det}\left(\not{D}+m-\mu\gamma_0\right)$$
$$=\mathrm{Det}\left(\not{D}+ m\right)\mathrm{e}^{-\int_0^{\mu_I}\mathrm{d}\mu'\,\mathrm{Tr}\left(\gamma_0\left(\not{D}+m-\mu'\gamma_0\right)^{-1}\right)}$$

If we know the spectrum of this operator the trace can be evaluated and from that the determinant.

- Evaluate at finite temperature and take zero temperature limit at end: anti-periodic boundary conditions
- Operator is linear in time derivatives. Eigenvalues fall into families

$$\lambda_{jn} = \varepsilon_j - \mu + i\left(\frac{\phi_j}{\beta} + \frac{(2n+1)\pi}{\beta}\right) \qquad n \text{ integer}$$

- ε_j are pseudo-energies.
- The sum over the integer n can be explicitly evaluated.

$$\mathrm{Tr}\left((\gamma_0(\not{D}+m)-\mu)^{-1}\right)=$$

$$\sum_j \theta(\varepsilon_j)\tfrac{\beta}{2}\left\{\tanh\left(\tfrac{\beta}{2}(\varepsilon_j-\mu)+i\phi_j\right)-\tanh\left(\tfrac{\beta}{2}(\varepsilon_j-\mu)-i\phi_j\right)\right\}$$

Note we are interested in the zero temp limit:

$$\mathrm{Tr}\left((\gamma_0(\not{D}+m)-\mu)^{-1}\right)=$$

$$\sum_j \theta(\varepsilon_j)\,\mathrm{sign}(\mu)\left(-\beta\,\theta(|\mu|-\varepsilon_j)+i\tfrac{\phi_j}{2}\delta(|\mu|-\varepsilon_j)\right)+O(e^{-\beta\Lambda})$$

where Λ is a typical QCD scale. Thus

$$\frac{\mathrm{Det}(\not{D}+m-\mu\gamma_0)}{\mathrm{Det}(\not{D}+m)} = \exp\left(\sum_j i\tfrac{\phi_j}{2}\theta(\varepsilon_j)\theta(|\mu|-\varepsilon_j)\right)$$

$$\times\ \exp\left(\beta\sum_j \theta(\varepsilon_j)\theta(|\mu|-\varepsilon_j)(|\mu|-\varepsilon_j)+O(e^{-\beta\Lambda})\right)$$

- This form helps explain the isospin “Silver Blaze” problem.
- It is solved if a gap exists in the quasi-energy spectrum for those configurations which contribute with nonzero measure at T=0: $\varepsilon_j > m_\pi/2$ for all j.
- The step functions then ensure that nothing happens---the system is identical to $\mu=0$ until $\mu_I = m_\pi/2$.
- This holds for the baryon “Silver Blaze Problem” as well. The phases also don’t contribute due to the step functions. Thus $G_I(\mu)=G_B(\mu)$ as is known phenomenologically.

The $m_\pi/2<\mu<\mu_{crit}$ regime

- A conspiracy in which the summing over phases exactly cancels the growth in the functional determinant is needed for baryon problem.
- The nature of this conspiracy:

$$Det(\mu) \equiv |Det(\mu)| e^{i\vartheta(\mu)}$$

$$Z_I(\mu) = \left\langle \frac{|Det(\mu)|^2}{Det(0)^2} \right\rangle_{\mu=0} \qquad Z(\mu)_B = \left\langle \frac{|Det(\mu)|^2 \cos(2\vartheta(\mu))}{Det(0)^2} \right\rangle_{\mu=0}$$

$$Z_B(\mu) = Z_I(\mu) \langle \cos(2\vartheta(\mu)) \rangle_\mu^{\text{isospin matter}}$$

- Look at spectral density for quasi-energies
- One can show from form of functional integral.

$$\frac{\partial\langle\hat{\rho}(\varepsilon,\mu_I)\rangle}{\partial\mu_I} = \left\langle \hat{\rho}(\varepsilon,\mu_I) \int_0^{\mu_I} d\varepsilon' \, \hat{\rho}(\varepsilon',\mu_I) \right\rangle$$

- The positivity of the spectral density operator implies that if the quasi energy density is zero for some value of the isospin chemical potential at zero temperature, its derivative with respect to the chemical potential is necessarily zero.
- Thus a gap at zero chemical potential ensures a gap at all chemical potentials below the gap. This in turn implies free energy independent of chemical potential.

Silver Blaze Problem solution requires

$$\langle \cos(2\vartheta(\mu)) \rangle_\mu^{\text{isospin matter}} = \frac{Z_0}{Z_I(\mu)} \quad \text{for } \tfrac{m_\pi}{2} < \mu < \mu_{crit}$$

- Ideally there we could find some clever way to organize the problem which makes manifest how this conspiracy works. No clear path to find this.
- **Can large Nc QCD give us insight into this?**

- Clear phenomenological problems with this
  - Scales suggest that 1/Nc Expansion is hopeless phenomenologically.
  - Nuclear scales are radically smaller than typical hadronic scales for essentially unknown reason--- reasons that have nothing to do with Nc.
  - *Eg. Binding energy per nucleon is formally of order N_c^1 and is 16 MeV ; The N-Δ mass splitting is N_c^{-1} and is 300 MeV .*
  - *Eg. Fermi momentum which is formally of order N_c^0 and is 270 MeV.*
  - ***Clearly large scales and small scales from large Nc are mixed; there is no clean scale separation based on large Nc.***

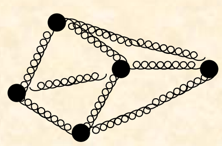
- At best large Nc can be used as diagnostic tool. We can track how things scale with Nc to test our understanding of important issues of principle---we cannot use it to predict numerical values. (At worst, we are doing mathematical physics only.)
- **Focus here: How can we understand the difference of QCD at nonzero baryon chemical potential (which leads to nuclear matter) from QCD at nonzero isospin chemical potential (which leads to pion condensation)?**

A large Nc puzzle

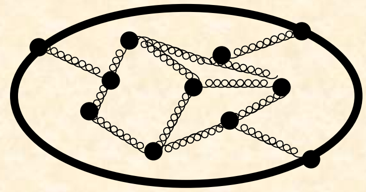
- Consider (two flavor) QCD with a chemical potential at large Nc.
- Follow the `t Hooft strategy of working in the limit $N_c \to \infty \quad g \to 0 \quad g^2 N \text{ fixed}$; study classes of Feynman diagrams.
- Key point: if one works with μ of order N_c^0 then the μ=0 analysis of the N_c dependence of diagrams goes through without change.

- Leading order: N_c^2
  - Pure glue with planar diagrams

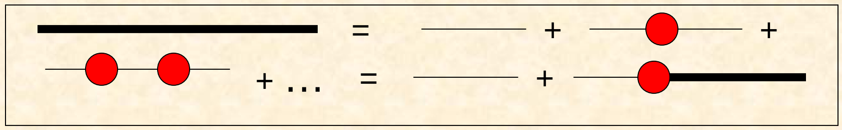
  Eg.



- Leading order containing quark line: N_c^1
  - One quark loop with planar glue fully interior
    Eg.



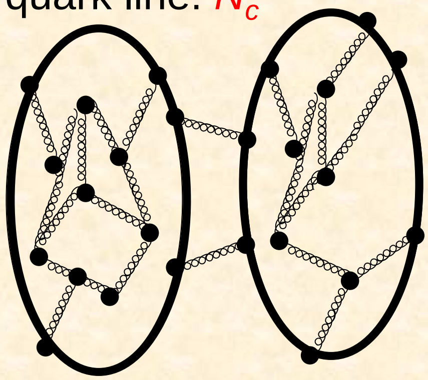
– Quark line contains $\mu\gamma_0$ to all orders



(μ=0 Propagator) $= \mu\gamma_0$

- Leading order containing both an up quark and a down quark line: N_c^0

Eg.



Leading order graphs which depend on μ

- Come from 1-loop graphs and are order N_c^1
- Up quark loop and down quark loops add.

$$G(\mu_u,\mu_d) = N_c\big(f(\mu_u) + f(\mu_d)\big) + \mathcal{O}(N_c^0)$$

- $f(\mu)$ is the sum of the one-quark-loop contributions at N_c^1 .
- Functional forms the same for up and down (assume isospsin invariance).

$$G_B(\mu) = N_c\big(f(\mu) + f(\mu)\big) + \mathcal{O}(N_c^0)$$

$$G_I(\mu) = N_c\big(f(\mu) + f(-\mu)\big) + \mathcal{O}(N_c^0)$$

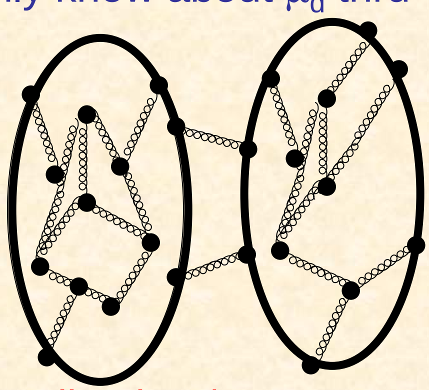
Charge conjugation requires

$$G_B(\mu) = G_B(-\mu) \quad \text{thus} \quad f(\mu) = f(-\mu)$$

which implies $G_B(\mu) = G_I(\mu) + \mathcal{O}(N_c^0)$

- But this is totally crazy!!
- QCD with an isospin chemical potential is phenomenonologically completely different than QCD with a baryon chemical potential.

Problem: up quarks need to know about down quark chemical potential in order to distinguish nuclear matter from pion condensation but only know about μ_d thru graphs like



Such graphs are subleading in 1/Nc.

Basic Issue

- Large Nc QCD is quenched.
- Quenched QCD is known to have spurious behavior for $\mu_b > m_\pi/2$.
- This was seen numerically.
- Misha Stephanov explained why a decade ago in the context of random matrix approach.
- However, unlike quenched QCD,large Nc QCD is a theory.

- *This suggests that description of nuclear matter as opposed to isospin matter is the culprit in the large Nc problem.*
- Is it?

Possible Theory Issues

- Effect due to chiral symmetry?
- Break down of 1/Nc expansion due infrared effects associated with the breaking of isospin symmetry in the region $\mu_I > m_\pi/2$?
  - Interesting possibility but raises a number of subsidiary issues.

Effect due to chiral symmetry?

- It is well known that the large Nc and chiral

$$Z_I(\mu) = -4 f_\pi^2 (2\mu - m_\pi)^2 \theta(2\mu - m_\pi)$$

Break down of 1/Nc expansion due to a phase transition associated with spontaneous U(1) isospin (third component) symmetry breaking?

- Simplest explanation---the large Nc expansion breaks down for the free energy.
- This is the simplest and most natural explanation and is presumably correct.
- However this explanations raises a number of interesting problems.

Third component of isospin *is* spontaneously broken

- In region where μ_I is slightly above $m_\pi/2$ chiral expectation valueas calculable via χPT.

$$\left\langle i\bar{u}\gamma_5 d\right\rangle = e^{i\phi}\,\theta(\mu-\tfrac{m_\pi}{2})\,\tfrac{m_\pi^2 f_\pi^2}{2}\sqrt{1-\tfrac{m_\pi^4}{16\mu^4}}\left(1+\mathrm{O}(\tfrac{m_\pi^2,\mu^2}{\Lambda^2})\right)$$

$$\left\langle i\bar{d}\gamma_5 u\right\rangle = e^{-i\phi}\,\theta(\mu-\tfrac{m_\pi}{2})\,\tfrac{m_\pi^2 f_\pi^2}{2}\sqrt{1-\tfrac{m_\pi^4}{16\mu^4}}\left(1+\mathrm{O}(\tfrac{m_\pi^2,\mu^2}{\Lambda^2})\right)$$

- The arbitrary phase is symptomatic of spontaneous symmetry breaking.
- Clear evidence that up and down quarks “talk to” each other at leading order with isospin chemical potential in conflict with naïve diagrammatic analysis.

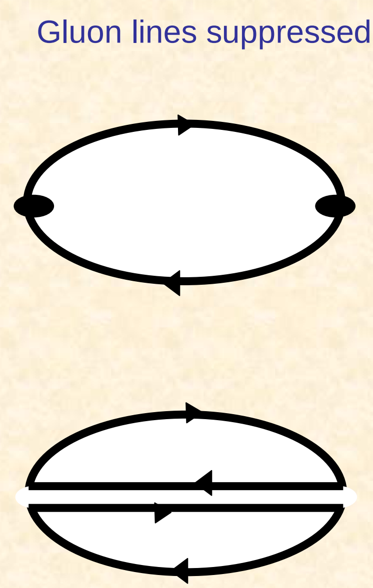
Connection to diagrammatics

- Spontaneous symmetry breaking is associated with a diverging susceptibility.
- In the present context it is the chiral susceptibility.

$$\chi_\chi = \int d^4x \left\langle (i\, m\, \bar{q}\gamma_5\tau_x q)(x)\; (i\, m\, \bar{q}\gamma_5\tau_x q)(0) \right\rangle$$

- This diverges for $\mu_I > m_\pi/2$.
- A diverging susceptibility means and infinitesimal perturbation has a finite effect .
- Diagrammatically at leading order in 1/Nc this is a quark loop with two insertions (with any number of gluons)

- Top diagram: chiral susceptibility at leading order in 1/Nc.
- Bottom diagram: Sources replaced with quark-antiquark pair with same quantum #s.
- This lower diagram is 1/Nc suppressed. However, diverging susceptibility means even an infinitesimal perturbation yields finite results.



Apparent problem: There is no obvious breakdown of the expansion

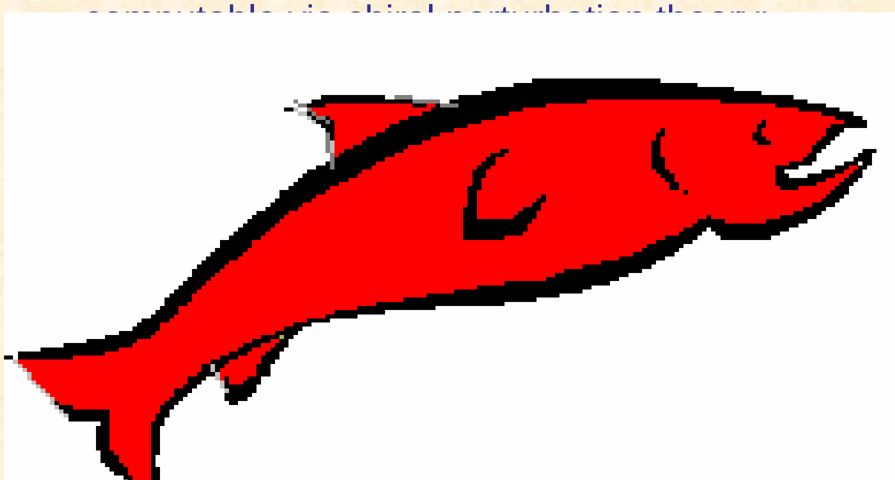
- Consider $\frac{m_\pi}{2} < \mu < \frac{M_N - B/A}{N_c}$ at $T = 0$

A breakdown in the expansion is associated with a failure of the Taylor expansion. Yet, both G_B and G_I have well defined Taylor series:

The 1/Nc expansion clearly converges for baryon free energy density: it is zero for any Nc in this regime. A Taylor series with all terms zero describes this perfectly.

Isospin case also has a well described large Nc expansion.

computable via chiral perturbation theory:



also has a well defined Taylor expansion

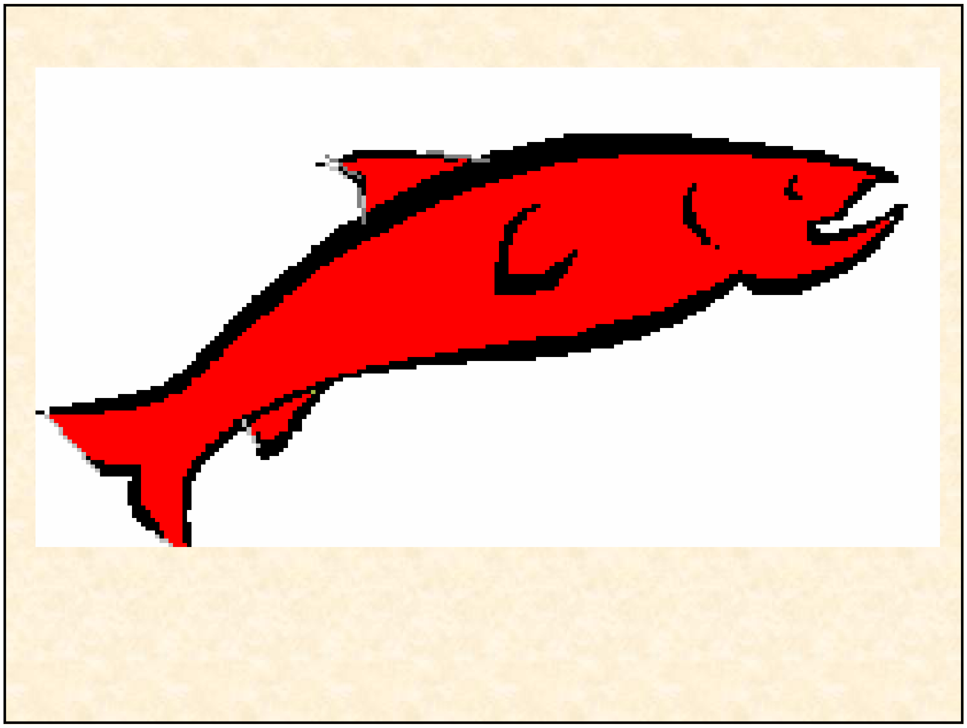
- Problem can be evaded if there are *two* distinct 1/Nc expansions for G_I.
  - One holds in the regime $\mu_I < m_\pi/2$. It is consistent with the diagrammatic analysis and matches on to G_B.
  - There is a second expansion valid in the regime $\mu_I > m_\pi/2$.
  - There is a breakdown of the 1/Nc expansion at the border of the two regimes.
- In fact this is what appears to happen. The step function in the isospin matter free energy induces nonanalytic behavior at the transition.

Possible problem: The behaviour at low temperature and chemical potential

- Consider regime $0 < \mu, T \ll m_\pi$ (with μ,T formally taken at $\mathcal{O}(N_c^0)$. In this regime one can apply χPT to the problem of isospin matter---system looks like weakly interacting pion gas:

$$G_I(T,\mu) = \frac{G_I(T,0)}{3}\left(1 + 2\cosh\left(\frac{\mu}{T}\right)\right) + O\left(e^{-2m_\pi/T}\right)$$

- G_I *varies* with μ at $\mathcal{O}(N_c^0)$. This variation holds down to $\mu=0$ at nonzero temperature.



In fact no contradiction

- The typical free energies in the problem are of order Nc for μ of order unity.
  - Easy to see in diagrammatics.
  - Easy to see in explict χPT calculations at zero temperature and $\mu_I > m_\pi/2$.
- In the regime $0<\mu, T<< m_\pi$ the free energy is of order Nc^0 .
- That means that the difference between the baryon and isospin free energies is at the first subleading order in 1/Nc---exactly as expected diagramatically.

This Illustrates a general point

- Large Nc continuum reduction (see Neuberger's talk) implies that below the QCD phase transition quantities are equal to their T=0 values plus relative order 1/Nc corrections. (TDC PRL 93 (2004) 201601)
- Only novelty here is that the leading order value happened to be zero.

Large Nc Paradox Seems Resolved

- A break down of the large Nc description of isospin matter due to spontaneous isospin symmetry breaking.
- Somewhat surprising: *a priori* one might have expected the baryon case to be the culprit. Large Nc is quenched & quenched QCD gets isospin matter basically right and screws up badly for nuclear matter.

An Open Issue

- The baryon “Silver Blaze” problem remains unsolved.
- It remains plausible that large Nc analysis might provide a clue.
  – A useful place to look is right at the point $\mu_I < m_\pi/2$. Just above this point the magnitude of the functional determinants grow but are exactly canceled by phases.
- All suggestions are welcomed!!!

Summary

- The treatment of cold nuclear matter is one of the most important challenges in LQFT.
- The “Silver Blaze” problem cuts to the core of this issue.
  – Isospin version is basically understood
  – Baryon version is open---involves cancellations due to phase of functional det.
- Large Nc QCD is potentially an important source of insight.
  – The different behavior of the isopin and baryon cases is largely understood from a large Nc perspective
  – This does not help solve the larger problem
  – Clever new ideas are sought---please provide them!