Dynamic universality class of the QCD critical point

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QCD critical point

![Graph showing the QCD critical point](image)
Locating the QCD critical point

Signatures: event-wise fluctuations.

Susceptibilities diverge $\Rightarrow$ fluctuations grow towards the critical point.
Magnitude of fluctuations

Scaling and universality of critical phenomena: \( \chi \sim \xi^{\text{power}} \).

How big can \( \xi \) grow?

Limits:
- Proximity of the critical point
- Finite size of the system \( \xi < 6 \text{ fm} \).
- Finite time: \( \tau \sim 10 \text{ fm} \).

\[ \xi \sim \tau^{1/z} \]

\( z \) – dynamical critical exponent.
Dynamic scaling and universality

Hohenberg, Halperin Rev. Mod. Phys. 49, 435 (1977)

Near the c.p. Typical relaxation time scale in the system diverges (critical slowing down):

\[ \tau \sim \xi^z \]

(dynamic scaling).

The exponent \( z \) is determined by the dynamic universality class.

Systems with equivalent static critical behavior are not always in the same dynamic universality class.

Example: Ising model and liquid-gas phase transition.

Simple reason: the order parameter (density) in the latter is a conserved quantity \( \Rightarrow \) relaxes slower at large distance scales (by diffusion), then in the Ising model (local relaxation).
Dynamic universality class of QCD critical point

Son, MS, hep-ph/0401052

Statics: symmetry and dimensionality → Ising model (same as liquid-gas).

Dynamic universality class depends on relevant hydrodynamic modes.

Modes which relax arbitrarily slowly: densities of conserved quantities, order parameter.

- The fluctuations of the energy and momentum densities: $\varepsilon \equiv T^{00} - \langle T^{00} \rangle$, and $\pi^i \equiv T^{0i}$;
- The fluctuations of the baryon number density, $n \equiv \bar{q} \gamma^0 q - \langle \bar{q} \gamma^0 q \rangle$;
- The chiral condensate $\sigma \equiv \bar{q} q - \langle \bar{q} q \rangle$.

Is it liquid gas (H), or is it Ising model (A), or is it another universality class altogether? (Berdnikov, Rajagopal: model C)
Statics

\( \sigma \) and \( n \) mix.

\[
F[\sigma, n] = \int d\mathbf{x} \left[ V(\sigma, n) + \frac{a}{2} (\nabla \sigma)^2 + b(\nabla \sigma)(\nabla n) + \frac{c}{2} (\nabla n)^2 \right],
\]

\[
V(\sigma, n) = \frac{A}{2} \sigma^2 + B\sigma n + \frac{C}{2} n^2 + \text{higher orders}.
\]

At the critical point \( \Delta \equiv AC - B^2 \to 0 \).

One flat direction: \((\sigma, n) \sim (-B, A)\). Only one critical mode.

Susceptibilities, e.g.:

\[
\langle \sigma_{q \to 0}^2 \rangle = \frac{TC}{\Delta}, \quad \langle n_{q \to 0}^2 \rangle = T\chi_B = \frac{TA}{\Delta}.
\]

Either \( \sigma \) or \( n \) can be used as an order parameter (both jump across 1st order phase transition).
Dynamic equations

\begin{align*}
\dot{\sigma} &= -\Gamma \frac{\delta F}{\delta \sigma} + \tilde{\lambda} \nabla^2 \frac{\delta F}{\delta n} + \xi_{\sigma}, \\
\dot{n} &= \tilde{\lambda} \nabla^2 \frac{\delta F}{\delta \sigma} + \lambda \nabla^2 \frac{\delta F}{\delta n} + \xi_n.
\end{align*}

with \( P[\sigma, n] \sim \exp(-F[\sigma, n]/T) \))

\begin{align*}
\langle \xi_{\sigma}(x) \xi_{\sigma}(y) \rangle &= 2T \Gamma \delta^4(x - y), \\
\langle \xi_{\sigma}(x) \xi_n(y) \rangle &= -2T \tilde{\lambda} \delta(t - t') \nabla^2 \delta^3(x - y), \\
\langle \xi_n(x) \xi_n(y) \rangle &= -2T \lambda \delta(t - t') \nabla^2 \delta^3(x - y).
\end{align*}

Using \( F \) (and to leading order in \( \nabla \)):

\begin{align*}
\dot{\sigma} &= -\Gamma A \sigma - \Gamma B n, \\
\dot{n} &= (\tilde{\lambda} A + \lambda B) \nabla^2 \sigma + (\tilde{\lambda} B + \lambda C) \nabla^2 n.
\end{align*}

\(+ \text{ noise}\)
Modes

\[
\det \begin{vmatrix}
\Gamma A - i\omega & \Gamma B \\
(\tilde{\lambda}A + \lambda B)q^2 & (\tilde{\lambda}B + \lambda C)q^2 - i\omega
\end{vmatrix} = 0.
\]

Near critical point:

\[
\begin{align*}
\omega_1 & = -i\lambda \frac{\Delta}{A} q^2. \\
\omega_2 & = -i\Gamma A.
\end{align*}
\]

Diffusive: \((\sigma, n) \sim (-B, A)\), relaxational: \((\sigma, n) \sim (1, 0)\).

Critical mode (flat direction) is the diffusive mode, \(\sigma = (-B/A)n\).

Sigma alone \((n = 0)\) relaxes on a finite time scale even at c.p \((\Gamma \text{ is finite})\).

Conclusion: only one hydrodynamic mode after \(\sigma\) and \(n\) mixing, and it is diffusive.

Diffusion constant:

\[
D = \lambda \frac{\Delta}{A} = \lambda C - \lambda \frac{B^2}{A}.
\]

\(D \to 0\) at c.p.
Diffusion constant and critical indices

\[ D = \lambda \frac{\Delta}{A} \text{ and } \chi_B = \frac{A}{\Delta}: \]

\[ D = \lambda \chi_B^{-1} \]

\[ \mu(x) = \chi_B^{-1} n(x) \rightarrow E = -\nabla \mu \rightarrow j_n = \lambda E = -\lambda \chi_B^{-1} \nabla n \]

Typical relaxation (diffusion) time

\[ \tau \sim D^{-1} \xi^2, \]

but \( D^{-1} \sim \chi_B \sim \xi^{2-\eta} \rightarrow \tau \sim \xi^{4-\eta} \rightarrow z = 4 - \eta. \]

This is model B.
Coupling/mixing with energy-momentum

Modes to consider:

- $\varepsilon$, $\pi$
- $n$
- $\sigma$

Only one combination of $n$ and $\sigma$ is truly hydrodynamic $\rightarrow \varepsilon$, $\pi$ and $n$.

Same as in the liquid-gas dynamic universality class.

This is model H.
Model H

Review results (Hohenberg, Halperin):

\[ D = \lambda \chi_B^{-1} \sim \xi^{\lambda \chi_B^{-1}}, \]
\[ \bar{\eta} \sim \xi^{x\eta}. \]

\[ x_\lambda + x_\eta = 4 - d - \eta. \]

\[ x_\eta = \frac{1}{19} \epsilon (1 + 0.238 \epsilon + \cdots) \approx 0.065, \quad (1) \]
\[ x_\lambda = \frac{18}{19} \epsilon (1 - 0.033 \epsilon + \cdots) \approx 0.916, \quad (2) \]

\[ D \sim \xi^{-2 + \eta + x_\lambda}, \]

\[ z = 4 - \eta - x_\lambda \approx 3. \]
Conclusion

- Mixing between $\sigma$ and $n$ (and $\varepsilon$, $\pi_{||}$) leaves only one hydrodynamic mode – diffusive.
- Dynamic universality class of QCD c.p. is that of model H.
- $z \approx 3$ ($>2$ of model A, but $<4$ of model B).
- Finite time constraint is rather strong: $\xi_{\text{max}} \sim \tau^{1/z}$ is not too large.
  Observability: fluctuations are $\chi \sim \xi^2$. 
Divergence of $\lambda \bar{\eta}$

\[ E = -\nabla \mu \quad \rightarrow \quad j_n = \lambda E \quad \text{(in model B)} \]

\[ f_{\text{appl}} \sim nE L^d \]

\[ f_{\text{visc}} + f_{\text{appl}} = 0 \]

\[ f_{\text{visc}} \sim -\bar{\eta} \nu L^{d-2} \]

\[ j_n = n\nu \sim \left(\frac{n^2}{\bar{\eta}}\right) L^2 E. \quad \text{(in model H)} \]

$L \to \infty$ divergence is cut off at $L \sim \xi$, because $n$ is correlated only up to this scale.

\[ \langle n^2 \rangle = T\chi_B / L^d \text{ for } L \gg \xi \]

\[ \lambda \bar{\eta} \sim \langle n^2 \rangle \xi^2 \sim \chi_B \xi^{2-d} \sim \xi^{4-d-\eta}, \]
Real time correlation functions

\[ \langle \sigma_{\omega q}^2 \rangle = \frac{2T \Gamma}{\omega^2 + \Gamma^2 A^2} + \frac{2TB^2 \lambda q^2}{A^2 (\omega^2 + D^2 q^4)}, \]

\[ \langle n_{\omega q}^2 \rangle = \frac{2T \lambda q^2}{\omega^2 + D^2 q^4}. \]