’t Hooft-Polyakov Monopoles on the Lattice

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Rajantie, in progress
Introduction

- 't Hooft-Polyakov monopoles
  - Pointlike magnetic charges
  - Georgi-Glashow model: SU(2)+adjoint Higgs

- Confinement in QCD and Yang-Mills
  - Monopole condensation?
  - Abelian projection?

- Predicted by all GUTs
  - Produced in the early universe
  - Greatly diluted by inflation
  - Constantly searched, none found yet
    - (or possibly one on Valentine’s Day 1982 (Cabrera 1982))

- Theoretical interest
  - SUSY models
  - Dualities
Topological Solitons

- Localized, topologically stable field configurations
- Order parameter $\phi$ at spatial infinity $|\vec{r}| \rightarrow \infty$:
  - Finite energy $\Rightarrow$ Must approach vacuum
  - Possibly different vacuum in different directions
  - Defines a map from $S^{d-1}$ to the vacuum manifold $\mathcal{M} \cong G/H$
- Solitons exist if $\pi_n(G/H) \neq 0$ for $n < d$
  - $n = 0$: Domain walls (kinks)
  - $n = 1$: Vortices (strings)
  - $n = 2$: Monopoles
  - Winding number $N_W \in \pi_n(G/H)$
- Convenient theoretical probes of phase properties
  - Dualities
  - Confinement $\leftrightarrow$ Monopole condensation? (’t Hooft, Mandelstam)
Classical Kink

- 1+1D real scalar field

\[ \mathcal{L} = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \phi'^2 - \frac{\lambda}{4} (\phi^2 - v^2)^2 \]

- Two vacua \( \phi = \pm v \Rightarrow \pi_0 = \mathbb{Z}_2 \), winding number 0 or 1
- Kink: Choose \( \phi(\pm \infty) = \pm v \)
- Exact stationary solution: \( \phi(x) = v \tanh(\lambda v^2/2)^{1/2} x \)

Energy \( M_{\text{kink}} = \frac{2}{3} \sqrt{2\lambda v^3} \)
Continuum:

\[ \mathcal{L} = -\frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \text{Tr} [D_\mu, \Phi] [D_\nu, \Phi] - m^2 \text{Tr} \Phi^2 - \lambda (\text{Tr} \Phi^2)^2 \]

- SU(2) gauge field \( A_\mu = A_\mu^a \sigma^a / 2 \), where \( a \in \{1, 2, 3\} \)
- Adjoint Higgs field \( \Phi = \Phi^a \sigma^a / 2 \)

Euclidean lattice action (lattice spacing = 1)

\[ \mathcal{L}_E = 2 \sum_\mu \left[ \text{Tr} \Phi(\vec{x})^2 - \text{Tr} \Phi(\vec{x}) U_\mu(\vec{x}) \Phi(\vec{x} + \hat{\mu}) U_\mu^\dagger(\vec{x}) \right] + \frac{2}{g^2} \sum_{\mu < \nu} \left[ 2 - \text{Tr} U_{\mu\nu}(\vec{x}) \right] + m^2 \text{Tr} \Phi^2 + \lambda (\text{Tr} \Phi^2)^2 \]

- Link variables \( U_\mu \in \text{SU}(2), U_\mu \sim \exp(igA_\mu) \)
- Plaquette \( U_{\mu\nu} = U_\mu(x) U_\nu(x + \hat{\mu}) U_\mu^\dagger(x + \hat{\nu}) U_\nu^\dagger(x) \)
\( m^2 < 0 \): Symmetry breaking SU(2) \( \rightarrow \) U(1)

- Vacuum manifold \( \{ \text{Tr} \Phi^2 = v^2 = |m^2|/\lambda \} \cong S^2 \)
- \( \pi_2(S^2) = \mathbb{Z} \Rightarrow \) Monopoles (\textquoteleft t Hooft, Polyakov)

\[
\Phi^a(\vec{r}) = \frac{r_a}{gr^2} H(gvr)
\]
\[
A_i^a(\vec{r}) = -\epsilon_{aij} \frac{r_j}{gr^2} [1 - K(gvr)]
\]

Broken phase: U(1) symmetry \( \Rightarrow \) Electrodynamics

- Field strength \( F_{\mu\nu} = \text{Tr} \hat{\Phi} F_{\mu\nu} + (2i g)^{-1} \text{Tr} \hat{\Phi} [D_\mu, \hat{\Phi}][D_\nu, \hat{\Phi}] \)
  - Unitary gauge \( \hat{\Phi} = \sigma_3 \): Reduces to \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \)
- Magnetic field \( B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} \):
  - If \( \Phi \neq 0 \), then \( \vec{\nabla} \cdot \vec{B} = 0 \)
  - For a smooth configuration \( \vec{\nabla} \cdot \vec{B}(\vec{x}) = (4\pi/g) \sum_i \pm \delta(\vec{x} - \vec{x}_i) \)
  \( \Rightarrow \) Magnetic monopoles with charge \( \pm 4\pi/g \)
Magnetic Field on the Lattice

Discretized version of $\mathcal{F}_{\mu\nu}$:
- Define projection $\Pi_+ = \frac{1}{2}(1 + \hat{\Phi})$
- Projected link $u_\mu(x) = \Pi_+(x)U_\mu(x)\Pi_+(x + \hat{\mu})$
- U(1) field strength tensor

\[ \alpha_{\mu\nu} = \frac{2}{g} \arg \text{Tr} \ u_\mu(x)u_\nu(x + \hat{\mu})u_\mu^\dagger(x + \hat{\nu})u_\nu^\dagger(x) \]

Magnetic field $\hat{B}_i = \frac{1}{2}\epsilon_{ijk}\alpha_{jk}$
- Magnetic charge in a lattice cell
\[ \hat{\rho}_M = \sum_i \left[ \hat{B}_i(x + \hat{i}) - \hat{B}_i(x) \right] \in (4\pi/g)\mathbb{Z} \]
\[ \Rightarrow \text{Stable monopoles} \]
Classical Monopole Mass

- Continuum result
  \[ M = (4\pi m_W/g^2)f(m_H/m_W) \]
  \[ f(x) \approx 1 + x/2 + (x^2/2)(\ln x + \sqrt{2}) \]
  (Kirkman & Zachos 1981)

- Example: \( \lambda = 0.1, g = 1/\sqrt{5} \)

- Finite size effects
  - Coulomb force \(|m^2| \gg 1/L^2|\):
    \[ \Delta E(L) \approx 11.0/g^2 L \]
  - Symmetry restoration
    \[ \Delta E(L) \approx V(0)L^3 = (\lambda v^4/4)L^3 \]

- Infinite-volume extrapolation:
  \[ f(x) \approx 1.10 \]
Perturbative Quantum Corrections

- Find lowest energy eigenvalue $E(N_W)$ with a given winding number $N_W$
  - Soliton mass $M = E(1) - E(0)$

- Perturbative approach: (Dashen et al. 1974)
  - Loop expansion around classical solution $\phi_0(x)$
    - Write $\phi(t, x) = \phi_0(x) + \delta(t, x)$
    - Quantize $\delta(t, x)$: Field in a $x$-dependent potential
    - Order $\delta^2$: Harmonic potential $U(\delta) = \frac{1}{2} V''(\phi_0(x))\delta^2$
    - Diagonalize:
      \[
      \left[ -\vec{\nabla}^2 + V''(\phi_0(x)) \right] \delta_k(x) = \omega_k^2 \delta_k(x)
      \]
      $\Rightarrow$ Frequencies $\omega_k$
  - One-loop level: $\Delta E = \sum_k (\omega_k^1 - \omega_k^0)/2$
  - Higher-order corrections: Difficult
One-loop Kink Mass

Equation for $\omega_k$:

$$\left[ -\frac{\partial^2}{\partial x^2} + \lambda v^2 \left( 3 \tanh^2 \sqrt{\frac{\lambda v^2}{2x}} - 1 \right) \right] \delta_k(x) = \omega_k^2 \delta_k(x)$$

Can be solved exactly:

$\omega_0^2 = 0$, $\omega_1^2 = 3\lambda v^2 / 2$ and a continuum $\omega_q^2 = (q^2 / 2 + 2)\lambda v^2$

Caveats: Zero mode, measure for $q$, UV regularisation

Result: (Dashen et al. 1974)

$$M_{\text{kink}} \approx \frac{2}{3} \sqrt{2\lambda} v^3 + \left( \frac{1}{2\sqrt{6}} - \frac{3}{\sqrt{2\pi}} \right) \sqrt{\lambda} v$$
Leading-log Monopole Mass

- Same principles, many extra complications
  - Gauge fixing
  - Two coupled fields
  - Higher dimensionality
  - Renormalisation issues
- Only leading log in the $m_H/m_W \to 0$ limit has been calculated (Kiselev&Selivanov 1988)

$$M = \frac{4\pi m_W}{g^2} \left( 1 + \frac{g^2}{8\pi^2} \ln \frac{m_H^2}{m_W^2} + O(g^2) \right)$$

- Infrared divergence as $m_H/m_W \to 0$
- Related to Coleman-Weinberg effect:
  $$m_H/m_W \gg g$$ due to quantum fluctuations
- Difficult to test: Need small $m_H/m_W \to 0$
  $$\Rightarrow$$ Small $g \Rightarrow$ Small quantum correction
Non-perturbative Soliton Masses

- Soliton creation and annihilation operators $\psi^\dagger$ and $\psi$ (Kadanoff&Ceva 1971)
  - $\langle 0 | \psi^\dagger(t_1) \psi(t_2) | 0 \rangle \propto e^{iM(t_2-t_1)}$

- Path integral formulation (integrate over $\varphi$ with $N_W = 0$)
  
  $e^{-M(t_2-t_1)} \propto Z^{-1} \int_0 D\varphi \psi^\dagger(t_1) \psi(t_2) e^{-S[\varphi]}$

- Easy to do in simple cases: Kinks, vortices

- Less straightforward for monopoles:
  - Magnetic field $\Rightarrow \psi$ necessarily non-local
  - Compact QED: Duality maps to an integer-valued gauge theory (Polley&Wiese)
    $\Rightarrow$ Becomes much simpler

- Non-Abelian theories: Several attempts (Frohlich&Marchetti, Di Giacomo et al.)
  - Idea: Add a classical monopole configuration between $t$ and $t + \delta t$
    (Dirac string with an endpoint, BPS monopole . . .)
  - Boundary conditions problematic
Removing Start and Endpoints

Take $t_2 \rightarrow t_1 + T$, where $T$ is temporal size

- $\langle \psi^\dagger(t_1) \psi(t_2) \rangle \rightarrow Z_1/Z_0 = \exp(-MT)$
- $M = -\ln(Z_1/Z_0)/T$

Define $Z_1$ using appropriate boundary conditions

Monte Carlo: Cannot calculate $Z_1$ or $Z_0$ directly
- Only expectation values: Derivatives or differences
Mass Derivatives

- \( M = -\frac{\ln Z_1/Z_0}{T} \), but cannot calculate \( Z_1 \) or \( Z_0 \) directly
- Calculate derivative with respect to some parameter \( \lambda \):
  \[
  \frac{\partial M}{\partial \lambda} = \frac{1}{T} \left( \frac{1}{Z_0} \frac{\partial Z_0}{\partial \lambda} - \frac{1}{Z_1} \frac{\partial Z_1}{\partial \lambda} \right)
  \]
- Express in terms of expectation values:
  \[
  \frac{1}{Z_{NW}} \frac{\partial Z_{NW}}{\partial \lambda} = -\frac{1}{Z_{NW}} \int_{NW} D\varphi \left( \frac{\partial S}{\partial \lambda} \right) e^{-S} = -\left\langle \frac{\partial S}{\partial \lambda} \right\rangle_{NW}
  \]
- Can be calculated with Monte Carlo simulations
- Integrate to obtain \( M(\lambda) \)
- Start in symmetric phase: No integration constant
Non-perturbative Kink Mass

- Comparison of one-loop, operator and twist results (Ciria&Tarancon 1994)
  - Twist: Simply antiperiodic b.c. $\phi(L) = -\phi(0)$

- Non-perturbative results agree with each other
- Twist has much smaller errors
  - Also true for monopoles in compact QED (Vettorazzo&de Forcrand 2004)
- Slightly above one-loop result
• Fix the field to the classical solution at the boundary (Smit&van der Sijs 1994, Cea&Cosmai 2000)
• Boundary effects?
Twisted Boundary Conditions

Most common choice: Periodic boundary conditions

- No boundary effects: Consequence of translation invariance
- Magnetic Gauss law $\vec{\nabla} \cdot \vec{B} = \rho_M \Rightarrow$ Magnetic charge $Q_M = 0$

Translation invariance only requires periodicity up to symmetries

- C-periodic: (Kronfeld&Wiese 1991)
  
  \[
  U_\mu(x + N\hat{j}) = U_\mu^*(x) = \sigma_2 U_\mu(x) \sigma_2 \\
  \Phi(x + N\hat{j}) = \Phi^*(x) = -\sigma_2 \Phi(x) \sigma_2
  \]

  - Charge conjugation: Avoid Gauss law problem
  - Restricts $Q_M$ to even values $\Rightarrow$ Use this to define $Z_0$

- Twisted b.c.:
  
  \[
  U_\mu(x + N\hat{j}) = \sigma_j U_\mu(x) \sigma_j \\
  \Phi(x + N\hat{j}) = -\sigma_j \Phi(x) \sigma_j
  \]

  - Locally gauge equivalent to C-periodic - but not globally!
  - Always gives odd $Q_M \Rightarrow$ Use this to define $Z_1$ (JHEP 2000)
Derivative of Monopole Mass

- Choose \( m^2 \) as the integration variable
  - Start at high enough \( m^2 \) ⇒ Symmetric phase
  - Measure \( \langle \text{Tr}\Phi^2 \rangle_{NW} \) at many values of \( m^2 \) using lattice Monte Carlo
  - Integrate:

\[
M = L^3 \int_{m_0^2}^{m^2} dm^2 \left( \langle \text{Tr}\Phi^2 \rangle_1 - \langle \text{Tr}\Phi^2 \rangle_0 \right)
\]

- Better: Finite differences

\[
M = \frac{1}{T} \sum_n \left( \langle e^{\Delta m^2 T L^3 \text{Tr} \Phi^2} \rangle_{1,m_n^2} - \langle e^{\Delta m^2 T L^3 \text{Tr} \Phi^2} \rangle_{0,m_n^2} \right)
\]
Derivative of Monopole Mass: Results

\[- \frac{dM}{dm} \]

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Direct Calculation

- Problems: – Must go through a phase transition
  – Errors accumulate

- Direct way of calculating $M$ at given $m^2$

  - Gauge transformation $\rightarrow$ C-periodic except

  \[
  U_3(t, x, L, L - 1) = -U_3^*(t, x, 0, L - 1) \\
  U_1(t, L - 1, y, L) = -U_1^*(t, L - 1, y, 0) \\
  U_1(t, L - 1, L, z) = -U_1^*(t, L - 1, 0, z)
  \]
Direct Calculation

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U_1(t, L - 1, L, z) = -U_1^*(t, L - 1, 0, z)
$$

- Change of variables

$$
U_3(t, x, L, L - 1) \rightarrow -U_3(t, x, L, L - 1)
$$
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Direct Calculation

- Problems: – Must go through a phase transition
  – Errors accumulate

- Direct way of calculating $M$ at given $m^2$
  - Gauge transformation
  - Change of variables

$$Z_1 = \int_{C\text{--per}} DU_\mu D\Phi \exp(-S - \Delta S) = \langle \exp(-\Delta S) \rangle_0 Z_0$$

where

$$\Delta S = \beta \sum_{t,x=0}^{L-1} [\text{Tr } U_{23}(x, y_0, z_0) + \text{Tr } U_{13}(x_0, y, z_0) + \text{Tr } U_{12}(x_0, y_0, z)]$$

- Three orthogonal 't Hooft lines crossing each other at $(x_0, y_0, z_0)$
Direct Calculation

- Problems: – Must go through a phase transition
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- Direct way of calculating $M$ at given $m^2$
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\]

- Three orthogonal ’t Hooft lines crossing each other at $(x_0, y_0, z_0)$
Non-Integer Twists

- Difficult to calculate $\langle \exp(-\Delta S) \rangle$: Poor overlap
- Define for $\epsilon \in [0, 1]$

\[
Z_\epsilon = \int_{C-\text{per}} DU_\mu D\Phi \exp(-S - \epsilon \Delta S')
\]

- Unphysical for non-integer $\epsilon$
- Still well-defined

- Differentiate with respect to $\epsilon$

\[
\frac{dM}{d\epsilon} = -\langle \Delta S' \rangle_\epsilon
\]
Non-Integer Twists

\[ \beta = 18, \ x = 0.35, \ y = -0.5, \ L = 16 \]

From 3D simulation (PRD65(2002))
Renormalisation

- Comparison with classical results?
  - $m^2$, $\lambda$, $g$ bare couplings
  - Must renormalise

- Scheme dependence

- Perturbative renormalisation
  - Monopole mass only to the same order in perturbative expansion

- Non-perturbative approach:
  - Measure three different quantities (say $g$, $m_H$, $m_W$)
  - Use them to fix the classical couplings

- For the moment, simply ignore logs and finite terms
  - Shift $m^2$ axis by a constant amount
Comparison with Classical Mass

- $m^2$ shifted by 0.268
- Quantum masses generally lower (renormalisation?)
Effective Couplings

- Classical simulation \( \Rightarrow \) Finite size effect \( \Delta E(L) = 11.0/g^2 L \)
- Fit quantum finite size effect to determine \( g_R \)
  - Gives \( g_R \approx 0.44(5) \) vs bare \( g \approx 0.447 \)
- Masses \( m_H \) and \( m_W \)
  - from correlation functions
  - Difficult to measure \( m_W \)
- Expectations: As \( m^2 \rightarrow m_c^2 \)
  - Triviality: \( \lambda_R \rightarrow 0 \)
  - Asymptotic freedom: \( g_R \) becomes large
  - \( m_H/m_W = \sqrt{\lambda_R/g_R} \rightarrow 0 \)
  - \( M/m_W = (4\pi/g_R^2)f(m_H/m_W) \rightarrow 0? \)
  - Will \( W^\pm \) decouple?
    \( \Rightarrow \) Charged scalar + photon (+ neutral scalar)
Near the critical point, $M_{\text{vort}} \propto (m_c^2 - m^2)^{0.671 \pm 0.038}$.

- Vortex becomes the lightest particle: $m_\gamma, m_s \propto (m_c^2 - m^2)^{1/2}$
- Dual to complex scalar field theory?

Numerical evidence: XY model critical exponent
Speculation: Asymptotic Duality in Georgi-Glashow Model?

<table>
<thead>
<tr>
<th>Georgi-Glashow model</th>
<th>Abelian Higgs model</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Higgs phase</strong></td>
<td><strong>Coulomb phase</strong></td>
</tr>
<tr>
<td>electric/magnetic field</td>
<td>magnetic/electric field</td>
</tr>
<tr>
<td>magnetic monopole</td>
<td>charged scalar</td>
</tr>
<tr>
<td>massless photon</td>
<td>massless photon</td>
</tr>
<tr>
<td><strong>Confining phase</strong></td>
<td><strong>Higgs phase</strong></td>
</tr>
<tr>
<td>confinement</td>
<td>superconductivity</td>
</tr>
<tr>
<td>confining string</td>
<td>vortex line</td>
</tr>
</tbody>
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- Puts the ’t Hooft-Mandelstam dual superconductor idea on firm footing
- Same duality is known to exist in supersymmetric theories
Hints for Monopole Duality

- Phase diagram for $\lambda \to \infty$ (Greensite et al. 2004)

- Limit $\kappa \to \infty = \text{compact QED}$
  - Exactly dual to 4D frozen superconductor (Peskin 1978)
  - Frozen superconductor $= \lambda, \kappa \to \infty$ limit of Abelian Higgs model
  - Duality maps electric and magnetic field to each other

- Will duality survive near critical point even for finite $\lambda, \kappa$?
Conclusions

- Monopole mass using twisted boundary conditions
  - Well defined even on the lattice
  - No cooling needed
  - No reference to any specific field configs

- Integrating the derivative
  - Derivative with respect to $m^2$
    - Straightforward
    - Growing errors
  - Derivative with respect to non-integer twist $\epsilon$
    - Non-integer values unphysical
    - Direct measurement of $M$ at given couplings

- Comparison with classical result
  - Significant correction in terms of bare couplings
  - Renormalisation: Perturbative/Non-perturbative

- Critical behaviour: Duality?