Nonequilibrium quantum field theory and the lattice

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(hep-ph/0409233)

Content

I. Motivation

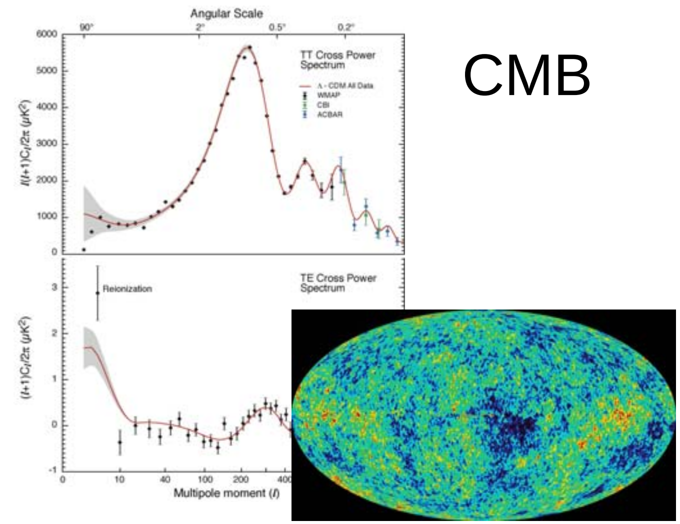
1) Early universe

2) Heavy-ion collisions

3) Ultra-cold quantum gases

1) Early Universe

End of Inflation | (P)reheating | CMB



far-from-equilibrium `initial´ state

`entropy´ production

thermal spectrum with fluctuations

time

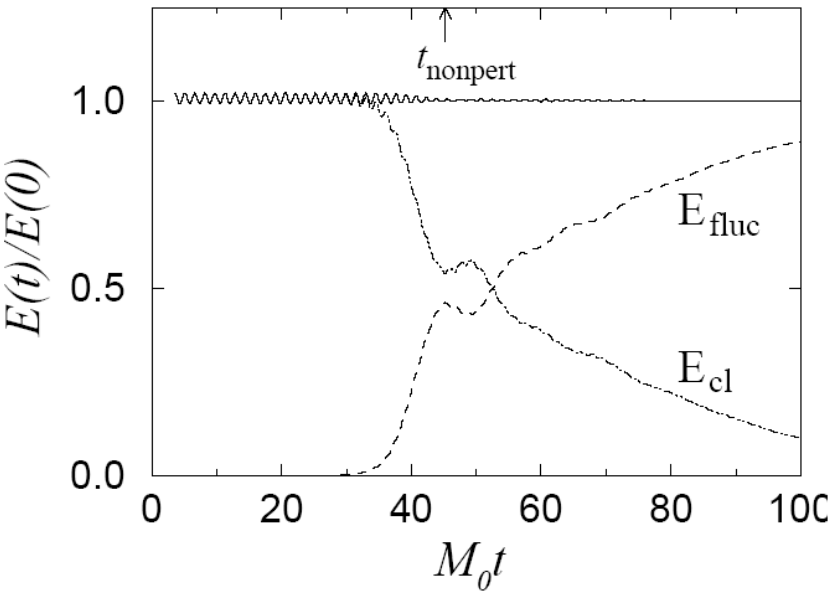
(P)reheating

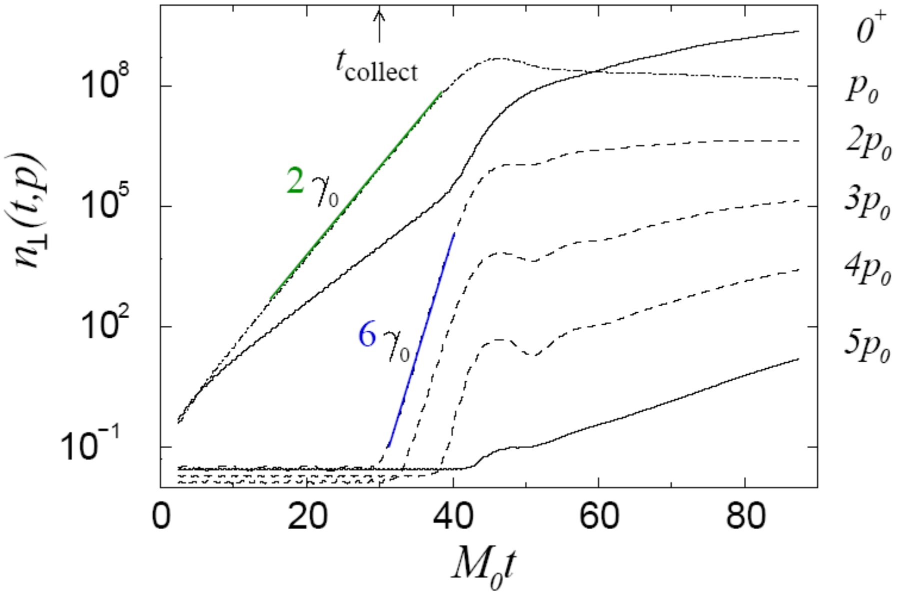
- *Far-from-equilibrium particle production* in quantum field theory

Traschen, Brandenberger, PRD 42 (1990) 2491; Kofman, Linde, Starobinsky, PRL 73 (1994) 3195

CLASSICAL: Khlebnikov, Tkachev, PRL 77 (1996) 219

QUANTUM: Berges, Serreau, PRL 91 (2003) 111601:





2) Heavy-ion collisions

- Heavy-ion collisions (BNL,CERN,GSI) explore strong interaction matter starting from a transient *nonequilibrium* state

➨ Thermalization ?

Properties of the *equilibrium phase diagram* of QCD ?

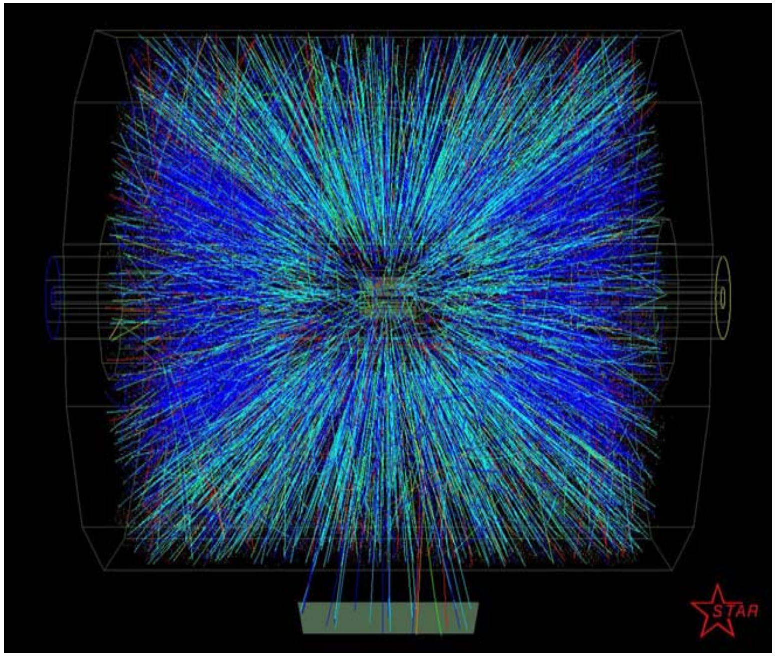
Braun-Munzinger et al., nucl-ex/0411053

Prethermalization ?

Berges, Borsanyi, Wetterich, PRL93 (2004)142002

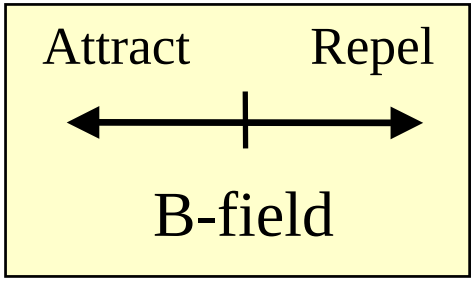
➨ \`Slowing´ out of equilibrium near the *QCD critical point*

Rajagopal, Berdnikov, PRD61 (2000) 105017; ...

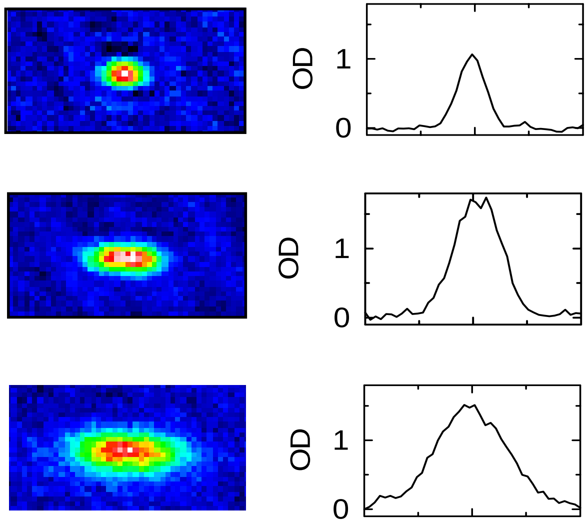


3) Ultra-cold quantum gases

- Tunable BEC self-interaction!
  (Feshbach resonance)



- Measure BEC size, shape:
  (500 *ms*)



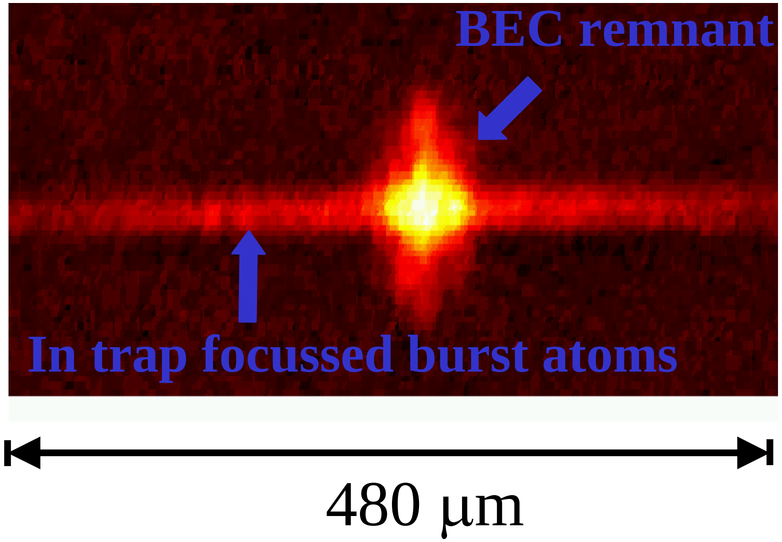
$a = 70\ a_0$

$550\ a_0$

$3000\ a_0$

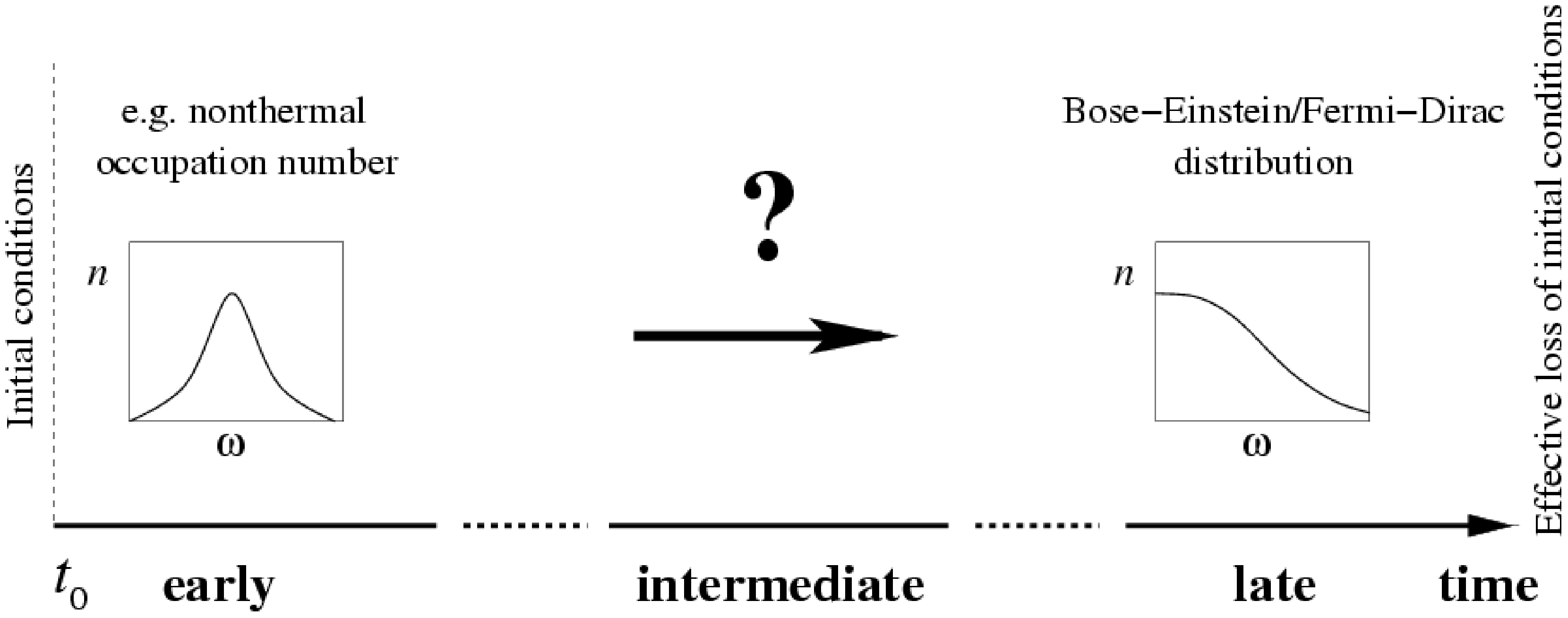
Cornish et al. PRL 85 (2000) 1795

- B(t) faster than atom motion:
  ($\Delta t \ll 1/\omega_{trap} \sim 10\ ms$)



Thermalization

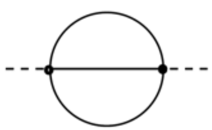
- Process of thermalization leads to loss of details about the initial conditions: late-time *`universality´*
- Approach to thermal equilibrium requires *quantum evolution* (*classical* equilibration times are functions of Rayleigh-Jeans cutoff)



Standard QFT techniques fail out of equilibrium

`Secularity´

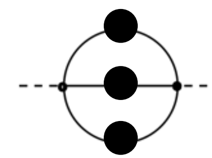
- uniform approximations in time require infinite pert. orders



$\sim \gamma t$

`Universality´

- nonlinear dynamics necessary for late-time thermalization



$\rightarrow e^{-\gamma t}$

Two-particle irreducible generating functionals

- systematic 2PI loop-, coupling- or 1/N-expansions available
- *far-from-equilibrium* dynamics as well as late-time *thermalization* in QFT

Berges, Cox ´01; Aarts, Berges ´01; Berges ´02; Cooper, Dawson, Mihaila ´03; Berges, Serreau ´03; Berges, Borsanyi, Serreau ´03; Cassing, Greiner, Juchem ´03; Arrizabalaga, Smit, Tranberg ´04 ...

II. Nonequilibrium QFT

1) Nonequilibrium generating functional

2) 2PI effective action

3) Time evolution equations

Nonequilibrium QFT

$\rho_D \equiv \rho_D^{(eq)}$

$\overset{\text{e.g.}}{\sim} e^{-\beta H}$

$\rho_D(t{=}t_0) \neq \rho_D^{(eq)}$

"Close-to-equilibrium QFT"

"Far-from-equilibrium QFT"

- *Thermal equilibrium* correlations in real-time

- *Nonequilibrium* correlations (not necessarily far-from-eq.)

e.g. L.G. Yaffe, NP Proc.Suppl. 106 (2002) 117

- linear response
- transport coefficients
  ...

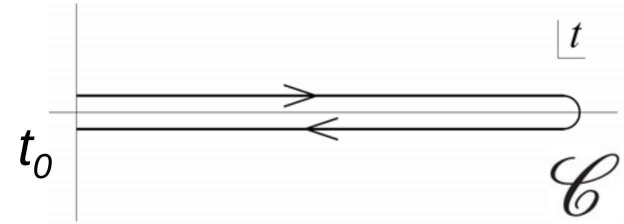
THIS TALK

Nonequilibrium generating functional

$$Z[J,R;\rho_D] = \underbrace{\int d\varphi^{(1)} d\varphi^{(2)} \langle \varphi^{(1)} | \rho_D(0) | \varphi^{(2)} \rangle}_{\text{initial conditions}} \underbrace{\int_{\varphi^{(1)}}^{\varphi^{(2)}} \mathscr{D}'\varphi \, e^{i\left(S[\varphi] + \int_x J(x)\varphi(x) + \frac{1}{2}\int_{xy} R(x,y)\varphi(x)\varphi(y)\right)}}_{\text{quantum dynamics}}$$

Finite (!), closed real-time contour: (*NO " $i\varepsilon$ " !* No imaginary part)

$$\int_x \equiv \int_{\mathscr{C}} dx^0 \int d^d x$$



$$\langle \varphi^{(1)} | \rho_D(0) | \varphi^{(2)} \rangle = \mathscr{N} e^{i f_{\mathscr{C}}[\varphi]}$$

$$f_{\mathscr{C}}[\varphi] = \alpha_0 + \int_x \alpha_1(x)\varphi(x) + \frac{1}{2}\int_{xy} \alpha_2(x,y)\varphi(x)\varphi(y) + \frac{1}{3!}\int_{xyz} \alpha_3(x,y,z)\varphi(x)\varphi(y)\varphi(z) + \ldots$$

$$\boxed{Z[J,R;\rho_D] = \int \mathscr{D}\varphi \, e^{i\left(S[\varphi] + \int_x J(x)\varphi(x) + \frac{1}{2}\int_{xy} R(x,y)\varphi(x)\varphi(y) + \frac{1}{3!}\int_{xyz} \alpha_3(x,y,z)\varphi(x)\varphi(y)\varphi(z) + \ldots\right)}}$$

Initial-time sources $\alpha_i \equiv 0$ for $t \neq t_0$

Two-particle irreducible effective action

- Gaussian *initial* density matrix $\rho_D(0)$: (i.e. $\alpha_i = 0$ for i > 2)

$$Z\left[J, R; \rho_D(0)\right] \longrightarrow Z[J, R]$$

(no approximation, just constrains class of initial conditions)

- 2PI effective action by double Legendre transform:

$$Z[J,R] = \exp\left(iW[J,R]\right)$$

$$\Gamma[\phi, G] = W[J,R] - \int_x \frac{\delta W[J,R]}{\delta J_a(x)} J_a(x) - \int_{xy} \frac{\delta W[J,R]}{\delta R_{ab}(x,y)} R_{ab}(x,y)$$

$$= W[J,R] - \int_x \phi_a(x) J_a(x) - \frac{1}{2} \int_{xy} R_{ab}(x,y) \phi_a(x) \phi_b(y) - \frac{1}{2} \mathrm{Tr}\, G R$$

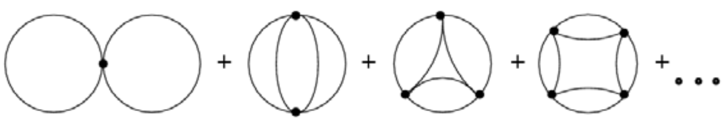
<u>2PI effective action:</u> Luttinger, Ward ´60; Baym ´62; Cornwall, Jackiw, Tomboulis ´74

$$\Gamma[\phi,G] = S[\phi] + \frac{i}{2}\mathrm{Tr}\ln G^{-1} + \frac{i}{2}\mathrm{Tr}\, G_0^{-1}(\phi)\, G + \Gamma_2[\phi,G]$$

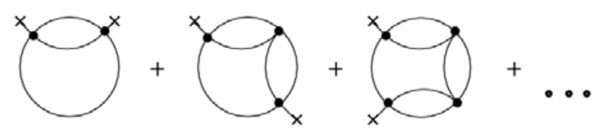
- Parametrized by macroscopic field: $\phi(x) = \langle \Phi(x) \rangle$ and
- exact connected propagator: $G(x,y) = \langle T\Phi(x)\Phi(y)\rangle - \phi(x)\phi(y)$
- $\Gamma_2[\phi,G]$ contains only *two-particle irreducible* (2PI) diagrams

E.g. scalar *N*-component field theory to NLO:

$$\Gamma_2[\phi,G] = -\frac{\lambda}{4!N}\int_x G_{aa}(x,x)G_{bb}(x,x) + \frac{i}{2}\,\mathrm{Tr}\ln \mathbf{B}(G)$$



$$+\frac{i\lambda^2}{(6N)^2}\int_{xyz} \mathbf{B}^{-1}(x,z;G)G^2(z,y)\phi_a(x)G_{ab}(x,y)\phi_b(y)$$



$$\mathbf{B}(x,y;G) \equiv \delta(x-y) + \frac{i\lambda}{6N}G^2(x,y)$$

Berges ´02; Aarts, Ahrensmeier, Baier, Berges, Serreau ´02

Time evolution equations

Equations of motion: (1) $\frac{\delta\Gamma[\phi,G]}{\delta\phi(x)} = 0$, (2) $\frac{\delta\Gamma[\phi,G]}{\delta G(x,y)} = 0$

spectral function $\sim \langle[\Phi,\Phi]\rangle$

$$G(x,y) = F(x,y) - \frac{i}{2}\rho(x,y)\,\mathrm{sign}_{\mathscr{C}}(x^0 - y^0)$$

statistical propagator $\sim \langle\{\Phi,\Phi\}\rangle$

$$\left[\Box_x \delta_{ac} + M^2_{ac}(x)\right]\rho_{cb}(x,y) = -\int_{y^0}^{x^0} \mathrm{d}z\,\Sigma^{\rho}_{ac}(x,z)\rho_{cb}(z,y)$$

$$\left[\Box_x \delta_{ac} + M^2_{ac}(x)\right]F_{cb}(x,y) = -\int_{0}^{x^0} \mathrm{d}z\,\Sigma^{\rho}_{ac}(x,z)F_{cb}(z,y) + \int_{0}^{y^0} \mathrm{d}z\,\Sigma^{F}_{ac}(x,z)\rho_{cb}(z,y)$$

$$\left(\left[\Box_x + \frac{\lambda}{6N}\phi^2(x)\right]\delta_{ab} + M^2_{ab}(x;\phi=0,F)\right)\phi_b(x) = -\int_0^{x^0} \mathrm{d}y\,\Sigma^{\rho}_{ab}(x,y;\phi=0,F,\rho)\,\phi_b(y)$$

Nonequilibrium:

$$F \not\sim \rho$$

Equilibrium/Vacuum:
(fluct.-diss. relation)

$$F \sim \rho$$

- In terms of spectral and statistical components the equations for fermionic or gauge fields have very similar structure as well:

$\rightsquigarrow$ In contrast to bosons, for fermions the field anti-commutator corresponds to the spectral function:

$$\begin{aligned} \rho^{(f)}(x,y) &= i\langle\{\psi(x),\bar{\psi}(y)\}\rangle \\ F^{(f)}(x,y) &= \frac{1}{2}\langle[\psi(x),\bar{\psi}(y)]\rangle \end{aligned}$$

Evolution equations are obtained from above by LHS replacement:

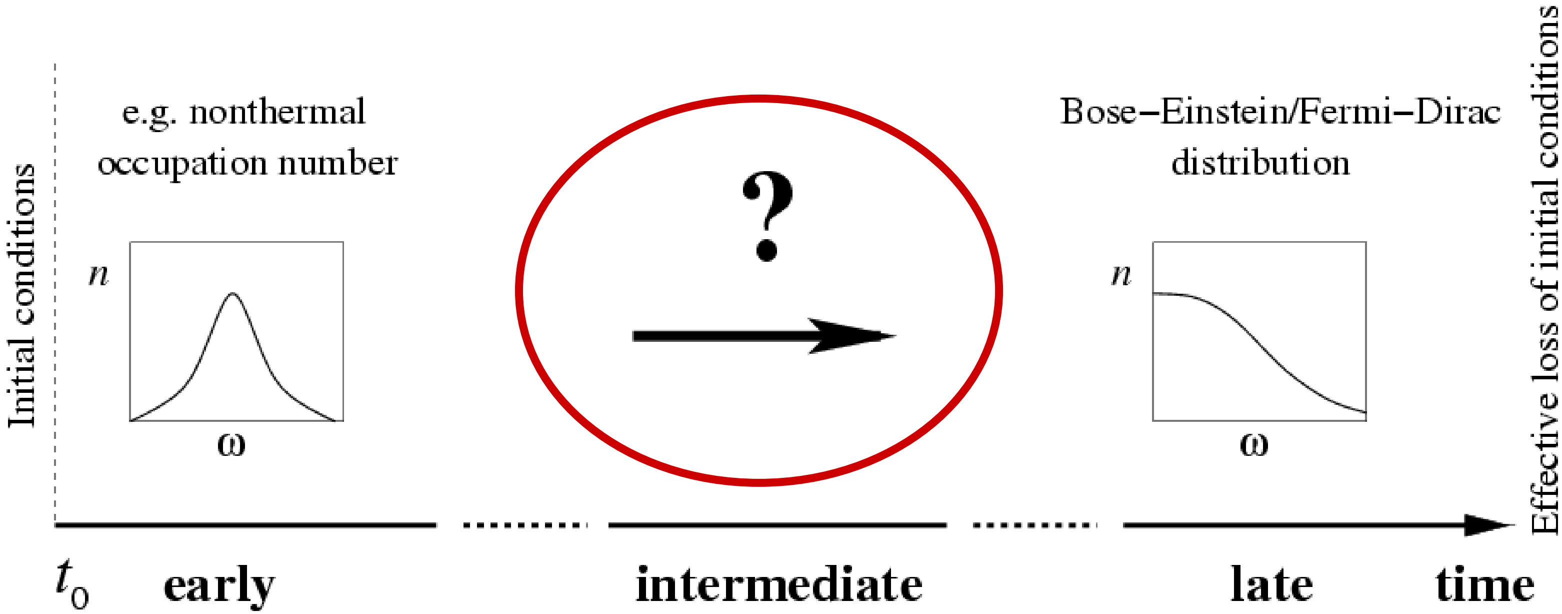
$$\left[\partial^2 + M^2\right]_x \rho(x,y) \longrightarrow -[i\partial\!\!\!/_x - m_f]_x\, \rho^{(f)}(x,y)$$

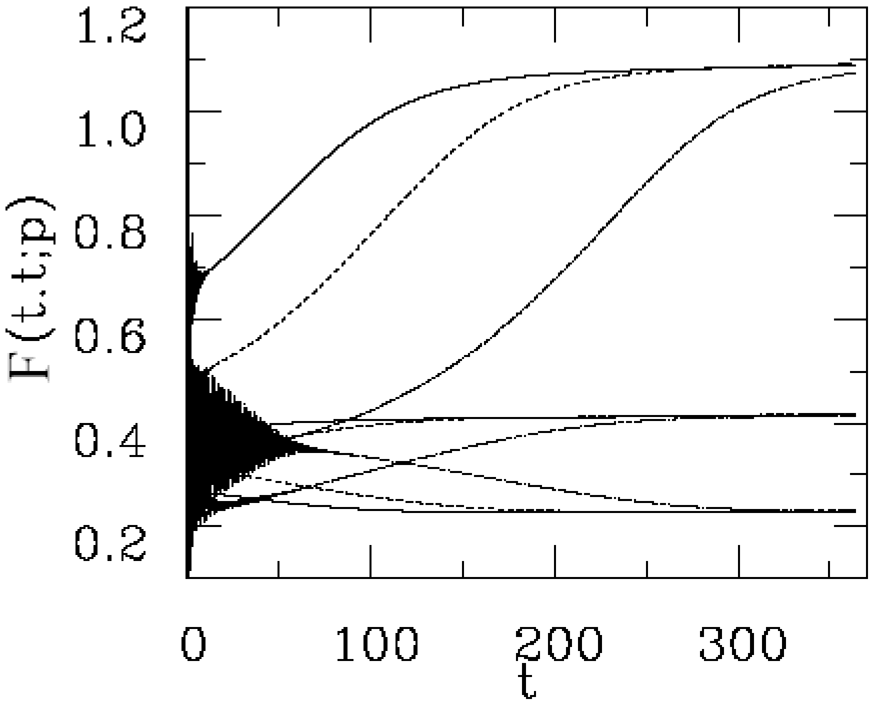
$\rightsquigarrow$ Gauge fields: $D^{\mu\nu}(x,y) = F_D^{\mu\nu}(x,y) - \frac{i}{2}\rho_D^{\mu\nu}(x,y)\,\mathrm{sign}_{\mathcal{C}}(x^0-y^0)$

$$\left[\partial^2 + M^2\right]_x \rho(x,y) \longrightarrow -\left[g^{\mu}{}_{\gamma}\partial^2 - (1-\xi^{-1})\partial^\mu\partial_\gamma\right]_x \rho_D^{\gamma\nu}(x,y)$$

e.g. for covariant gauges and vanishing ‘background’ fields

III. Far-from-equilibrium dynamics



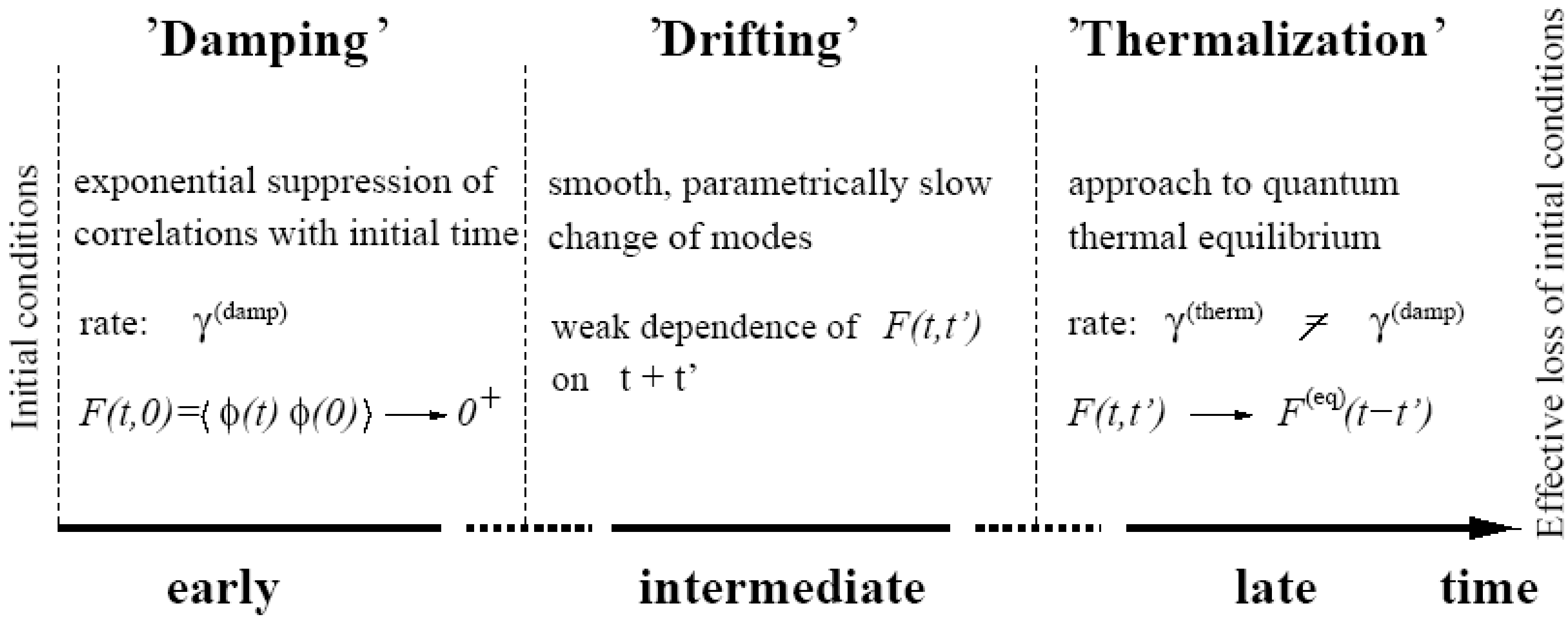


Propagator for (very) different initial conditions with *same* $\langle E \rangle$

(momenta $p = 0, 3, 5$; all in initial mass units, $\phi = 0$)

scalar ϕ^4 in 1+1 d (2PI 3-loop)

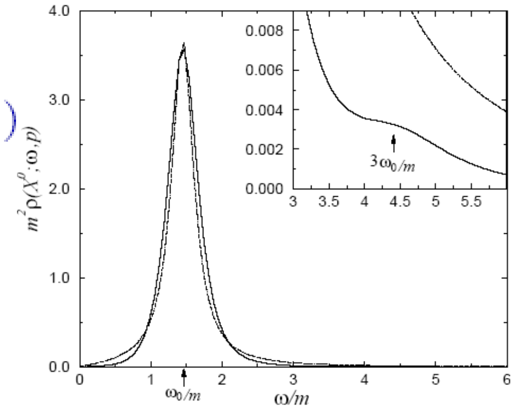
Berges, Cox, PLB 517 (2001) 369



Characteristic time scales for modes are associated to

1. rapid oscillations of correlation functions with period $\sim 1/\omega_{\mathbf{p}}$

$\rightsquigarrow$ *described by 'peak' of spectral function*



2. damping of oscillations with inverse rate $1/\gamma_{\mathbf{p}}^{(\mathrm{damp})}$

$\rightsquigarrow$ *described by nonzero 'width'* $\Gamma_{\mathbf{p}} = 2\gamma_{\mathbf{p}}^{(\mathrm{damp})}$

(dynamical)

3. 'late-time' thermalization with inverse rate $1/\gamma_{\mathbf{p}}^{(\mathrm{therm})}$

$\rightsquigarrow$ *because of 'off-shell' number changing processes:*

Number changing processes require nonzero 'width' $\sim \lambda^2/N$ effect in $\mathcal{O}(\lambda^2/N)$ evolution equation:

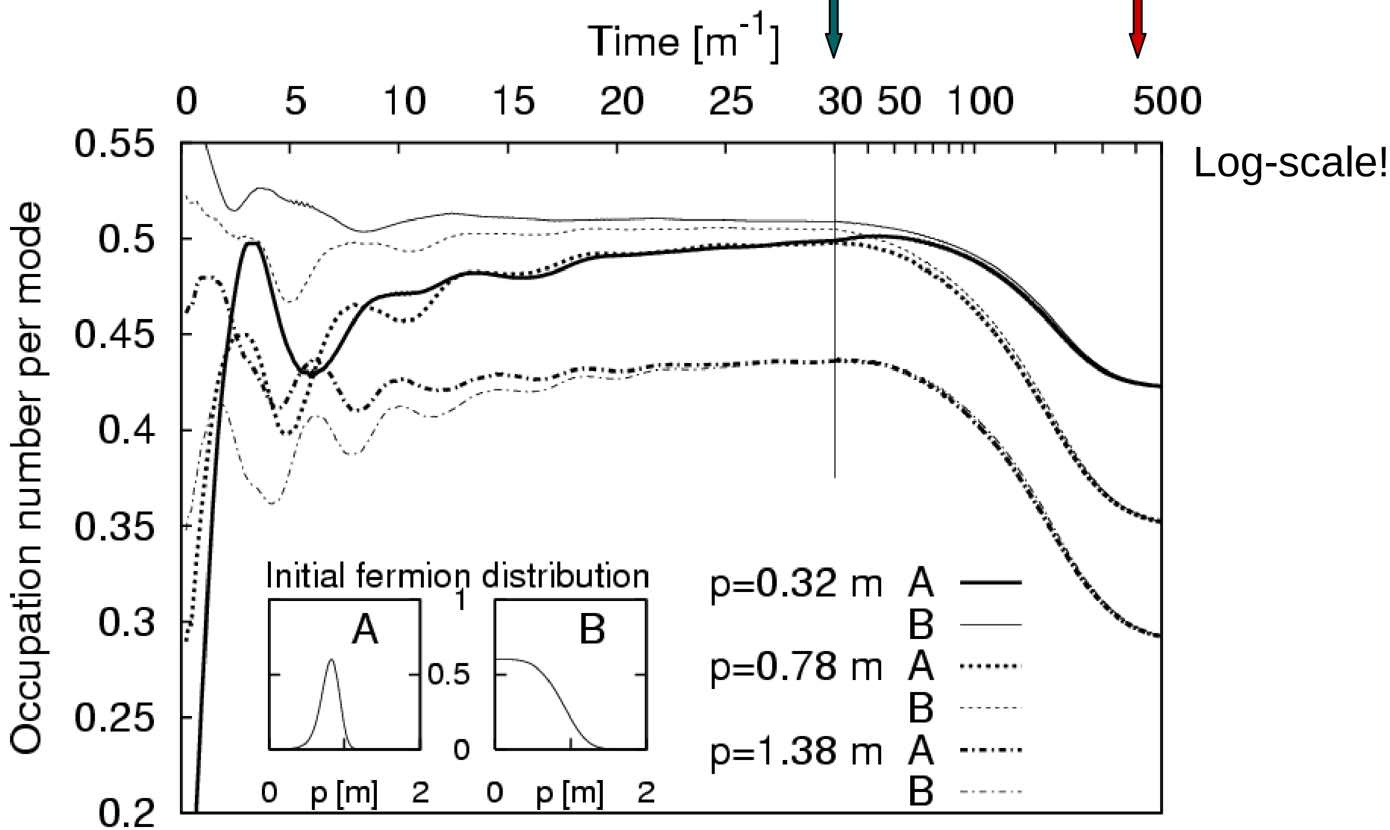
$\rightsquigarrow$ parametrically of order λ^4/N^2 ("slow!")

Compare to e.g.

chiral Yukawa model in 3+1 d (2PI *1/(N_F=2)* to NLO)

Loss of details of initial conditions after t_{damp}

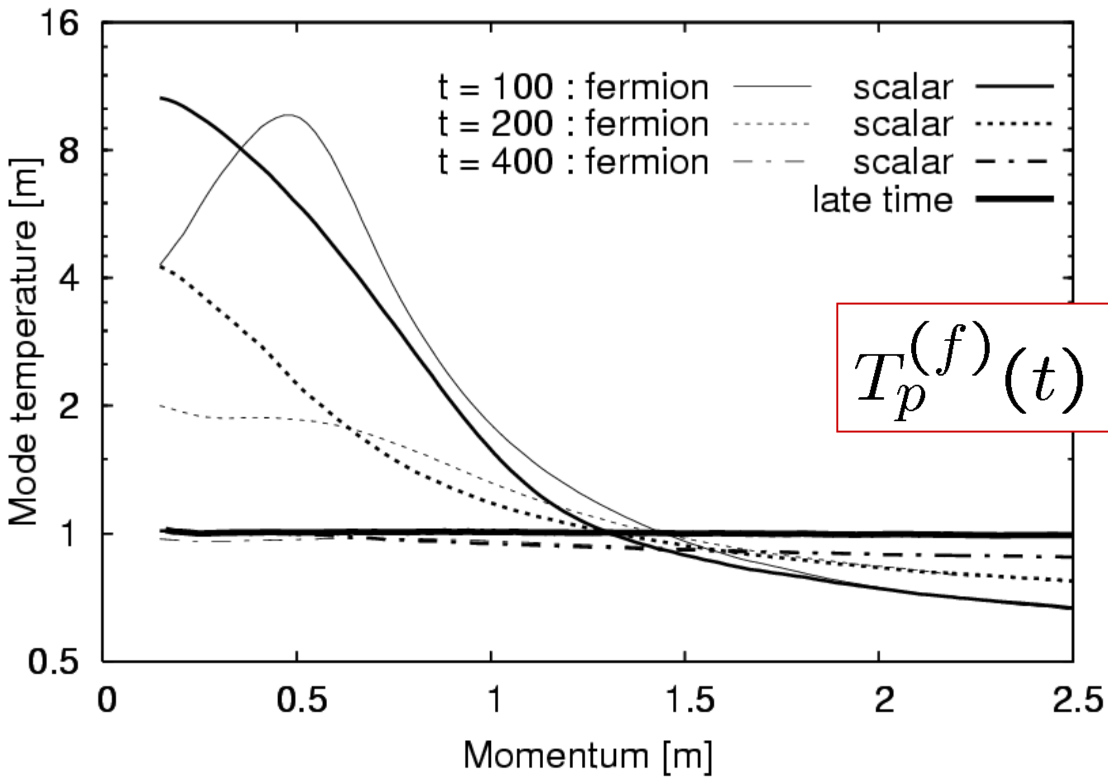
Thermalization t_{therm}



Berges, Borsányi, Serreau NPB 660 (2003) 51

Thermalization

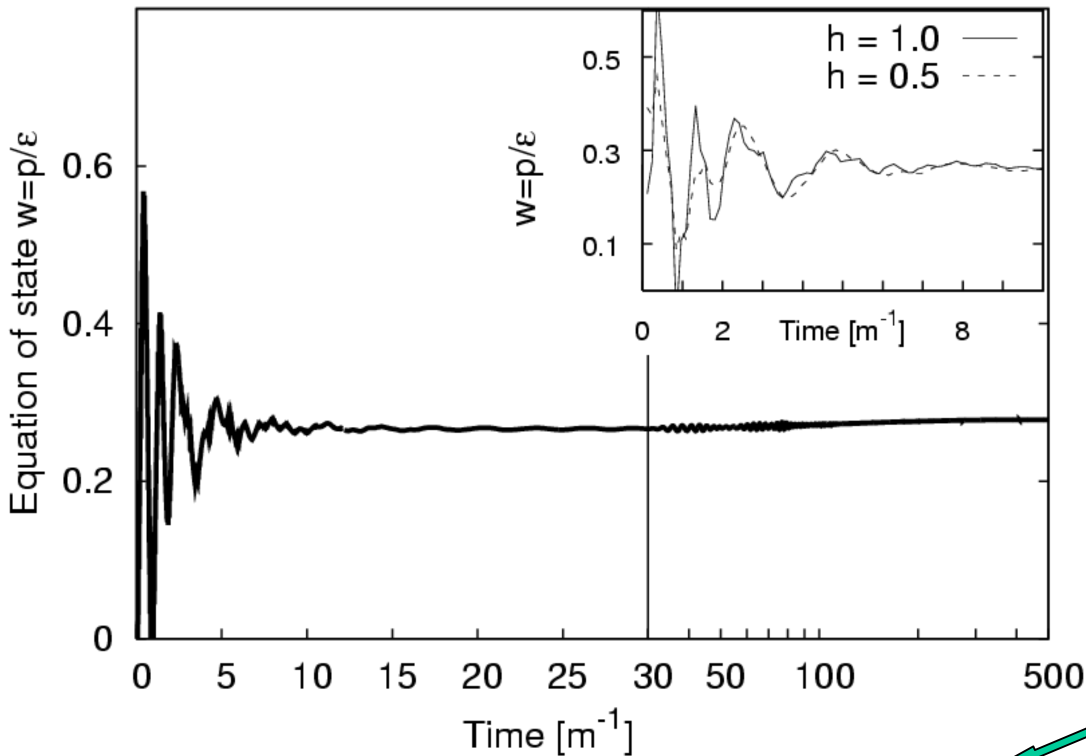
- 'Mode temperature' $T_p(t)$: $n_p(t) \stackrel{!}{=} \frac{1}{\exp[\omega_p(t)/T_p(t)] \pm 1}$
  $\left(n_p \sim \mathrm{tr}\, \frac{p^i \gamma^i}{p} \langle [\psi, \bar{\psi}] \rangle_p \right)$



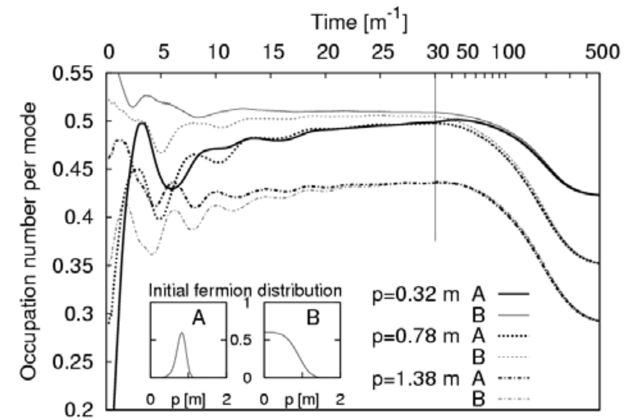
$$T_p^{(f)}(t) = T_p^{(s)}(t) = T_{\mathrm{eq}}$$

Emergence of BE/FD distribution at late times

Prethermalization



compare:



Prerequisite for hydrodynamics

- Almost time-independent EOS builds up very early, though distributions are far from equilibrium!
- Prethermalization-time independent of interaction details

$$t_{\mathrm{pt}} \ll t_{\mathrm{damp}} \ll t_{\mathrm{eq}}$$

Application

Estimate of prethermalization time in view of heavy-ion collisions:

- t_{pt} is rather independent of details of the model like particle content, couplings etc. (`dephasing´)
- If `temperature´, i.e. average kinetic energy per mode, sets the relevant scale we find:

$$T\, t_{\mathrm{pt}} \simeq 2 - 2.5$$

- For $T \simeq 400\mathrm{MeV}$:

$$\boxed{t_{\mathrm{pt}} \sim 1\mathrm{fm/c}}$$

Consistent with observed very early hydrodynamic behavior

IV. Precision tests on a lattice

1) Nonequilibrium *classical field simulations*

2) Comparison of *simulations* with *classical 2PI*

3) Comparison with *quantum 2PI*

4) Nonequilibrium *quantum field simulations?*

to 1) Exact classical equation of motion for N-component field $\phi_a(x)$:

$$\left[\Box_x + m^2 + \lambda\phi_b(x)\phi_b(x)/6N\right]\phi_a(x) = 0$$

Define classical 'statistical' two-point function:

$$F_{ab,\mathrm{cl}}(x,y) = \langle\phi_a(x)\phi_b(y)\rangle_{\mathrm{cl}} \equiv \int D\pi D\phi\, W[\pi,\phi]\phi_a(x)\phi_b(y)$$

$W[\pi,\phi]$: normalized probability functional at initial time with canonical momentum $\pi_a(x) = \partial_t\phi_a(x)$, $\phi_a(0,\mathbf{x}) \equiv \phi_a(\mathbf{x})$, $\pi_a(0,\mathbf{x}) \equiv \pi_a(\mathbf{x})$

Integration over classical phase-space: $\int D\pi D\phi = \int \prod_{a=1}^{N} \prod_{\mathbf{x}} \mathrm{d}\pi_a(\mathbf{x})\mathrm{d}\phi_a(\mathbf{x})$

Similarly, define classical spectral function:

$$\rho_{ab,\mathrm{cl}}(x,y) \;=\; -\langle\, \{\phi_a(x),\phi_b(y)\}_{\mathrm{PoissonBracket}}\, \rangle_{\mathrm{cl}}$$

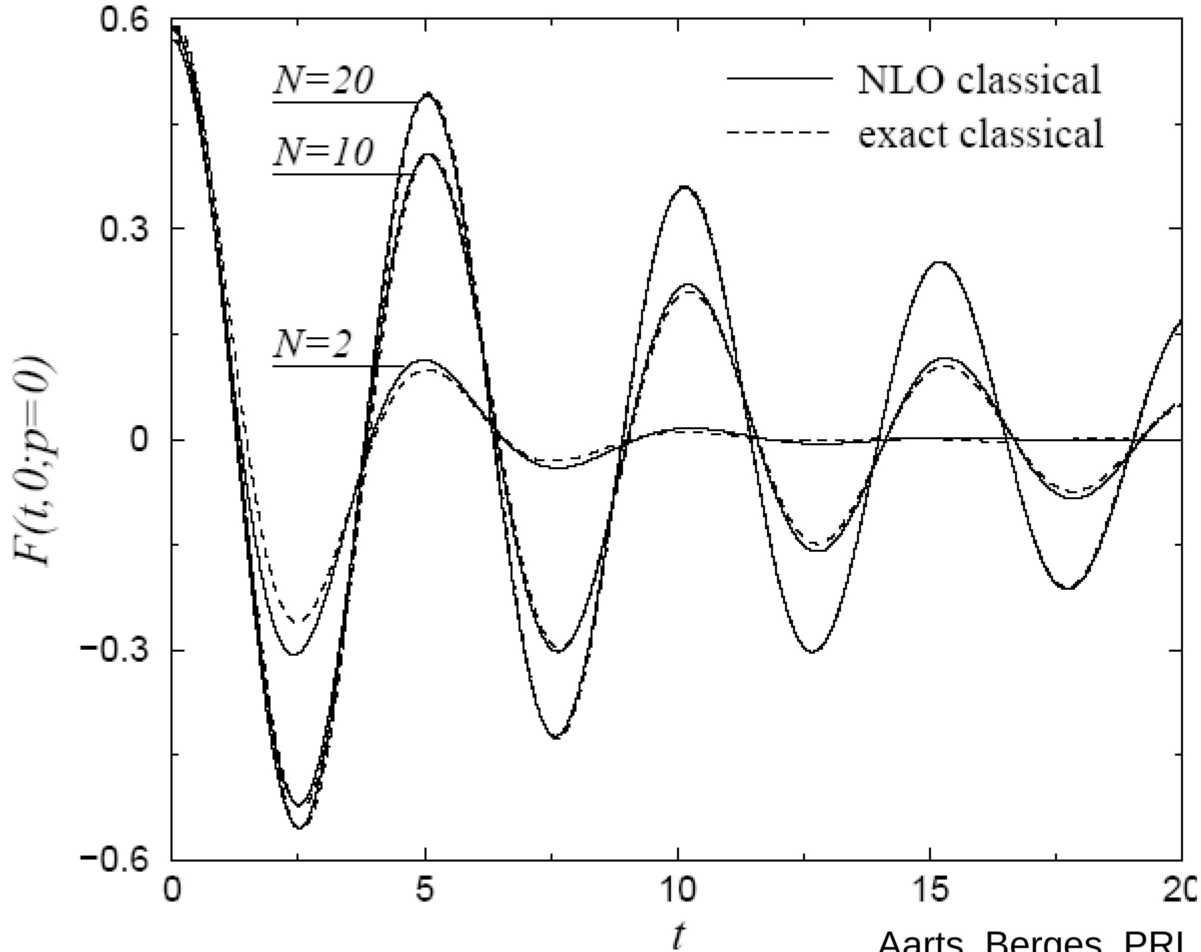
$$\Rightarrow \;\rho_{ab,\mathrm{cl}}(x,y)|_{x^0=y^0} \;=\; 0, \qquad \partial_{x^0}\rho_{ab,\mathrm{cl}}(x,y)|_{x^0=y^0} = \delta_{ab}\delta(\mathbf{x}-\mathbf{y})$$

$\{A(x),B(y)\}_{\mathrm{PoissonBracket}} = \sum_{a=1}^{N}\int \mathrm{d}\mathbf{z}\left[\frac{\delta A(x)}{\delta\phi_a(\mathbf{z})}\frac{\delta B(y)}{\delta\pi_a(\mathbf{z})} - \frac{\delta A(x)}{\delta\pi_a(\mathbf{z})}\frac{\delta B(y)}{\delta\phi_a(\mathbf{z})}\right]$

"Early-time" damping (effective loss of details of initial conditions)

$(1+1)\ d$

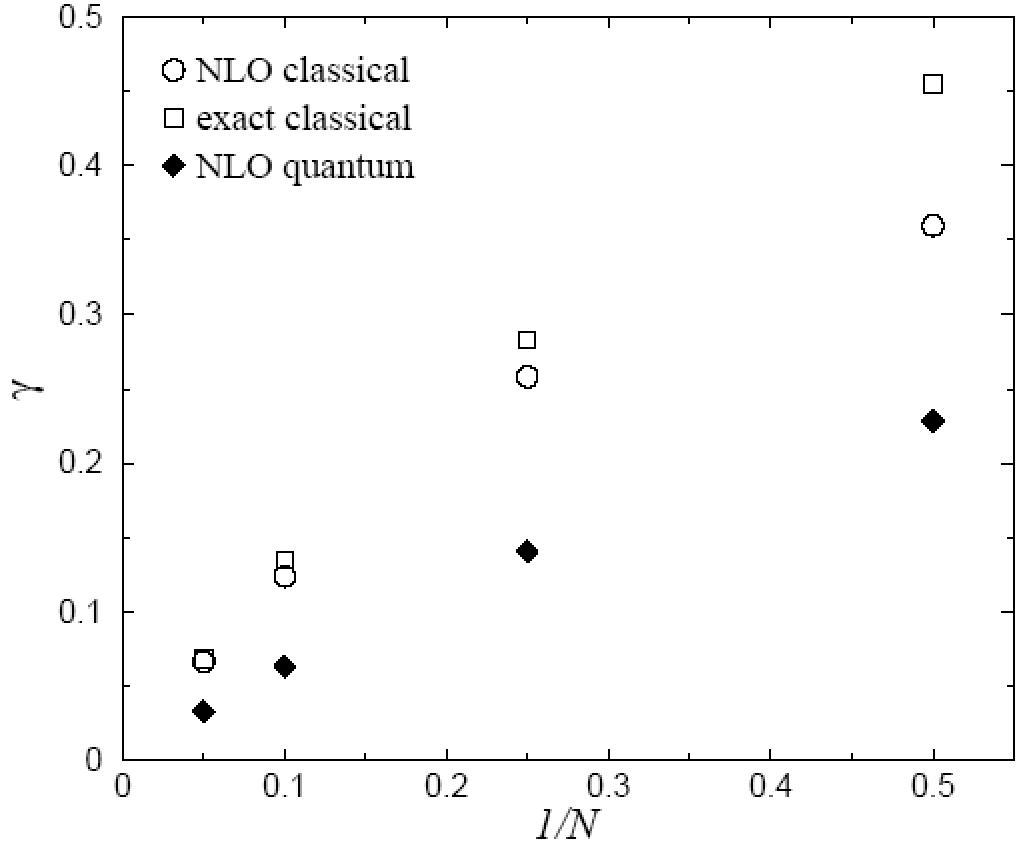
$$\frac{\lambda}{M_0^2} = 30$$



Aarts, Berges, PRL 88 (2002) 041603

$\Rightarrow$ Convergence of classical *NLO* and *exact* (MC) results already for moderate values of N (!)

Parametric behavior/comparison with quantum theory:



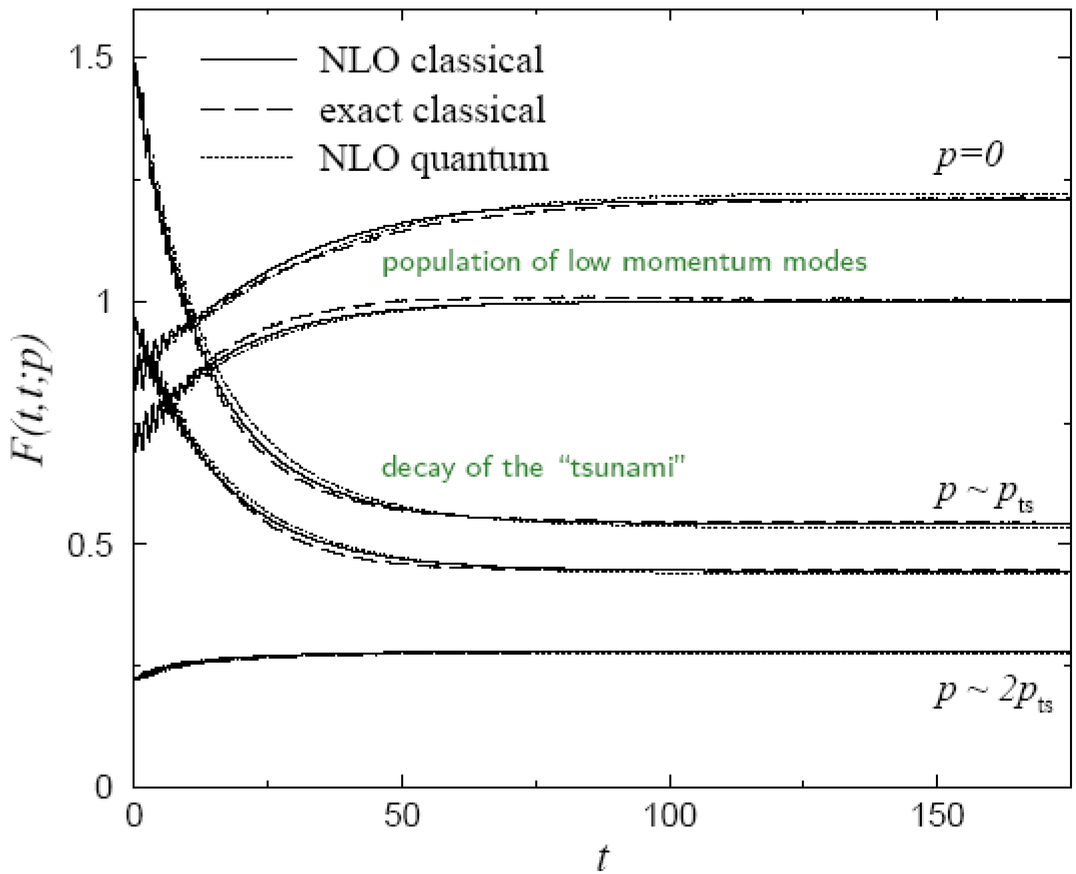
- Inverse damping rate $1/\gamma$ scales $\sim N$ for large N
  $\rightsquigarrow$ zero damping at LO ($N \to \infty$)
- Damping *enhanced* if quantum corrections are neglected

Here: $n_0(p) \ll \frac{1}{2}$ $\rightsquigarrow$ quantum corrections important!

"Late-time" (i.e. for quantum theory) behavior:

High initial particle number n_0 peaked around $p = |p_{\mathrm{ts}}| \simeq 4M_0$

"Tsunami":



N=4 (!)

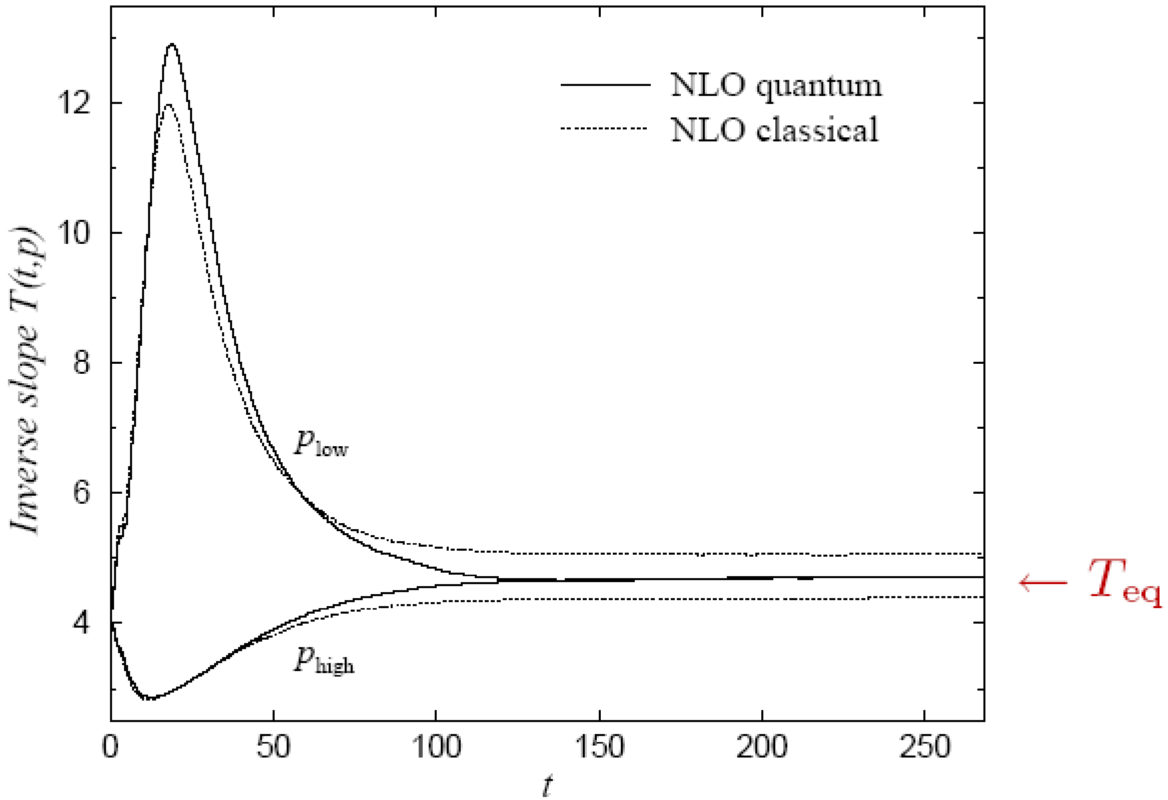
$\lambda = 12\, M_0^{-2}$

- *NLO* quantum evolution well approximated by *exact* classical result!

Here: $n_0(p_{\mathrm{ts}}) \gg \frac{1}{2}$ and $n_0(p) \gtrsim 0.35$ for $p \lesssim 2p_{\mathrm{ts}}$

$\rightsquigarrow$ small quantum corrections for displayed modes

Quantum vs. classical thermalization:

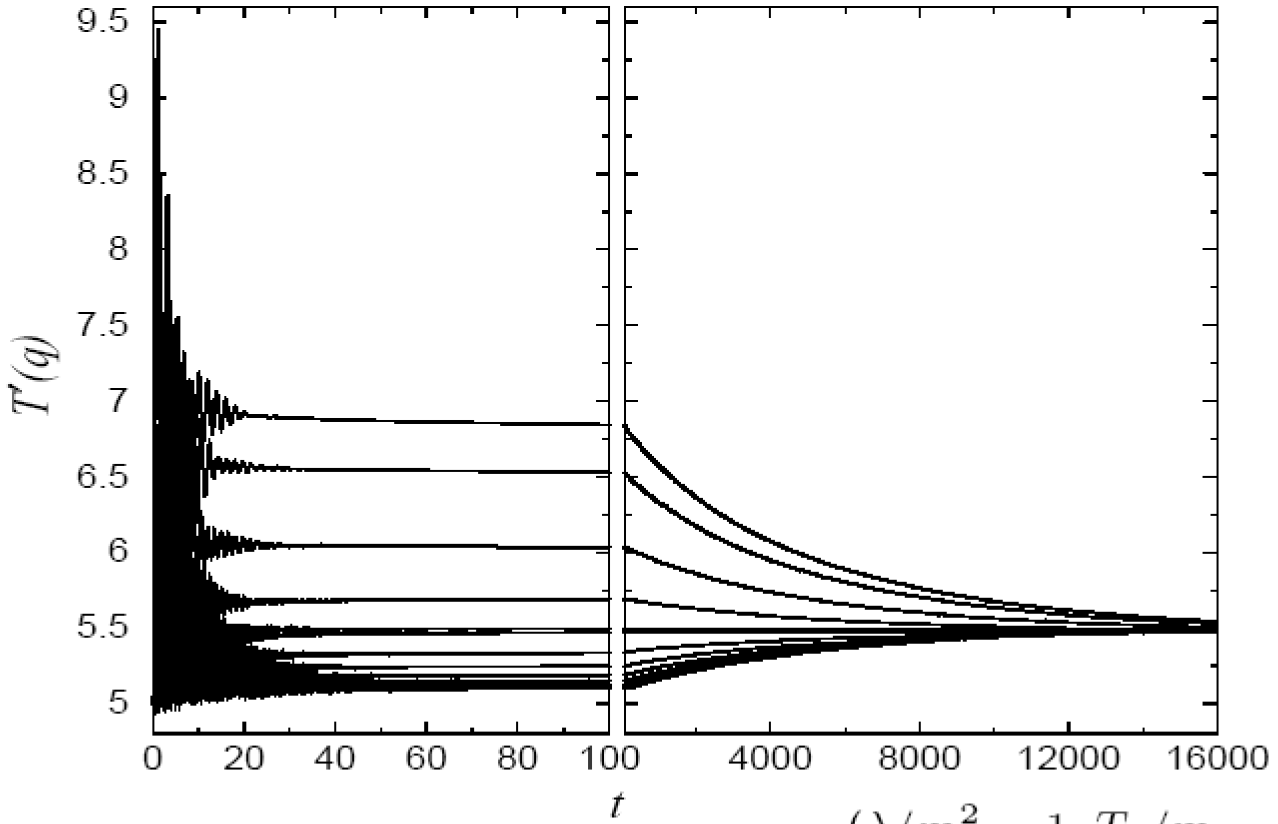


Inverse slope parameter : $T(t,p) \equiv -n(t,\epsilon_p)[n(t,\epsilon_p)+1](dn/d\epsilon_p)^{-1}$

$\Rightarrow$ constant for $n(t,\epsilon_p) = 1/[e^{\epsilon_p/T_{\text{eq}}} - 1]$ (Bose-Einstein)

- Classical theory does (of course) not reach Bose-Einstein distribution
- Typically classical thermalization time $\gg$ quantum thermalization time

Classical thermalization example:



$(\lambda/m^2 = 1,\ T_0/m = 5,\ ma = 0.4)$

Here: classical 2PI *3-loop* approximation J.B., NPA**699** (2002) 847

$\rightsquigarrow$ compares well with exact results! Aarts, Bonini, Wetterich, PRD**63** (2001) 025012

$$T'(t,p) \;=\; \partial_t \partial_{t'} F_{\rm cl}(t,t';p)|_{t=t'}$$

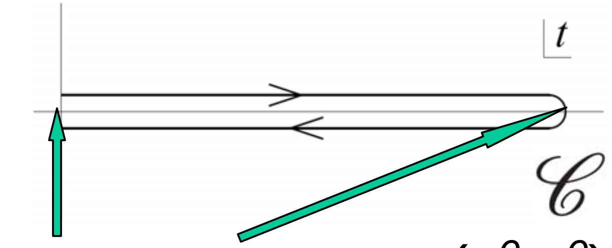
$$\langle \pi_a(t,\mathbf{x})\pi_b(t,\mathbf{y})\rangle_{\rm cl}^{\rm (eq)} = \partial_t \partial_{t'} F_{\rm cl}^{\rm (eq)}(t,t';p)|_{t=t'}\delta_{ab} = T_{\rm cl}\delta(\mathbf{x}-\mathbf{y})\delta_{ab}$$

Simulations of nonequilibrium quantum fields?

- Non-positive definite probability measure

  - note again: no " $i\varepsilon$ " , finite times

$$\langle T_{\mathscr{C}}\Phi(x)\Phi(y)\rangle = \frac{\delta^2 Z[J,R;\rho_D]}{i\delta J(x)\, i\delta J(y)}\Big|_{J=R=0}$$



- Similar to positivity problems at nonzero chemical potential for baryon number in QCD

  - reweighting? Cf. e.g. Fodor & Katz hep-lat/0104001

    very likely not a good idea for nonequilibrium…

    (but: interesting short-time/intermediate-time behavior requires relatively small volumes (physical IR-cutoff, cf. UV): enough statistics?)

Stochastic quantization (complex Langevin equation)

Parisi, Wu '81; …

- in principle very well suited, however, not 'safe'
  (not applied so far to nonequilibrium lattice simulations (?))

<u>Idea</u>

Standard Langevin: $$\frac{\partial\phi}{\partial\tau} = -\frac{\partial S(\phi)}{\partial\phi} + \eta(\tau)$$

with white noise $\langle\eta(\tau)\rangle = 0 \quad , \quad \langle\eta(\tau)\eta(\tau')\rangle = 2\delta(\tau - \tau')$

Stochastic process described by probability distribution $P(\phi,\tau)$,

Fokker-Planck: $$\frac{\partial P(\phi,\tau)}{\partial\tau} = \frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial\phi} + \frac{\partial S(\phi)}{\partial\phi}\right) P(\phi,\tau)$$

$P(\phi,\tau) \;\rightarrow\; e^{-S(\phi)}$ for $\tau \rightarrow \infty$:

$$\langle A\rangle_P = \frac{\int \mathcal{D}\phi A(\phi)P(\phi,\tau)}{\int \mathcal{D}\phi P(\phi,\tau)} \;\rightarrow\; \frac{\int \mathcal{D}\phi A(\phi)e^{-S(\phi)}}{\int \mathcal{D}\phi e^{-S(\phi)}}$$

Consider: $\tau \to it \quad , \quad \phi(\tau) \to \phi(t) \quad , \quad \eta(\tau) \to -i\eta(t)$

$$\frac{\partial\phi}{\partial t} = -i\frac{\partial S(\phi)}{\partial\phi} + \eta(t)\,, \quad \langle\eta(t)\rangle = 0\,, \; \langle\eta(t)\eta(t')\rangle = 2i\delta(t-t')$$

i.e. ϕ, η *complex!*

$$R(\phi,t) \equiv Re\left(i\frac{\partial S}{\partial\phi}\right)\,,\; I(\phi,t) \equiv Im\left(i\frac{\partial S}{\partial\phi}\right)$$

$$\frac{\partial\phi_R}{\partial t} = -iR(\phi,t) + \eta_R(t)\,, \;\; \frac{\partial\phi_I}{\partial t} = -iI(\phi,t) + \eta_I(t)$$

$$\langle\eta_R(t)\rangle = \langle\eta_I(t)\rangle = 0 \qquad (\star)$$

$$\langle\eta_R(t)\eta_R(t')\rangle = \langle\eta_I(t)\eta_I(t')\rangle = (A+B)\delta(t-t')$$

$$\langle\eta_R(t)\eta_I(t')\rangle = \delta(t-t') \qquad , A-B=1\,, \; A>0\,, \; B>0$$

(Define: $\eta \equiv (\eta_1 - \eta_2) + i\,(\eta_1 + \eta_2)$ with η_1, η_2 real: $\langle \eta_1 \rangle = \langle \eta_2 \rangle = 0$, $\langle \eta_1 \eta_1 \rangle = A\,\delta$, $\langle \eta_2 \eta_2 \rangle = B\,\delta$, $\langle \eta_1 \eta_2 \rangle = 0$)

Stochastic process $(\star)$ described by real positive $P(\phi_R,\phi_I,t)$:

$$\frac{\partial P(\phi_R,\phi_I,t)}{\partial t} = \frac{1}{2}\left[(A+B)\frac{\partial^2}{\partial\phi_R^2}+\frac{\partial^2}{\partial\phi_R\partial\phi_I}+(A+B)\frac{\partial^2}{\partial\phi_I^2}\right]P(\phi_R,\phi_I,t)$$
$$+\frac{\partial}{\partial\phi_R}\left[R(\phi_R,\phi_I,t)P(\phi_R,\phi_I,t)\right]+\frac{\partial}{\partial\phi_I}\left[I(\phi_R,\phi_I,t)P(\phi_R,\phi_I,t)\right]$$

Note that averages are given by area integral in the complex plane:

$$\langle A\rangle_P = \frac{\int\mathcal{D}\phi_R\mathcal{D}\phi_I A(\phi_R+i\phi_I)P(\phi_R,\phi_I,t)}{\int\mathcal{D}\phi_R\mathcal{D}\phi_I P(\phi_R,\phi_I,t)} = \frac{\int\mathcal{D}\phi_R A(\phi_R)P_{eff}(\phi_R,t)}{\int\mathcal{D}\phi_R P_{eff}(\phi_R,t)}$$

↑ $\phi_R \to \phi_R - i\phi_I$

with complex $P_{eff}(\phi_R,t)=\int\mathcal{D}\phi_I P(\phi_R-i\phi_I,\phi_I,t)$ governed by *analytic continuation of Fokker-Planck for real variables:*

$$\frac{\partial P_{eff}(\phi_R,t)}{\partial t} = i\frac{\partial}{\partial\phi_R}\left(\frac{\partial}{\partial\phi_R}+\frac{\partial S(\phi)}{\partial\phi_R}\right)P_{eff}(\phi_R,t)$$

➨ Convergence? Reliability? Not well understood so far…

Conclusions

- Nonequilibrium real-time evolution in QFT crucial for wide range of phenomena in particle physics and cosmology

- Loop-, coupling- or 1/N-expansions of 2PI effective action so far uniquely suitable to resolve secularity and universality
  - apparently good convergence properties, even for rather strong couplings
  - excellent agreement of *classical simulations* with *2PI quantum* results for large occupation numbers and not too late times

- Non-positive definite probability measure for *quantum simulations*
  - however: no “ $i\varepsilon$ “ , finite times
  - complex Langevin in principle suitable for nonequilibrium, but not well understood so far

Self-energies from 3-loop 2PI effective action (similarly for NLO $1/N$):

to 3) Quantum $\phi = 0$

$$\Sigma^F(t,t';\mathbf{p}) = -\frac{\lambda^2}{6}\int_{\mathbf{q},\mathbf{k}} F(t,t';\mathbf{p}-\mathbf{q}-\mathbf{k}) \left[F(t,t';\mathbf{q})F(t,t';\mathbf{k}) - \frac{3}{4}\rho(t,t';\mathbf{q})\rho(t,t';\mathbf{k})\right]$$

$$\Sigma^\rho(t,t';\mathbf{p}) = -\frac{\lambda^2}{2}\int_{\mathbf{q},\mathbf{k}} \rho(t,t';\mathbf{p}-\mathbf{q}-\mathbf{k}) \left[F(t,t';\mathbf{q})F(t,t';\mathbf{k}) - \frac{1}{12}\rho(t,t';\mathbf{q})\rho(t,t';\mathbf{k})\right]$$

to 2) Classical

$$\Sigma^F_{\mathrm{cl}}(t,t';\mathbf{p}) = -\frac{\lambda^2}{6}\int_{\mathbf{q},\mathbf{k}} F(t,t';\mathbf{p}-\mathbf{q}-\mathbf{k})F(t,t';\mathbf{q})F(t,t';\mathbf{k})$$

$$\Sigma^\rho_{\mathrm{cl}}(t,t';\mathbf{p}) = -\frac{\lambda^2}{2}\int_{\mathbf{q},\mathbf{k}} \rho(t,t';\mathbf{p}-\mathbf{q}-\mathbf{k})F(t,t';\mathbf{q})F(t,t';\mathbf{k})$$

$$\Sigma^F \;\rightarrow\; \Sigma^F_{\mathrm{cl}} \;=\; \Sigma^F(F^2 \gg \rho^2)$$
$$\Sigma^\rho \;\rightarrow\; \Sigma^\rho_{\mathrm{cl}} \;=\; \Sigma^\rho(F^2 \gg \rho^2)$$

(LO large-N/Hartree approximations have $\Sigma^F = \Sigma^\rho \equiv 0$, i.e. quantum $\equiv$ classical)

$\leadsto$ *sufficient* condition for classical evolution:

$$\boxed{|F(t,t';\mathbf{q})F(t,t';\mathbf{k})| \;\gg\; \frac{3}{4}\,|\rho(t,t';\mathbf{q})\rho(t,t';\mathbf{k})|}$$

Estimate in terms of nonequilibrium 'quasi-particle' number:

$$\overline{F^2}(t,t';\mathbf{p}) \equiv \frac{\omega_{\mathbf{p}}}{2\pi}\int_{t-2\pi/\omega_{\mathbf{p}}}^{t} \mathrm{d}t'\, F^2(t,t';\mathbf{p}) \leadsto \frac{(n_{\mathbf{p}}(t)+1/2)^2}{2\omega^2_{\mathbf{p}}(t)}$$

and for time-averaged spectral function: $\overline{\rho^2}(t,t';\mathbf{p}) \leadsto 1/2\omega^2_{\mathbf{p}}(t)$

$$\leadsto \qquad \left[n_{\mathbf{p}}(t)+\frac{1}{2}\right]^2 \gg \frac{3}{4} \qquad \text{or} \qquad n_{\mathbf{p}}(t) \;\gg\; 0.37$$

2. For the source-dependent matrix element one has in complete analogy to the vacuum/equilibrium case:

$$\langle\varphi^{(2)}|T e^{i\left(\int_x J(x)\Phi(x)+\frac{1}{2}\int_{xy}R(x,y)\Phi(x)\Phi(y)\right)}|\varphi^{(1)}\rangle$$

$$= \int_{\substack{\varphi(0^+,\mathbf{x})=\varphi^{(1)}(\mathbf{x})\\ \varphi(0^-,\mathbf{x})=\varphi^{(2)}(\mathbf{x})}} \mathscr{D}'\varphi\, e^{i\left(S[\varphi]+\int_x J(x)\varphi(x)+\frac{1}{2}\int_{xy}R(x,y)\varphi(x)\varphi(y)\right)}$$

3. In contrast to the equilibrium case the density matrix, $\mathcal{D}(0) \not\sim e^{-\beta H}$, cannot be interpreted as an evolution operator in imaginary time!

Example: the most general *Gaussian* density matrix can be written as

$$\langle\varphi^{(1)}|\mathcal{D}(0)|\varphi^{(2)}\rangle =$$

$$\frac{1}{\sqrt{2\pi\xi^2}}\exp\left\{i\dot{\phi}_0(\varphi^{(1)}-\varphi^{(2)})-\frac{\sigma^2+1}{8\xi^2}\left[(\varphi^{(1)}-\phi_0)^2+(\varphi^{(2)}-\phi_0)^2\right]\right.$$

$$\left.+i\,\frac{\eta}{2\xi}\left[(\varphi^{(1)}-\phi_0)^2-(\varphi^{(2)}-\phi_0)^2\right]+\frac{\sigma^2-1}{4\xi^2}(\varphi^{(1)}-\phi_0)(\varphi^{(2)}-\phi_0)\right\}$$

...where we neglect the spatial dependencies for a moment.

The density matrix is equivalent to the set of *initial conditions:*

$$\phi_0 \equiv \phi(t)_{|t=0} = \mathrm{Tr}\left\{\mathcal{D}(0)\Phi(t)\right\}_{|t=0}$$

$$\dot{\phi}_0 \equiv \partial_t \phi(t)_{|t=0}$$

$$\xi^2 \equiv G(t,t')_{|t=t'=0} = \left[\mathrm{Tr}\left\{\mathcal{D}(0)\Phi(t)\Phi(t')\right\} - \phi(t)\phi(t')\right]_{|t=t'=0}$$

$$\xi\eta \equiv \frac{1}{2}\left[\partial_t G(t,t') + \partial_{t'} G(t,t')\right]_{|t=t'=0}$$

$$\eta^2 + \frac{\sigma^2}{4\xi^2} \equiv \partial_t \partial_{t'} G(t,t')_{|t=t'=0}$$

The equivalence between initial density matrix and initial conditions can be checked explicity:

$$\begin{aligned}
\mathrm{Tr}\mathcal{D}(0) &= \int_{-\infty}^{\infty} \mathrm{d}\varphi \, \langle\varphi|\mathcal{D}(0)|\varphi\rangle \\
&= \frac{1}{\sqrt{2\pi\xi^2}} \int_{-\infty}^{\infty} \mathrm{d}\varphi \exp\Big\{ -\frac{1}{2\xi^2}(\varphi - \phi_0)^2 \Big\} = 1 \\
\mathrm{Tr}\,\{\mathcal{D}(0)\Phi(0)\} &= \frac{1}{\sqrt{2\pi\xi^2}} \int_{-\infty}^{\infty} \mathrm{d}\varphi \, \varphi \exp\Big\{ -\frac{1}{2\xi^2}(\varphi - \phi_0)^2 \Big\} \overset{\varphi\to\varphi+\phi_0}{=} \phi_0
\end{aligned}$$

etc.

Similarly one finds

$$\mathrm{Tr}\,\mathcal{D}^2(0) = \int_{-\infty}^{\infty} \mathrm{d}\varphi \int_{-\infty}^{\infty} \mathrm{d}\varphi' \langle\varphi|\mathcal{D}(0)|\varphi'\rangle\langle\varphi'|\mathcal{D}(0)|\varphi\rangle = \frac{1}{\sigma}$$

$\leadsto$ for $\sigma > 1$ the initial conditions with $\eta^2 + \frac{\sigma^2}{4\xi^2} \equiv \partial_t\partial_{t'}G(t,t')_{|t=t'=0}$ describe a *mixed state*. For $\sigma = 1$ the "mixing term" in $\mathcal{D}(0)$ is absent and one obtains a pure-state density matrix.

HELICS

HEidelberg LInux Cluster System

- 512 AMD Athlon MP processors (~1 Teraflops)
- 2 Gbit Myrinet
- MPI-Standard