Note Title

# S-duality in Vafa-Witten Theory

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Gauge Theory and Langlands duality
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Outline:

I. instanton numbers and discrete electric & magnetic fluxes

II. S-duality for simply laced and non-simply laced gauge groups

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II. S-duality for simply laced and non-simply laced gauge groups

References:

- Miniscule representations, Gauss sum and modular invariance, in: Proc. of 4th Int. Congr. Chinese Mathematicians, 2007 arXiv: 0802.2038 [math.RT]
- S-duality in Vafa-Witten theory for non-simply laced gauge groups, JHEP 05 (2008) 009 arXiv: 0802.2047 [hep-th]

# I. Instanton numbers and discrete fluxes

- Some notations of Lie groups

$G$ compact connected simple Lie group

$\underline{g} = \mathrm{Lie}(G)$ its Lie algebra

$T \subset G$ maximal torus

$\underline{t} = \mathrm{Lie}(T)$ its Lie algebra

$\ell \subset \sqrt{-1}\,\underline{t}$ lattice such that $T = \underline{t}/2\pi\sqrt{-1}\,\ell$.

roots and weights

$\Delta \subset \sqrt{-1}\,\underline{t}^*$ root system

$\Lambda = \mathrm{span}_{\mathbb{Z}} \Delta$ root lattices

$\Delta^\vee = \{\check{\alpha} \mid \alpha \in \Delta\} \subset \sqrt{-1}\,\underline{t}$ coroot system $\check{\alpha} = \frac{2\nu^{-1}(\alpha)}{(\alpha|\alpha)}$

$\Lambda^\vee = \mathrm{span}_{\mathbb{Z}} \check{\Delta}$ coroot lattice

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$(\Lambda^\vee)^* \subset \sqrt{-1}\,\underline{t}^*$ weight lattice

$\Lambda^* \subset \sqrt{-1}\,\underline{t}$ coweight lattice

$$\boxed{\begin{array}{l} \Lambda^\vee \subset l \subset \Lambda^* \subset \sqrt{-1}\,\underline{t} \\ \Lambda \subset l^* \subset (\Lambda^\vee)^* \subset \sqrt{-1}\,\underline{t}^* \end{array}}$$

$\tilde{G}$ universal covering group of $G$

$\mathcal{Z} = Z(\tilde{G}) \cong \Lambda^*/\Lambda^\vee$, its centre

$\pi_1(G) \cong \ell/\Lambda^\vee \subset \mathcal{Z}$, $Z(G) \cong \mathcal{Z}/\pi_1(G) \cong \Lambda^*/\ell$.

$G_{ad} = \tilde{G}/\mathcal{Z}$, $Z(G_{ad}) = 1$, $\pi_1(G_{ad}) \cong \mathcal{Z}$.

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$G_{ad} = \tilde{G}/\mathcal{Z}$, $Z(G_{ad}) = 1$, $\pi_1(G_{ad}) \cong \mathcal{Z}$.

${}^LG$ Langlands dual ${}^L\mathfrak{g} = \mathrm{Lie}({}^LG)$ has root system $\check{\Delta}$.

$Z(\widetilde{{}^LG}) \cong (\Lambda^\vee)^*/\Lambda \cong \mathcal{Z}^*$ i.e. $\mathrm{Hom}(\mathcal{Z}, U(1))$.

$\pi_1({}^LG) \cong \ell^*/\Lambda \cong Z(G)^*$

$Z({}^LG) \cong (\Lambda^\vee)^*/\ell^* \cong \pi_1(G)^*$.

$\underline{g}$ is simply laced if all roots are of the same length.

$\underline{g}$ is non-simply laced if $n_{\underline{g}} = \frac{\|\text{long root}\|^2}{\|\text{short root}\|^2} = 2, 3.$

Choose $(\cdot|\cdot)$ on $\sqrt{-1}\,\underline{t}^*$ such that $\|\text{long root}\|^2 = 2$.

Then the Killing form $K(x,y) = 2\check{h}_{\underline{g}}\,(x|y)$, $x, y \in \sqrt{-1}\,\underline{t}$.

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$h_{\underline{g}}$ Coxeter number

$\check{h}_{\underline{g}}$ dual Coxeter number

If $\underline{g}$ is simply laced, then $h_{\underline{g}} = \check{h}_{\underline{g}}$

In general, $h_{{}^L\underline{g}} = h_{\underline{g}}$, $\check{h}_{\underline{g}} + \check{h}_{{}^L\underline{g}} = (1 + \frac{1}{n_{\underline{g}}})\,h_{\underline{g}}$.

- Instanton numbers and discrete fluxes

$X$ compact smooth orientable 4-manifold

$P \to X$ principal $G$-bundle

characteristic classes of $P$:

$p_1(\mathrm{ad}P) \in H^4(X,\mathbb{Z})$, $\underbrace{w_2(P)}_{\text{discrete flux}} \in H^2(X, \pi_1(G))$.

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$k(P) \overset{\text{def}}{=} -\frac{1}{2h_{\mathfrak{g}}^{\vee}} \langle p_1(\mathrm{ad}P), [X] \rangle$

If $G = \tilde{G}$ is simple connected, then $w_2(P) = 0$ and $k(P) \in \mathbb{Z}$ is the only characteristic number.

Take $G = G_{ad}$, then $w_2(P) \in H^2(X, \mathcal{Z})$.

Recall $\mathcal{Z} = \Lambda^* / \Lambda^\vee \Rightarrow$ long exact sequence

$$\cdots \to H^2(X, \check{\Lambda}) \to H^2(X, \Lambda^*) \to H^2(X, \mathcal{Z}) \to H^3(X, \check{\Lambda}) \to \cdots$$

Assume all elements in $H^2(X, \mathcal{Z})$ can be lifted to $H^2(X, \Lambda^*)$.

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Assume all elements in $H^2(X, \mathcal{Z})$ can be lifted to $H^2(X, \Lambda^*)$.

Denote $\quad v \in H^2(X, \Lambda^*) \longmapsto w_2(P) \in H^2(X, \mathcal{Z})$.

Then $\quad \boxed{k(P) = -\frac{1}{2}(v|v) \mod 1}$

Example $\quad \underline{g} = su(2)$, $v = v_0 \otimes \check{\lambda}_1$, $v_0 \in H^2(X, \mathbb{Z}_2)$, $\check{\lambda}_1$ fund. cowt.

$k(P) = -\frac{1}{4} v_0^2 \mod 1$.

- quantisation of gauge theories

fields $\begin{cases} \text{gauge field} & A = \text{connection on } P \\ \text{matter fields} & \psi = \text{sections of associated bundles.} \end{cases}$

action $S[A,\psi] = \int_X \frac{1}{e^2} F\wedge * F + \frac{\sqrt{-1}\,\theta}{8\pi^2} F\wedge F + \cdots$

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$$= \int_X \frac{\sqrt{-1}}{4\pi}\left[\bar{\tau}\, F^+\wedge F^+ + \tau\, F^-\wedge F^-\right] + \cdots$$

where $\boxed{\tau = \frac{\theta}{2\pi} + \frac{4\pi\sqrt{-1}}{e^2}} \in \mathbb{H}$.

path integral: integrate over $A, \psi$

Do we sum over $k(P)$ and $w_2(P)$ ?

Yes for $k(P)$ $\Leftarrow$ Hamiltonian picture

$$Z_v(\tau) = \sum_{k \in \mathbb{Z} - \frac{1}{2}(v|v)} \frac{1}{\mathrm{vol}(\mathcal{G}_{k,v})} \int_{\substack{k(P)=k \\ w_2(P)=v}} \mathcal{D}A\, \mathcal{D}\psi \; e^{-S[A,\psi]}, \quad v \in H^2(X, \mathbb{Z}).$$

How about $w_2(P) = v$ ?

Yes for $k(P)$ $\Leftarrow$ Hamiltonian picture

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How about $w_2(P) = v$ ? follow Kapustin-Witten

For $X = S^1 \times Y^3$, $H^2(X,\mathbb{Z}) = H^1(Y,\mathbb{Z}) \oplus H^2(Y,\mathbb{Z})$

$$v = (\,a\,,\; m\,)$$

$m$ discrete magnetic flux 't Hooft

$e \in H^1(Y,\mathbb{Z})^* = \mathrm{Hom}(H^1(Y,\mathbb{Z}), U(1))$

discrete electric flux

Canonical quantisation: Sum over $a$ but fix $m$.

$$Z_{e,m}(\tau) = \sum_{a \in H^1(Y,\mathbb{Z})} e(a)\, Z_{a,m}(\tau) \Rightarrow \text{Hilbert space } \mathcal{H}_{e,m}.$$

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If the gauge group $G$ is not $G_{ad}$, any $G$-bundle is a finite cover of $G_{ad}$-bundle

$Z_G(\tau)$ is a sum of some $Z_{e,m}(\tau)$.

Choose $$\boxed{Z_G(\tau) = \sum_{\substack{e|_{H^1(Y,\pi_1(G))}=1 \\ m \in H^2(Y,\pi_1(G))}} Z_{e,m}(\tau)}$$

Justifications:

- admitting a Hilbert space $\mathcal{H}_G = \bigoplus_{e|_{H^1(Y,\pi_1(G))}=1,\; m\in H^2(Y,\pi_1(G))} \mathcal{H}_{e,m}$

discrete Fourier transf $\Downarrow$

- $Z_G(\tau) = |Z(G)|^{-1+b_1(X)} \sum_{v\in H^2(X,\pi_1(G))} Z_v(\tau)$

relativistic : manifestally 4-dimensional

- When $G \longleftrightarrow {}^LG$, $\{e\} \longleftrightarrow \{{}^Lm\}$ $\Rightarrow$ possible
  $\{m\} \longleftrightarrow \{{}^Le\}$. S-duality

## II. S-duality for simply laced and non-simply laced gauge groups.

- Vafa-Witten theory

Some history: 1977 Goddard-Nuyts-Olive

$\{$elec charges in $G$-theory$\} \longleftrightarrow \{$mag. charges in ${}^LG$-theory$\}$

1977 Montonen-Olive

electric-magnetic duality: at qtm level $G$-theory $\cong {}^LG$-theory

Montana-Olive duality cannot be exact without supersymmetry.

Need $N=4$ supersymmetric gauge theory.

twist $\Rightarrow$ topological qft. $\Rightarrow$ mathematics

For $N=2$ gauge theory, $\exists$ one twist

$\Rightarrow$ Donaldson theory, Seiberg-Witten.

For $N=4$ gauge theory, $\exists$ three different twists

twist $\Rightarrow$ topological qft. $\Rightarrow$ mathematics

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For $N=4$ gauge theory, $\exists$ three different twists

Yamron (88) $\begin{cases} 1. \Rightarrow ? \\ 2. \Rightarrow \text{Vafa-Witten theory (1995). Euler number of} \end{cases}$

Donaldson's moduli space of ASD connections

Marcus (95) 3. $\Rightarrow$ geometric Langlands programme

Kapustin-Witten, Gukov-Witten, Frenkel-Witten, ...

Lagrangian of twisted theory, formal path integral, Mathai - Quillen formalism (for tangent bundles)

$\Rightarrow$ the partition function, fixing $v = w_2(P) \in H^2(X, \mathbb{Z})$.

$$Z_v(\tau) = q^{-s} \sum_{k \in \mathbb{Z} - \frac{1}{2}(v|v)} \chi(\overline{\mathcal{M}}_{k,v})\, q^k .$$

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Here: $q = e^{2\pi\sqrt{-1}\tau}$

$\mathcal{M}_{k,v}$ : moduli space of ASD connections

$\chi(\overline{\mathcal{M}}_{k,v})$ : Euler number of certain compactification.

$q^{-s}$ : to ensure S-duality in curved space

$s = a\,\chi(X) + b\,\sigma(X)$.

- S-duality for simply laced gauge groups.

Vafa-Witten's sharpened S-duality conjecture:

$$Z_v\left(-\frac{1}{\tau}\right) = \pm \frac{1}{|Z|^{b_2/2}} \left(\frac{\tau}{\sqrt{-1}}\right)^{w/2} \sum_{u \in H^2(X,\mathbb{Z})} e^{2\pi\sqrt{-1}(u|v)} Z_u(\tau).$$

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Let $\hat{Z}_v(\tau) = \eta(\tau)^{-w} Z_v(\tau)$,

$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)$ Dedekind $\eta$-function.

Then

$$\begin{cases} \hat{Z}_v(\tau+1) = e^{-\pi\sqrt{-1}\frac{c}{12} - \pi\sqrt{-1}(v|v)} \hat{Z}_v(\tau) \\ \hat{Z}_v\left(-\frac{1}{\tau}\right) = \pm \frac{1}{|Z|^{b_2/2}} \sum_{u \in H^2(X,\mathbb{Z})} e^{2\pi\sqrt{-1}(u|v)} \hat{Z}_u(\tau). \end{cases}$$

$$c = 24s + w = a'\chi(X) + b'\sigma(X).$$

The constant $c = r\,\chi(X)$ and $\pm = (-1)^{\frac{r(\chi+\sigma)}{2}}$ are determined by

- having a representation of $\Gamma = SL(2,\mathbb{Z})$

  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \tau \mapsto \tau+1$, $\quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \tau \mapsto -\frac{1}{\tau}$

  $S^4 = 1$, $\quad S^2 = (ST)^3 \in Z(\Gamma)$ should be satisfied by

  the matrices $\mathbb{T}_{uv} = e^{-\pi\sqrt{-1}\frac{c}{12} - \pi\sqrt{-1}(v|v)}\,\delta_{uv}$

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- known mathematical data when $X = K3$, $G = SU(2)$ or $SO(3)$.

  $\Rightarrow Z_{SU(2)}(\tau) = \frac{1}{8} G(q^2) + \frac{1}{4} G(q^{1/2}) + \frac{1}{4} G(-q^{1/2})$

  $Z_{SO(3)}(\tau) = \frac{1}{4} G(q^2) + 2^{21} G(q^{1/2}) + 2^{10} G(-q^{1/2})$, $\quad G(q) = \frac{1}{\eta(\tau)^{24}}$.

Further evidence:

- implies the original Montonen-Olive duality.

  If $\hat{Z}_G(\tau) = |Z(G)|^{-1+b_1} \sum_{v \in H^2(X, \pi_1(G))} \hat{Z}_v(\tau)$

  then $\hat{Z}_G(-\frac{1}{\tau}) = \pm |Z(G)|^{-\chi/2} |Z({}^LG)|^{\chi/2} \hat{Z}_{{}^LG}(\tau)$.

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  then $\hat{Z}_G(-\frac{1}{\tau}) = \pm |Z(G)|^{-\chi/2} |Z({}^LG)|^{\chi/2} \hat{Z}_{{}^LG}(\tau)$.

- physics calculation of $Z(\tau)$ when $G = SU(2)$ or $SO(3)$

  $X$ is Kähler, $\exists\, \omega \in H^{2,0}(X)$. $\omega \not\equiv 0$. holomorphic 2-form

  so that $\omega^{-1}(0)$ = disjoint union of Riemann surfaces.

  (add mass supersymmetrically, use mass gap.)

- blow up formula

$\tilde{X} = X \# \overline{\mathbb{C}P^2}$ $\quad H^2(\tilde{X},\mathbb{Z}) = H^2(X,\mathbb{Z}) \oplus H^2(\overline{\mathbb{C}P^2},\mathbb{Z})$

alg. surface $\qquad \tilde{v} = (v,\ a \otimes e)$ $\quad$ exc. div. $\quad e^2 = -1$.

factorisation: $\hat{Z}_{\tilde{X},\tilde{v}}(\tau) = \hat{Z}_{X,v}(\tau)\,\hat{Z}_a(\tau)$

Yoshioka (94), Li-Qin (99) if $G = SU(2)$ or $SO(3)$.

X projective, $a=0$ $\qquad$ general

- blow up formula

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Yoshioka (94), Li-Qin (99) if $G = SU(2)$ or $SO(3)$.

The repr. of $SL(2,\mathbb{Z})$ on $\{\tilde{v}\}$ is the tensor product of the repr's on $\{v\}$ and on $\{a\}$.

$\hat{Z}_a(\tau)$ does not depend on $X$, but its transformation under $SL(2,\mathbb{Z})$ gives evidence of S-duality.

In fact $\hat{Z}_a(\tau) = \eta(\tau)^{-r} \sum_{\xi \in \Lambda + a} e^{\pi\sqrt{-1}(\xi|\xi)\tau}$ $\left[\begin{array}{l}\text{Li-Qin when } G=SO(3) \\ \text{Kapranov when } a=0\end{array}\right]$

transform under $SL(2,\mathbb{Z}) \Rightarrow$ consistency relation.

$$\boxed{\frac{1}{\sqrt{|Z|}} \sum_{\substack{\mu \text{ miniscule} \\ \text{or } 0 \text{ weight}}} e^{\pi\sqrt{-1}(\mu|\mu)} = e^{\frac{\pi\sqrt{-1}\, r}{4}}}.$$

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e.g. for $G = SU(n)/\mathbb{Z}_n$, $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{\pi\sqrt{-1}\frac{k(n-k)}{n}} = e^{\frac{\pi\sqrt{-1}}{4}(n-1)}$.

Compare Gauß sum $\sum_{k=0}^{n-1} e^{2\pi\sqrt{-1}\frac{k^2}{n}} = \sqrt{n}\, \frac{1+(-\sqrt{-1})^n}{1-\sqrt{-1}}$.

quadratic reciprocity $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{4}}$, $p \neq q$ odd prime.

For details, See arXiv: 0802.2038 [math.RT].

- S-duality for non-simply laced gauge groups

| $\underline{g}$ | $B_r$ | $C_r$ | $F_4$ | $G_2$ |
|---|---|---|---|---|
| $n_{\underline{g}}$ | 2 | 2 | 2 | 3 |
| $h_{\underline{g}}$ | $2r$ | $2r$ | 12 | 6 |
| $\check{h}_{\underline{g}}$ | $2r-1$ | $r+1$ | 9 | 4 |
| $Z$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 1 | 1 |

The lattices $\quad {}^L\Lambda = \frac{1}{\sqrt{n_{\underline{g}}}}\check{\Lambda}$, $({}^L\Lambda)^\vee = \sqrt{n_{\underline{g}}}\,\Lambda$, $(({}^L\Lambda)^\vee)^* = \frac{1}{\sqrt{n_{\underline{g}}}}\Lambda^*$, $({}^L\Lambda)^* = \sqrt{n_{\underline{g}}}\,\check{\Lambda}^*$.

to ensure $\|\text{long root}\|^2 = 2$.

The blow up factor is related to

Kapranov, when $a=0$

$$\hat{\vartheta}_a(\tau) = \eta(\tau)^{-r_{long}}\,\eta(n_{\underline{g}}\tau)^{-r_{short}} \sum_{x\in\check{\Lambda}+a} e^{\pi\sqrt{-1}(x|x)\tau}, \quad a\in\Lambda^*/\check{\Lambda}$$

$$\hat{\vartheta}_\alpha(\tau) = \eta(\tau)^{-r_{short}}\,\eta(n_{\underline{g}}\tau)^{-r_{long}} \sum_{\xi\in({}^L\Lambda)^\vee+\alpha} e^{\pi\sqrt{-1}(\xi|\xi)\tau}, \quad \alpha\in({}^L\Lambda)^*/({}^L\Lambda)^\vee$$

$r_{\substack{long\\short}}$ = number of $\substack{\text{long}\\\text{short}}$ simple roots.

The blow up factor is related to Kapranov, when $a=0$

$$\hat{\vartheta}_a(\tau) = \eta(\tau)^{-r_{\mathrm{long}}}\,\eta(n_{\mathfrak{g}}\tau)^{-r_{\mathrm{short}}} \sum_{x\in\Lambda^\vee+a} e^{\pi\sqrt{-1}(x|x)\tau}, \quad a\in\Lambda^*/\Lambda^\vee$$

$$\hat{\vartheta}_\alpha(\tau) = \eta(\tau)^{-r_{\mathrm{short}}}\,\eta(n_{\mathfrak{g}}\tau)^{-r_{\mathrm{long}}} \sum_{\xi\in({}^L\Lambda)^\vee+\alpha} e^{\pi\sqrt{-1}(\xi|\xi)\tau}, \quad \alpha\in({}^L\Lambda)^*/({}^L\Lambda)^\vee$$

$r_{\substack{\mathrm{long}\\\mathrm{short}}}$ = number of $\substack{\text{long}\\\text{short}}$ simple roots.

They transform under $T=\begin{pmatrix}1&1\\0&1\end{pmatrix}: \tau\mapsto\tau+1$

$S=\begin{pmatrix} & -\frac{1}{\sqrt{n_{\mathfrak{g}}}}\\ \sqrt{n_{\mathfrak{g}}} & \end{pmatrix}: \tau\mapsto -\frac{1}{n_{\mathfrak{g}}\tau}$.

explicit formula known

S, T generate the Hecke group $G(\sqrt{n_{\mathfrak{g}}})$

relations: $S^4=1$, $S^2=(ST)^{2n_{\mathfrak{g}}}$

$$\{\hat{\vartheta}_a(\tau)\} \overset{S}{\rightleftarrows} \{\hat{\vartheta}_\alpha(\tau)\}$$

$T$ $\circlearrowleft$ $\{\hat{\vartheta}_a(\tau)\}$, $T$ $\circlearrowleft$ $\{\hat{\vartheta}_\alpha(\tau)\}$

$S, T$ generate the Hecke group $G(\sqrt{n_g})$

relations: $S^4 = 1$, $S^2 = (ST)^{2n_g}$

$$\begin{tikzcd} \{\hat{\vartheta}_a(\tau)\} \arrow[r, bend left, "S"] \arrow[loop below, "T"] & \{\hat{\vartheta}_\alpha(\tau)\} \arrow[l, bend left] \arrow[loop below, "T"] \end{tikzcd}$$

Compare
$$\begin{tikzcd} \{\hat{Z}_v(\tau)\} \arrow[r, bend left, "S"] \arrow[loop below, "T"] & \{\hat{Z}_\mu(\tau)\} \arrow[l, bend left] \arrow[loop below, "T"] \end{tikzcd}$$

Factorisation $\hat{Z}_{\tilde{X},\tilde{v}}(\tau) = \hat{Z}_{X,v}(\tau)\,\hat{\vartheta}_a(\tau)$

suggests how $\hat{Z}_{X,v}(\tau)$ should transform.

The modular properties of $\hat{\vartheta}_a(\tau)$, $\hat{\vartheta}_\alpha(\tau)$ ( $a \in \Lambda^*/\check{\Lambda} \overset{\cong Z}{}$, $\alpha \in ({}^L\Lambda)^*/({}^L\Lambda)^\vee \overset{\cong {}^LZ}{}$ )

can be obtained explicitly by Poisson summation:

$$\begin{cases} \hat{\vartheta}_a(\tau+1) = e^{-\frac{\pi\sqrt{-1}}{12} n_g \frac{\check{h}({}^L\mathfrak{g})}{h(\mathfrak{g})} r + \pi\sqrt{-1}(a|a)}\, \hat{\vartheta}_a(\tau), \\ \hat{\vartheta}_\alpha(\tau+1) = e^{-\frac{\pi\sqrt{-1}}{12} n_g \frac{\check{h}(\mathfrak{g})}{h(\mathfrak{g})} r + \pi\sqrt{-1}(\alpha|\alpha)}\, \hat{\vartheta}_\alpha(\tau), \\ \hat{\vartheta}_a\left(-\frac{1}{n_g\tau}\right) = \frac{1}{|Z|^{1/2}} \sum_{\alpha \in {}^LZ} e^{-\frac{2\pi\sqrt{-1}}{\sqrt{n_g}}\langle\alpha, a\rangle}\, \hat{\vartheta}_\alpha(\tau), \\ \hat{\vartheta}_\alpha\left(-\frac{1}{n_g\tau}\right) = \frac{1}{|Z|^{1/2}} \sum_{a \in Z} e^{-\frac{2\pi\sqrt{-1}}{\sqrt{n_g}}\langle\alpha, a\rangle}\, \hat{\vartheta}_a(\tau). \end{cases}$$

new phase factors

arXiv: 0802.2038 [math.RT]

We propose a modification of Vafa-Witten conjecture when the gauge group is non-simply laced.

$$\begin{cases} \hat{Z}_\nu(\tau+1) = e^{-\frac{\pi\sqrt{-1}}{12} n_{\mathfrak{g}} \frac{h^\vee({}^L\mathfrak{g})}{h(\mathfrak{g})} r\chi - \pi\sqrt{-1}(\nu|\nu)} \hat{Z}_\nu(\tau) \\ \hat{Z}_\nu\left(-\frac{1}{n_{\mathfrak{g}}\tau}\right) = \frac{1}{|Z|^{b_2/2}} \sum_{\mu\in H^2(X,{}^LZ)} e^{\frac{2\pi\sqrt{-1}\langle\mu|\nu\rangle}{\sqrt{n_{\mathfrak{g}}}}} \hat{Z}_\mu(\tau) \\ \hat{Z}_\mu(\tau+1) = e^{-\frac{\pi\sqrt{-1}}{12} n_{\mathfrak{g}} \frac{h^\vee(\mathfrak{g})}{h(\mathfrak{g})} r\chi - \pi\sqrt{-1}(\mu|\mu)} \hat{Z}_\mu(\tau) \\ \hat{Z}_\mu\left(-\frac{1}{n_{\mathfrak{g}}\tau}\right) = \frac{1}{|Z|^{b_2/2}} \sum_{\nu\in H^2(X,Z)} e^{\frac{2\pi\sqrt{-1}\langle\mu|\nu\rangle}{\sqrt{n_{\mathfrak{g}}}}} \hat{Z}_\nu(\tau) \end{cases}$$

new phase factors

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$\Rightarrow$ original Montonen-Olive duality $Z_G\left(-\frac{1}{n_{\mathfrak{g}}\tau}\right) \sim Z_{{}^LG}(\tau)$

Check that it is indeed a representation of $G(\sqrt{n_{\mathfrak{g}}})$.

The contribution of the new phases

$$\left(e^{-\frac{\pi\sqrt{-1}}{12} n_{\mathfrak{g}} \frac{h^\vee({}^L\mathfrak{g})}{h(\mathfrak{g})} r\chi} e^{-\frac{\pi\sqrt{-1}}{12} n_{\mathfrak{g}} \frac{h^\vee(\mathfrak{g})}{h(\mathfrak{g})} r\chi}\right)^{n_{\mathfrak{g}}} = e^{-\frac{\pi\sqrt{-1}}{12} n_{\mathfrak{g}}(n_{\mathfrak{g}}+1) r\chi}.$$

Check that it is indeed a representation of $G(\Gamma_{n_{\mathfrak{g}}})$.

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For $F_4$, $Z=1$. $(ST)^4 = e^{-\frac{\pi\sqrt{-1}}{12} 2\cdot 3\cdot 4\chi} = 1$

$G_2$, $Z=1$. $(ST)^6 = e^{-\frac{\pi\sqrt{-1}}{12} 3\cdot 4\cdot 2\chi} = 1$

Check that it is indeed a representation of $G(\Gamma_{n_{\mathfrak{g}}})$.

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using Wu formula $x^2 = w_2 x \bmod 2$

For $B_r$ or $C_r$, $(ST)^2_{xy} = e^{-\frac{\pi\sqrt{-1}}{2} r\chi - \pi\sqrt{-1}\frac{r}{2} x^2}\, \delta_{x+y,\, w_2(X)}$. $\quad x, y \in H^2(X, \mathbb{Z}_2)$

$$(ST)^4_{xy} = e^{-\frac{\pi\sqrt{-1}}{2} r(\chi + w_2(X)^2)}\, \delta_{xy} = e^{-\frac{\pi\sqrt{-1}\, r}{2}(\chi+\sigma)}\, \delta_{xy} = \delta_{xy}$$

$\sigma = w_2(X)^2 \bmod 8$ $\qquad$ if $2 \mid \dim(\mathcal{M})$

mod 4 is enough $\qquad = -2p_1 - \dim G\, \frac{\chi+\sigma}{2}$

van der Blij, Rokhlin, anal. pf. by W. Zhang.

The blow-up formulae depend also on the $s$ in

$$Z_v(\tau) = q^{-s} \sum_{k\in\mathbb{Z}-\frac{1}{2}(v,v)} \chi(\overline{\mathcal{M}}_{k,v})\, q^k .$$

$s$ can not be fixed by $S$-duality.

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$s$ can not be fixed by $S$-duality.

One possible solution: $\tilde{v} = (v, \alpha\otimes e)$, $\tilde{\mu} = (\mu, \alpha\otimes e)$.

$$\sum_{k\in\mathbb{Z}-\frac{1}{2}(\tilde{v}|\tilde{v})} \chi(\overline{\mathcal{M}_{k,\tilde{v}}(\tilde{X})})\, q^k = \left(\frac{q^{(r_{long}+n_g r_{short})/24}}{\eta(\tau)^{r_{long}}\,\eta(n_g\tau)^{r_{short}}}\right)^{1+\frac{1}{r}} \vartheta_a(\tau) \sum_{k\in\mathbb{Z}-\frac{1}{2}(v|v)} \chi(\overline{\mathcal{M}_{k,v}(X)})\, q^k$$

$$\sum_{k\in\mathbb{Z}-\frac{1}{2}(\tilde{\mu}|\tilde{\mu})} \chi(\overline{\mathcal{M}_{k,\tilde{\mu}}(\tilde{X})})\, q^k = \left(\frac{q^{(r_{short}+n_g r_{long})/24}}{\eta(\tau)^{r_{short}}\,\eta(n_g\tau)^{r_{long}}}\right)^{1+\frac{1}{r}} \vartheta_\alpha(\tau) \sum_{k\in\mathbb{Z}-\frac{1}{2}(\mu|\mu)} \chi(\overline{\mathcal{M}_{k,\mu}(X)})\, q^k .$$

fractional power $\Rightarrow$ more singular $\overline{\mathcal{M}}$ ?