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Stability conditions and Stokes matrices

(with Tom Bridgeland)

Joyce's holomorphic generating fns for CY_3 $\leftrightsquigarrow$ Stokes phenomena on $\mathbb{P}^1$

stability conditions on triang cat $\leftrightsquigarrow$ Stokes

§1. Stability conditions

M. Douglas, Π stability
T. Bridgeland

$\mathcal{D}$ - triangulated cat.

Def A central charge on $\mathcal{D}$ is a hom.

$$Z : K(\mathcal{D}) \to \mathbb{C}.$$

Ex ① $M = (M, I, \beta + i\omega)$ CY_3

$$\mathcal{D} = \mathcal{D}^b \mathrm{Fuk}(M, \beta + i\omega)$$

Central charge : $Z_I(L) = \int_L \Omega_I$

of phase ϕ

L special Lag if

$$\Omega|_L = e^{i\pi\phi}\, \mathrm{vol}_L$$

$$\Rightarrow Z_I(L) = e^{i\pi\phi} \int_L \mathrm{vol}_L \in e^{i\pi\phi}\mathbb{R}_{>0}$$

$\forall \phi \in \mathbb{R}$, $P(\phi) \subset \mathcal{D}$ subcat of SLAG of phase ϕ.

$\shortparallel$

\mathcal P

(ii) X smooth proj. curve $\mathbb{C}$

$$\mathcal{D} = \mathcal{D}\mathrm{coh}(X)$$

$$Z([E]) = -\deg E + i \operatorname{rank} E$$

$$E : \text{slope } \phi(E) := \frac{1}{\pi}\arg Z(E)$$

$\forall \phi$, $P(\phi) \subset \mathcal{D}$.

<u>Def</u> (Bridgeland)

A stability condition on $\mathcal{D}$ consists of

(I) a central charge $Z : K(\mathcal{D}) \to \mathbb{C}$

(II) $\forall \phi \in \mathbb{R}$ a full abelian subc. $P(\phi) \subset \mathcal{D}$

s.t.

(I) $E \in P(\phi)$, $Z(E) = e^{i\pi\phi} m(E)$, $m(E) > 0$, <u>mass</u>

(II) $P(\phi + 1) = P(\phi)[1]$

(III) $\phi_1 > \phi_2$, $E_j \in \mathcal{P}(\phi_j)$, $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$

(IV) Harder-Narasimhan cond:

$$\forall\, 0 \neq E \in \mathcal{D} \;\; \exists\, \phi_1 > \cdots > \phi_n \text{ and}$$

a sequence of triangles

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_n \cong E \arrow[d] \\
& A_1 \arrow[ul, dashed, "{[1]}"] & & A_n \arrow[ul, dashed, "{[1]}"]
\end{tikzcd}$$

$$A_j \in \mathcal{P}(\phi_j)$$

$\mathcal{P}(\phi) =$ semistable obj of phase ϕ

<u>Varying Z</u>:

$$\begin{tikzcd}
\mathrm{Stab}(\mathcal{D}) = \{(Z, \mathcal{P}) \mid \mathcal{P} \text{ loc-finite}\} \arrow[d, "Z"] \\
\mathrm{Hom}(K(\mathcal{D}), \mathbb{C})
\end{tikzcd}$$

<u>Thm</u> (Bridgeland)

For each conn. cpt. $\Sigma \subset \mathrm{Stab}(\mathcal{D})$

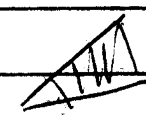
$\exists$ a linear subspace $V(\Sigma) \subset \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$

with a well-def linear top. s.t.

$$\begin{array}{l} \Sigma \subset \mathrm{Stab}(\mathcal{D}) \\ \downarrow Z \qquad\qquad \text{is a loc. hom.} \\ V \subset \mathrm{Hom}(K(\mathcal{D}), \mathbb{C}) \end{array}$$

<u>Slogan</u>
deformation of the central charge lift (uniquely) to defs of stability condition.

<u>Cor/Proof</u>: $\Sigma \subset \mathbb{C}$ convex cone centered at 0.


Σ

$$Z \to Z'$$

$$\mathcal{P}_Z(\Sigma) = \bigvee_{\phi : e^{i\pi\phi} \subset \Sigma} \mathcal{P}_Z(\phi) = \bigvee_{\phi : e^{i\pi\phi} \subset \Sigma'} \mathcal{P}_{Z'}(\phi)$$

if no line crosses Σ

<u>An alternative defn</u>

<u>Def</u>: a stability fn on an <u>abelian</u> cat $\mathcal{A}$ is a hom

$$\begin{array}{l} Z : K(\mathcal{A}) \longrightarrow \mathbb{C} \\ \quad \cup\!\mid \\ K_{>0}(\mathcal{A}) \longrightarrow \mathbb{H} = \{ z \mid \operatorname{Im} z > 0 \} \cup \mathbb{R}^*_- \end{array}$$

Def

(i) $E \neq 0 \in \mathcal{A}$, phase$(E) = \phi(E)$

$$= \frac{1}{\pi} \arg Z(E) \in (0,1]$$

(ii) $E \in \mathcal{A}$ semistable if $0 \neq F \subset E$

$$\phi(F) \leq \phi(E)$$

(iii) Z has the HN prop if $\forall\, 0 \neq E \in \mathcal{A}$,

$$0 = E_0 \subset E_1 \cdots \subset E_n = E$$

s.t. $F_i = E_i / E_{i+1}$ are semistable

$$\phi(F_i) > \phi(F_{i+1})$$

Def A stab cond. in $\mathcal{A}$ is given by a stab fn. for which the HN prop. holds

Thm (Bridgeland)
Giving a stab. cond on $\mathcal{D}$ is equiv to giving
(i) a bounded t-structure on $\mathcal{D}$ with heart $\mathcal{A}$
(ii) a stability condition in $\mathcal{A}$.

Sketch

$(Z, \mathcal{P}) \longrightarrow (\mathcal{A}, Z)$

- $\mathcal{A} = \mathcal{P}((0,1])$
- $Z = Z|_{\mathcal{A}}$

$(\mathcal{A}, Z) \to (Z, \mathcal{P})$

$$\mathcal{P}(\phi) = \begin{cases} \text{semistable objs of } \mathcal{A} \text{ of phase } \phi & \phi \in (0,1] \\ \mathcal{P}(\phi - [\phi])[\phi] & \text{if not} \end{cases}$$

Linearize the axioms - Function theory 101

$\mathcal{A}$ ab. cat.

$\mathcal{M}$ = stack of objects of $\mathcal{A}$

E.g. $\mathcal{A} = Rep(R)$, R f.d. alg/$\mathbb{C}$.

$$\mathcal{M} = \coprod_{d \geq 0} Rep_d(R)/GL_d(\mathbb{C})$$

$\mathcal{H}(\mathcal{A})$ = Hall alg. of $\mathcal{A}$

$:=$ {constructible fns on $\mathcal{M}$}.

product corr.

$$\mathcal{M}^{\times 3} \supset \mathcal{M}^{(3)} = \{0 \to A \to C \to B \to 0\}$$

$$\begin{tikzcd} & \mathcal{M} \arrow[dl] \arrow[d] \arrow[dr] & \\ \mathcal{M} & \mathcal{M} & \mathcal{M} \end{tikzcd}$$

product

$$f * g(C) = \int_{0 \to A \to C \to B \to 0} f(A)\, g(B) = \sum f(A) g(B)$$

$$= \int_{0 \subset A \subset C} f(A)\, g(C/A)$$

identity elt: $1_0 =$ char function of 0

More generally,

$$f_1 * \cdots * f_n(c) = \int_{0 \subseteq C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n = C} f_1(C_1/C_0) \cdots f_n(C_n/C_{n-1})$$

<u>Remarks</u>

- $\mathbb{1}_{\mathcal{A}} =$ char fn of <u>all</u> objects in $\mathcal{A}$
- $\forall\, l = e^{i\pi\phi}\mathbb{R}_{>0} \subseteq \mathbb{H}$,

  $\mathbb{S}_l =$ char. fn. of all semistable of phase ϕ

<u>Obs</u> (Reineke)

$$\text{HN prop} \equiv \mathbb{1}_{\mathcal{A}} = \overrightarrow{\prod_{l \in \mathbb{H}}} \mathbb{S}_l$$

Coproduct: $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$

$$\Delta f(M,N) = f(M \oplus N)$$

Δ cocomm $\Rightarrow$ $\mathcal{H} = U\mathfrak{n}$, $\mathfrak{n}$ Lie alg (pro nilpotent)

<u>Ex</u>: $\Delta(\mathbb{S}_l) = \mathbb{S}_l \otimes \mathbb{S}_l$.

<u>Stokes data</u>:

Birkhoff, Balser-Jurkat-Lutz, Boalch
(Lin) (G reductive/$\mathbb{C}$)

G red/$\mathbb{C}$

$\cup$

H max. torus

P = principal trivial G-bundle on $\mathbb{P}^1$

$$\nabla = d - \left(\frac{Z}{t^2} + \frac{f}{t}\right) dt$$

$$Z \in \mathfrak{h}_{reg} = \mathfrak{h} \setminus \bigcup_{\alpha} \ker(\alpha)$$

$$f \in \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

<u>Thm</u> $\exists!$ fund. soln to ∇ of form

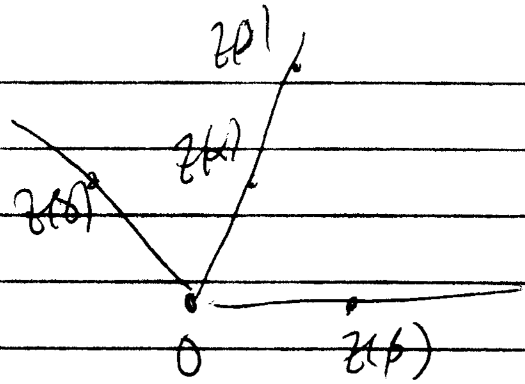
$$\phi = F e^{-Z/t}$$

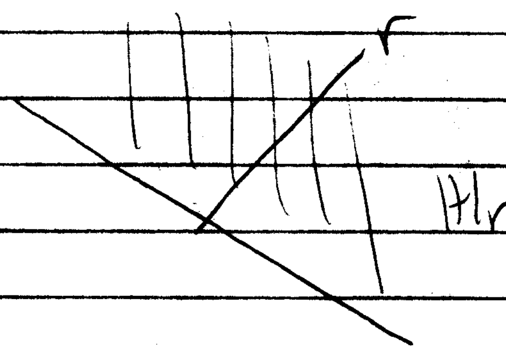
$$F \in G[[t]], \quad F(0) = 1.$$

<u>Pb</u> radius of convergence of $F = 0$.

<u>Defn</u> the Stokes rays of ∇ are the rays

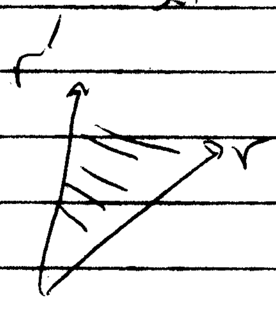
$$\mathbb{R}_{>0} Z(\alpha) \in \mathbb{C}^*$$





Thm If r is not a Stokes ray, $\exists !$ holo fn. soln ~~…~~ $\Phi_r : \mathbb{H}_r \to G$

s.t. $\Phi_r e^{Z/t} \to 1$ as $t \to 0$ in $\mathbb{H}_r$.



$$\Phi_r = \Phi_{r'} S_{\Sigma} \in G.$$

Prop $S_\Sigma = \exp(\mathcal{E}_\Sigma)$, $\mathcal{E}_\Sigma \in \bigoplus_{\alpha : Z(\alpha) \in \Sigma} \mathfrak{g}_\alpha$

Cor

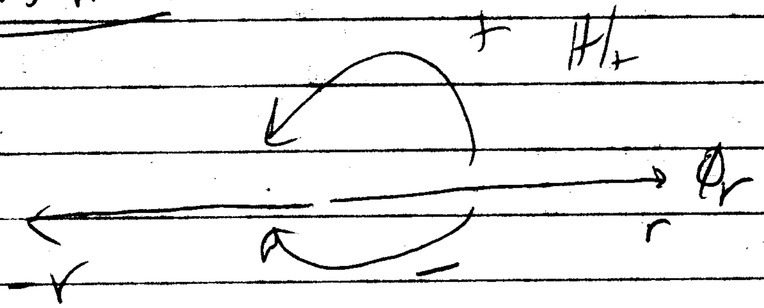
(i) If Σ does not contain Stokes rays, then $S_\Sigma = 1$.

(ii) If Σ contains a single Stokes ray ℓ, then

$$S_\ell := S_\Sigma = \exp(\mathcal{E}_\ell), \quad \mathcal{E}_\ell \in \bigoplus_{\alpha;\, Z(\alpha) \in \ell} \mathfrak{g}_\alpha$$

S_ℓ = Stokes factor corr. to ℓ.

Stokes matrices



$$\Phi_r = \Phi_{-r} \cdot S_\pm$$

$\nwarrow \in G$, Stokes matrices

$$S_+ = \overrightarrow{\prod_{\ell \in \mathbb{H}_+}} S_\ell$$

Thm The following are equiv. As Z varies (and f too)

(i) (S_+, S_-) do not change so long as r is not a Stokes ray

(ii) S_Σ does not change so long as Stokes rays do not cross $\partial\Sigma$

(iii) $df_\alpha = \sum_{\beta+\gamma=\alpha} [f_\beta, f_\gamma]\, d\log\left(\frac{\beta}{\gamma}\right)$

Dictionary

~~Hall(A)~~

Stab	Hall	ODE
$(\mathcal{A}, Z)$	$(\mathbb{1}_{\mathcal{A}}, Z)$	Stokes matrices S_+
(P, Z)	(SS_ℓ, Z)	Stokes factors S_ℓ
HN	$\mathbb{1}_A = \overset{\curvearrowright}{\prod_{\ell \in H}} SS_\ell$	$S_+ = \overset{\rightarrow}{\prod_{\ell \in H}} S_\ell$
Reineke inversion	$SS_\ell = F(\mathbb{1}_{\mathcal{A}})$	new!
Wall-crossing	$\overset{\curvearrowright}{\prod} SS_\ell^{Z} = \overset{\curvearrowright}{\prod} SS_\ell^{Z'}$	Isomonodromy

An application to Joyce's work

$\mathcal{A}$ a "good" abelian cat: $\mathrm{Rep}(R)$ ✓
$\mathrm{Coh}(X)$ $\ddot{\frown}$

Z stab function on $\mathcal{A}$

$\in \mathrm{Hom}(K(\mathcal{A}), \mathbb{C}) = \mathfrak{h}$

$\mathcal{H}$ = Hall alg of $\mathcal{A}$ = $U\mathfrak{n} \supset \mathfrak{h}$

B = pro-solvable group with Lie alg. $\mathfrak{n} \rtimes \mathfrak{h}$

(eg $\mathcal{A} = \mathrm{Rep}(Q)$, Q Dynkin quiver $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

Ringel: $H(\mathcal{A}) = U\mathfrak{n}_+$

$B = B_+$

$\nabla_{\mathcal{A},Z} = d - \left(\frac{Z}{t^2} + \frac{f}{t}\right)$ conn. on principal B-bundle over $\mathbb{P}^1$

$f \in \mathfrak{n}$

Thm (BTL)

The following are equivalent

(I) f is given by Joyce's gen. fns:

$$f_\alpha = \sum_{n \geq 1} \sum_{\alpha_1 + \dots + \alpha_n = \alpha} J(Z(\alpha_1), \dots, Z(\alpha_n))\, \epsilon_{\alpha_1} * \dots * \epsilon_{\alpha_n}$$

$\alpha \in K_{>0}(\mathcal{A})$

$\epsilon_\alpha \in H(\mathcal{A}) \rightsquigarrow$ char. fn. of Z-semistable of ~~class~~ class α

$J : (\mathbb{C}^*)^n \to \mathbb{C}$.

(II) The Stokes matrices of $\nabla_{\mathcal{A},Z}$ are

$S_+ = 1_{\mathcal{A}}$ ideally $S_- = 1_{\mathcal{A}[1]}$

$S_- = 1_0$

(III) The Stokes factors S_ℓ of $\nabla_{\mathcal{A},Z}$ is

S_ℓ = char. fn. of semistable with phase in ℓ

Cor $\mathcal{A}$, Z vari in $\mathrm{Stab}(\mathcal{A})$

$\nabla_{\mathcal{A},Z}$ is isomonodromic

$$\Downarrow$$

$$df_\alpha = \sum_{\beta+\delta=\alpha} [f_\beta, f_\delta]\, d\log\left(\frac{\beta}{\delta}\right)$$

For $G = GL_n$, these eqns written by Jimbo - Miwa - Ueno
reductive G - - - - - - - - - - - - Boalch.

G-Moore et al: BPS states in gauge quantum theory

$\xleftrightarrow[\text{Denef}]{}$ BPS states in quiver QM

$\longleftrightarrow$ above, for $\mathcal{A} = \mathrm{Rep}(Q)$, Q Pyrkin quivers

$$\mathcal{H}_q(\mathcal{A}) \xrightarrow{\ \sim\ } \mathbb{C}_q[T] \quad \downarrow\text{"}q\to 1\text{"}$$

$$\mathcal{H}(\mathcal{A}) \dashrightarrow \mathrm{SDiff}(\mathbb{T})$$

Joyce
BTL

Poisson tors

$\mathbb{C}[\mathbb{T}] = e^\alpha$

KS, Moore et al.