Teleman / Derived Geometric Langlands near the Opers

joint w/ Frenkel.

Σ proj. cplx curve
G cx simple gp $\quad {}^L G$ Langlands dual

Recall Geometric Langlands: bijection between

constructible Hecke eigensheaves on $\mathcal{M}_G$ moduli stack of holo G-bundles $\longleftrightarrow$ points on $\mathcal{M}^{flat}_{{}^L G}$-bundles

$GL(n)$ essentially known (Frenkel, Gaitsgory, Vilonen / Laumon - Drinfeld / Lafforgue)

Generalization: roughly equivalence (of derived categories)

$$D(D\text{-mod on } \mathcal{M}_G) \longleftrightarrow D(\mathcal{O}\text{-mod}(\mathcal{M}^{flat}_{{}^L G}))$$

1. True for GL_1 (Laumon, Polishchuk, Rothstein)
2. Something true on $Op \hookrightarrow \mathcal{M}^{flat}_{{}^L G}$ (opers) (smooth subvariety, $\cong$ to affine space)

Construct an (exact) functor from $Coh(Op) \longrightarrow \mathcal{D}_c\text{-mod on } \mathcal{M}_G$.

[$\mathcal{D}_c$ = differential operators on $\sqrt{K}$.] Used description of $Z(U_c(Lg))$

by Feigin - Frenkel. c = critical level.

They show $\Gamma(\mathcal{M}_G, \mathcal{D}_c) \cong \mathbb{C}[Op]$. Given $\mathcal{M} \in Coh(Op) \longrightarrow \widehat{\mathcal{M}} = \mathcal{D}_c \underset{\Gamma(\mathcal{M}_c, \mathcal{D})}{\otimes} \Gamma(Op, \mathcal{M})$

<u>Note</u>: Would like a kernel for a Fourier transform $\mathcal{M}_G \times \mathcal{M}^{flat}_{{}^L G}$,

a $\mathcal{D}_c - \mathcal{O}$ bimodule. BD construct a kernel in $\mathcal{M}_G \times Op$.

<u>Main result</u>: Extend kernel to a formal nbhd of Op.

Then can define the functor on full triangulated subcategories in the two sides.

Thm: The BD construction extends to an equivalence of (abelian) categories from Coh Sh on $\mathcal{M}^{flat}$, supported on Op, to $\mathcal{D}_c$-modules on $\mathcal{M}_G$ which are successive extensions of ~~modules~~ finitely presented $\mathcal{D}$-modules: $\mathcal{D}_c^{\oplus P} \to \mathcal{D}_c^{\oplus g} \to \tilde{\mathcal{M}}$. The Ext groups match.
So get derived geometric Langlands in a formal nbhd of Op.

Toy example: Abelian varieties

$A, A^\vee$ dual ab varieties

$P \to A \times A^\vee$ Poincaré

Laumon - Rothstein (Mukai): $D\,Coh(A) \xrightarrow{\sim} D\,Coh(A^\vee)$ defined by $R\pi_*(P \otimes p^*)$
is an equivalence
FM = Fourier - Mukai

Let $T^* = T_1^* A$. Repeat with $A \times T^*$, $A^\vee \times T^*$ over T^*. Again can $\cong$ of cat, but this deforms:

$\mathcal{O}(T^*A)$ deforms to $\mathcal{D}$ on A

$A^\vee \times T^*$ deforms to
$$\begin{array}{c} \tilde{A} \hookleftarrow T^* \\ \downarrow \\ A^\vee \ni 1 \end{array}$$

$\tilde{A}$ = moduli of flat holo line bundles on A.
P lifts to $A \times \tilde{A}$ and has a flat connection along A

LR: $D\,\mathcal{D}\text{-mod}(A) \xrightarrow{\sim} D\,Coh(\tilde{A})$ by FM.

$T^* \hookrightarrow \tilde{A}$, fiber over 1, "moduli" of flat holo connections on $\mathcal{O}$ over A.

$p \in Op \longleftrightarrow \mathcal{O}$ with some connection.

Coh sheaves on $Op \longrightarrow \mathcal{D}$-mod on A which are quots of $\mathcal{D}$.

$$\mathcal{O}_{op} \longrightarrow \mathcal{D}$$

Koszul res of $p \longrightarrow (\Lambda^\bullet T \otimes \mathcal{D}) \longrightarrow \underset{\text{w/ some flat conn}}{\mathcal{O}}$.

$$\Gamma(A;\mathcal{D}) = \mathrm{Sym}\, T = \mathbb{C}[Op]$$

Exercise: FM transform for sheaves on Op is given by $\mathcal{M} \longrightarrow \hat{\mathcal{M}} = \mathcal{D}_c \underset{\Gamma(M_c, \mathcal{D}_c)}{\otimes} \Gamma(\ldots)$

$M_G = A,\ q \hookrightarrow \hat{A} \longrightarrow \mathcal{M}^{flat}_{{}^L G}$.

$\mathrm{Ext}_{\hat{A}}(\mathcal{O}_{op}, \mathcal{O}_{op}) = \mathbb{C}[Op] \otimes \Lambda^\bullet T_1 A^\vee$ — perfect match on Ext.

$\mathrm{Ext}_{\mathcal{D}}(\mathcal{D}, \mathcal{D}) = H^*(A, \mathcal{D}) = \mathbb{C}[Op] \otimes \Lambda^\bullet T A^*$

Locally: From correspondence on Op and $\cong$ of Ext data can recover formal nbhd of Op.

Equiv. of categories: Coh(set-th support on Op) $\longleftrightarrow$ $\mathcal{D}$-mod on A

successive extensions of f.p. $\mathcal{D}$-mods

This last part applies to any G,

$\mathcal{D}$-side		
$M_G(\Sigma) \longleftrightarrow A$	$T^* M_G$	$T^* M_{{}^L G}$
$M^{flat}_{{}^L G}(\Sigma) \longleftrightarrow \hat{A}$	$T^* A$	$A^\vee \otimes T^*$

If A has pp, $T^*A \cong T^*\hat{A}$.

Hitchin system $\quad \Sigma$ proj. smooth curve

$$\begin{array}{ccc} T^* M_G & & T^* M_{{}^L G} \\ & {\scriptstyle \chi} \searrow \quad \swarrow {\scriptstyle {}^L\chi} & \\ & \mathcal{H} = \bigoplus_{d \in \exp(G)+1} \Gamma(\Sigma; K^{\otimes d}) = \bigoplus_{\text{for } GL(n)}^{n} \Gamma(\Sigma; K^d) & \end{array}$$

Hitchin, Faltings, BD, Hausel-Thaddeus, Donagi-Pantev,

Generically, fibers of X are ab varieties (GL_n: Jac of spectral curve)
in duality for $X, {}^L X$.

Away from discriminant, have Poincaré bundle $\mathcal{P} \to T^*\mathcal{M} \times_{\mathcal{H}} T^*\mathcal{M}$

$\mathcal{P}$ defines an equivalence of DCoh over $\mathcal{H} \setminus$ disc. [Pantev-Donagi, Hausel-Thaddeus].

Now $T^*\mathcal{M}_G \rightsquigarrow \mathcal{D}_c(\mathcal{M}_G)$

$T^*\mathcal{M}_{{}^LG} \rightsquigarrow \mathcal{M}_{{}^LG}^{flat}$

Observation: $\mathcal{P}$ can be extended ~~to a line bundle~~ a b.T: to a line bundle on $T^*\mathcal{M} \times T^*\mathcal{M} - T^*\mathcal{M}^{sing} \times T^*\mathcal{M}^{sing}$.

princ. bundle
P, $\theta \in H^0(\Sigma; ad_P \otimes K)$. $\chi(\theta) \in \mathcal{H}$ defines a spectral curve in K_Σ.

Fact: P_{SL}, E, θ together define a sheaf over spectral curve $Sp(\theta)$ whose projection to Σ is iso to E (associated to P), Call θ regular if This sheaf is a line bundle (generic condition).

Recall for GL, on $J_\Sigma \times J_\Sigma$ have $m: J_\Sigma \times J_\Sigma \to J_\Sigma$,

$\mathcal{P} = \Theta_1 \otimes \Theta_1 \otimes m^*\Theta^{-1}$, $\Theta = \det^{-1} H^0(\Sigma; \text{line bundle})$.

At any pair θ_1, θ_2 with same spectral curve S have $\mathcal{E}_1, \mathcal{E}_2$ over S. If one of them is a line bundle, can form $\det H^0(S; \mathcal{E}_1 \otimes \mathcal{E}_2) \otimes \det^{-1} \otimes \det^{-1}$

This defines $\mathcal{P}$.

Can define the complete FT of any sheaf supported inside $T^*\mathcal{M}^{reg}$.

Hitchin section: Given $(p_1, \dots, p_n) \in \hat{\bigoplus} \Gamma(\Sigma; K^{\otimes d_i})$ let

$E = \mathcal{O} \oplus K^{-1} \oplus \cdots \oplus K^{-n+1}$

$$\theta = \begin{bmatrix} p_1 & p_2 & \cdots & p_n \\ -1 & & & \\ & -1 & & 0 \\ 0 & & \ddots & -1 \end{bmatrix} \qquad \text{Regular.}$$

This is a section $\sigma : \mathcal{H} \longrightarrow T^*\mathcal{M}$.

Observe: $FM(\mathcal{O}_\sigma) = \mathcal{O}$ on $T^*\mathcal{M}$. (Analog of $T_1^* A^\vee$).

Expect FM is an equivalence of categories $DCoh$ (supported in $T^*\mathcal{M}^{reg}$) $\to$ image. Know it is true for formal nbhd of Hitchin section.

In particular, $Ext_{T^*\mathcal{M}}(\mathcal{O}_\sigma, \mathcal{O}_\sigma) = H^*(T^*\mathcal{M}; \mathcal{O})$; both are $\cong$ to

Thm: $Ext_{T^*\mathcal{M}}(\mathcal{O}_\sigma, \mathcal{O}_\sigma) = H^*(T^*\mathcal{M}; \mathcal{O})$ and both sides are $\cong$ to

$\mathbb{C}[\mathcal{H}] \otimes \Lambda^\bullet \mathcal{H}^\vee = \Omega^\bullet[\mathcal{H}]$. [poly diff forms on $\mathcal{H}$].

LHS obvious since σ is a smooth locus, normal bundle $\cong \mathcal{H}^\vee$.

H^1: Hitchin
gen'l: Frenkel-T, based on ...

Thm: "This quantizes". Hitchin section $\rightsquigarrow$ $Op \hookrightarrow \mathcal{M}^{flat}_{{}^LG}$.

$\mathcal{O}$ on $T^*\mathcal{M}_G \to \mathcal{D}_c$

Op has a natural structure of affine space over $\mathcal{H}$.

Thm (BD): $\Gamma(\mathcal{M}, \mathcal{D}_c) = \mathbb{C}[\mathcal{O}_p^{{}^LG}]$

Thm (Frenkel-T): $H^*(\mathcal{M}, \mathcal{D}_c) = \Omega^\bullet(\mathcal{O}_p^{{}^LG})$.