FOUR-DIMENSIONAL WALL-CROSSING FROM THREE-DIMENSIONAL FIELD THEORY

WORK DONE WITH

DAVIDE GAIOTTO & ANDY NEITZKE

KITP, JULY 31, 2008

BASED ON

arXiv: 0807.4723
OUTLINE

1. INTRODUCTION
2. REVIEW OF BPS WALL-CROSSING
3. THE KS-FORMULA
4. COMPACTIFICATION OF $\mathfrak{N}=2, D=4$ THEORIES ON $T^3 \times S^1$
5. TWISTOR SPACE
6. SINGLE PARTICLE Q.C.'S TO T.S.
7. MULTI-PARTICLE: RIEMANN-HILBERT
8. PHYSICAL PROOF OF THE KS FORMULA
9. TAKE-HOME SUMMARY
10. CONCLUSION
1. INTRODUCTION

This talk is about the BPS spectrum of \( W=2, D=4 \) field theories.

The BPS spectrum of the theory on \( \mathbb{R}^4 \) is a "piecewise constant" function of the boundary conditions at \( \infty \).

Recently there has been some progress in understanding precisely how the spectrum depends on boundary conditions.

These are called Wall-Crossing Formulae (WCF). This talk will give a physical interpretation and proof of a famous WCF of Kontsevich + Soibelman.
Consider a theory on $\mathbb{R}^4$ with $N = 2$ superPoincare' symmetry $\Delta$.

Let $\mathcal{H}$ be the one-particle Hilbert space.

As a representation of $\Delta$, $\mathcal{H}$ depends on the boundary conditions of fields at $\infty$.

These boundary conditions are valued in the moduli space of vacua: $\mathcal{M}_v$.

For $u \in \mathcal{M}_v$, write $\mathcal{H}_u$. 
For all \( u \in \mathfrak{m}_v \), there is an unbroken abelian gauge symmetry of rank \( r \), so \( \mathfrak{h} \) is graded by the symplectic lattice \( \Gamma \) of elec.+ mag. charges. (of rank \( 2r \)).

\[ \mathfrak{h}_u = \bigoplus_{\gamma \in \Gamma} \mathfrak{h}_{\gamma, u} \]

On each subspace \( \mathfrak{h}_{\gamma, u} \) the central charge operator \( \mathcal{Z} \in \Delta \) is a scalar.

Denote the value \( \mathcal{Z}_\gamma(u) \)
RECALL THE $W=2, D=4$ SUSY ALGEBRA

$$\Delta = \Delta_0 \oplus \Delta_1$$

$$\Delta_0 = (\text{Spin}(1,3) \times \mathbb{R}^4) \oplus u(2) \oplus \mathbb{R}$$

$$M_{\mu\nu}, P_\mu \quad \mathbb{Z}$$

$$\Delta_1 = \left( \text{Spinor} \otimes \mathbb{C}^2 \right)_{\mathbb{R}}$$

$$Q^{\alpha I}, \bar{Q}^{\dot{\alpha} I}$$

$$\{Q^{\alpha I}, \bar{Q}^{\dot{\beta} J}\} = 2P_\mu \sigma^{\mu}_{\alpha\dot{\beta}} S^{I J}$$

$$\{Q^{\alpha I}, Q^{\beta J}\} = 2\mathbb{Z} \epsilon_{\alpha\beta} \epsilon^{IJ}$$

UNITARY IRREPS SATISFY BPS BOUND:

$$M \geq |\mathbb{Z}|$$
DEF: $H^{bps}_{\gamma_1, \gamma_2} =$ SUBSPACE SATURATING THE BPS BOUND.

ON THIS SUBSPACE $E = \{ |Z_{\gamma}(u)| \}$

SOME BPS PARTICLES CAN BE VIEWED AS BOUNDSTATES OF OTHERS: DECAY WHEN BOUNDSTATE ENERGY $E(u) \to 0$.

[CECOTTI et. al.; SEIBERG & WITTEN]

$Z_{\gamma}(u)$ IS LINEAR IN $\gamma = \gamma_1 + \gamma_2$ so

$E(u) = |Z_{\gamma}(u)| - (|Z_{\gamma_1}(u)| + |Z_{\gamma_2}(u)|) \leq 0$

$\implies$ DECAY ONLY HAPPENS ALONG WALLS OF MARGINAL STABILITY:

$MS(\gamma_1, \gamma_2) := \{ u | \frac{Z_{\gamma_1}(u)}{Z_{\gamma_2}(u)} \in \mathbb{R}_+ \}$
When $\gamma_1, \gamma_2$ are primitive, Denef \& Moore described how $\mathcal{H}_{\gamma, u}^{bps}$ changes across a wall: $u_+ \mid u_-$.

\[
\mathcal{H}_{\gamma, u_+}^{bps} - \mathcal{H}_{\gamma, u_-}^{bps} = (J_{12}) \otimes \mathcal{H}_{\delta_1, u}^{b} \otimes \mathcal{H}_{\delta_2, u}^{b}
\]

\[
J_{12} = \frac{1}{2}(1 < \gamma_1, \gamma_2 > | - 1)
\]

$\langle \gamma_1, \gamma_2 \rangle = \text{Symplectic Product}$

Based on Denef's sugra construction of boundstate solutions:

\[
\begin{array}{ccc}
& \leftrightarrow & \\
R(u) & & \end{array}
\]

There is a generalization for decays of the form

\[
\gamma = \gamma_1 + N\gamma_2, \quad N > 1
\]
But for decays of the form
\[ \gamma = N_1 \gamma_1 + N_2 \gamma_2 \quad N_1, N_2 \geq 1 \]
The methods of DM are difficult to use.

Kontsevich & Soibelman proposed a remarkable formula for the change of the index:
\[ \Omega(\gamma; u) = -\frac{1}{2} \text{Tr}_{\mathfrak{g}^{bos, \gamma}} (2J_3)^2 (-1)^{2J_3} \]
which includes all cases.

[In supergravities arising from CY compactification \( \Omega(\gamma; u) = \) "Generalized Donaldson-Thomas in ut." ]
3. KS Formula

- Introduce the symplectic torus

\[ T = \Gamma^* \otimes \mathbb{C}^* \]

\( \gamma \in \Gamma \implies \text{function } \chi_\gamma : T \to \mathbb{C}^* \)

- Choosing a basis \( \gamma_i \), define:

\[ \omega = \frac{1}{2} \varepsilon_{ij} \frac{dx_i}{x_i} \wedge \frac{dx_j}{x_j}, \quad \varepsilon_{ij} = \langle \gamma_i, \gamma_j \rangle \]

- Choose a quadratic refinement

\[ \frac{\sigma(\gamma_1 + \gamma_2)}{\sigma(\gamma_1) \sigma(\gamma_2)} = (-1)^{\langle \gamma_1, \gamma_2 \rangle}, \quad \sigma = \pm 1 \]

- Define "KS Transformations"

\[ \mathcal{U}_\gamma : X_\gamma \mapsto X_\gamma \left( 1 - \sigma(\gamma)X_\gamma \right)^{\langle \gamma_1, \gamma \rangle} \]
To each $y \in \Gamma$ associate the "BPS ray" in the complex plane:

$$\ell_y := \{ s \mid s \in \mathbb{Z}_+(u) \cdot \mathbb{R}_- \}$$

Choose a convex cone $\nu$ in the $s$-plane:

Associate the symplectic TMN:

$$A_{\nu} = \prod_{\ell_y \subset \nu} \Omega(y; u)$$
For later convenience note that if $\gamma_1 = N \gamma_2 \neq 0$ then $l_{\gamma_1} = l_{\gamma_2}$, so define

\[ S_\gamma := \prod_{l_{\gamma'} = l_\gamma} \bigcup_{\gamma'} \Omega(\gamma; u) \]

and write:

\[ A_\gamma \rightarrow \prod_{\gamma \text{ prime} \atop l_\gamma \subset \gamma} S_\gamma \]

The $\Omega(\gamma; u)$ depend on $u$...
THE KS FORMULA STATES THAT

\[ A_\gamma = \prod_{\ell_\gamma < \gamma} \Upsilon_{\gamma} \Omega(\gamma; u) \]

IS CONSTANT IN U AS LONG AS NO BPS RAY ENTERS OR LEAVES THE SECTOR \( \gamma \).

IN PARTICULAR, ACROSS WALLS OF MARGINAL STABILITY

\[ \Rightarrow \text{WALL CROSSING FORMULA} \]
4. Compactification of $\mathbb{N}=2, D=4$ Field Theories

A. Seiberg-Witten Solution

$G$ - Compact S.S. Gauge Group, Rank $r$


$\implies D=4, N=2$ Field Theory

$\mathcal{N}_V = (\mathcal{Y}_G)^C = C^r \quad (u_2 = \text{Tr} \Phi^2, u_3 = \text{Tr} \Phi^3, \ldots)$

S&W gave formulae for

- $Z_g(u)$

- Low Energy Abelian Gauge Theory

in terms of

**Special Kähler Geometry**
Review Special Kähler Geom:

C.f. D. Freed, hep-th/9712042

View $\Gamma$ as a local system over $\mathcal{M}_v$

$\mathcal{G}_v \rightarrow \mathcal{M}_v$

$\downarrow$

$\mathcal{U}_v \hookrightarrow \mathcal{M}_v$

$\mathcal{F} = \Gamma^* \otimes_{\mathbb{Z}} (\mathbb{R}/2\pi \mathbb{Z}) \cong U(1)^{2r}$

Fibers = Abelian Varieties

In regions of $\mathcal{M}_v$ choose a duality frame:

$\Gamma = \Gamma_{el} \otimes \Gamma_{mag}$, $\Gamma_{mag} = \Gamma_{el}^*$

$= \text{Span}\{\alpha_i\} \oplus \text{Span}\{\beta_i\}$

$\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0$, $\langle \alpha_i, \beta_j \rangle = \delta_{i,j}$
Choosing a duality frame, $\mathcal{J}_u$ has period matrix $T_{ij}$

1. Low energy Lagrangian:

$$L = -\frac{1}{4\pi} \text{Im} \tau_{ij} \left( da^i \bar{d}a^j + F_i \bar{F}^j \right) + \frac{1}{4\pi} \text{Re} \tau_{ij} F_i \wedge F^j$$

$$a^i = Z_{\xi_i}(u), \quad i = 1, \ldots, r$$

Local coords on $\mathcal{M}_v$

2. Central charge function

$$Z_{\gamma}(u) = a \cdot \gamma_e + a_D \cdot \gamma_m$$

$$T_{ij} = \left. \frac{\partial a^i}{\partial a^j} \right| = \frac{\partial^2 F}{\partial a^i \partial a^j}$$
S W IDENTIFY $\Sigma_u$ AS JACOBIANS OF AN EXPLICIT FAMILY OF RIEMANN SURFACES

**Basic Example:** $G = SU(2)$

$$\Sigma_u: \quad y + \frac{\lambda^u}{y} = x^2 - 2u$$

$$a = \frac{c}{2} \times \frac{dy}{y} \quad a_D = \frac{c}{3} \times \frac{dy}{y}$$
\( \mathcal{H}_{\text{BPS}}^{\text{weak}} = \bigoplus_{n \in \mathbb{Z}} \text{HM}(2n,1) \oplus \text{VM}(2,0) \oplus \text{CONJUGATE} \)

\( \mathcal{H}_{\text{BPS}}^{\text{strong}} = \text{HM}(2,-1) \oplus \text{HM}(0,1) \oplus \text{CONJUGATE} \)

**KS Identity:**

\( U_{2,-1} U_{0,1} = U_{0,1} U_{2,1} U_{4,1} \cdots U_{-2} \cdots U_{6,1} U_{4,1} U_{2,1} \)

**IT IS TRUE !!!**
B. COMPACTIFY ON A CIRCLE.

- **NOW CONSIDER THE THEORY ON** \( R^2 \times S_\mathbb{R}^1 \).
- **LOW ENERGY THEORY IS A** 3D **\( \sigma \)-MODEL :** \( R^3 \rightarrow \mathcal{M} \)

\[
\alpha^\mathcal{I}(\mathcal{X},x^4) \rightarrow \alpha^\mathcal{I}(\mathcal{X})
\]

\[
\Phi^\mathcal{I}_e = \int_{S^1} A^\mathcal{I}_4 \, dx^4 \quad \text{PERIODIC!}
\]

\[
\Phi^\mathcal{I}_m = \int_{S^1} (A_{D,4})^\mathcal{I} \, dx^4
\]

- **SUPERSYMMETRY** \( \Rightarrow \)

\( \mathcal{M} \) **MUST CARRY A HYPERKÄHLER METRIC**

LET US TRY TO DESCRIBE IT
Topologically $U$ is a torus fibration over $U_v$:

$U_v: \{ \}$

$U_v: \{ \}$
THE SEMI-FLAT METRIC

LEADING $R \to \infty$ APPROXIMATION:
USE DIMENSIONAL REDUCTION + DUALIZATION OF 3D GAUGE FIELD:

$$L^{(3)} = \frac{-R}{2} \text{Im} \tau_{IJ} \, da^I \ast d\bar{a}^J$$

$$- \frac{1}{8\pi^2 R} \, (\text{Im} \tau)^{-1,l} \wedge_J \, dz^I \ast d\bar{z}^J$$

$$dz^I = dg_{m,I} - \tau_{IJ} d\sigma^J_e$$

THIS DEFINES THE SEMI-FLAT METRIC

$$g^S_{IJ} = R \, (\text{Im} \tau) \, |da|^2 + \frac{1}{4\pi^2 R} \, (\text{Im} \tau)^{-1} \, |dz|^2$$
C. The Key Idea

- The metric $g^s$ receives quantum corrections from BPS particle world-lines wrapping $S^1$.

- Therefore the quantum corrections depend on the BPS spectrum.

- The true metric $g$ should be a smooth metric on $M$ away from the locus in $M_0$ where BPS particles become $M=0$.

- Smoothness of $g$ across walls of $M_S$ implies a WCFS.
WE WILL USE HITCHIN’S THEOREM: KNOWING $(M,g)$ IS EQUIVALENT TO KNOWING TWISTOR SPACE $Z := M \times \mathbb{CP}^1$ AS A HOLOMORPHIC MANIFOLD.

**Theorem:** IF $M$ IS HK OF DIMENSION 4, THEN:
1. \( \exists \) Holo. Fibration

\[ p: \mathbb{P} \rightarrow \mathbb{CP}^1 \]

\[ \mathcal{M}_5 = p^{-1}(\Sigma) = \mathcal{M} \text{ in complex structure } \Sigma \]

2. \( \exists \) Holomorphic Section

\[ \tilde{\omega} \text{ or } \tilde{\omega}^2 \]

\[ \tilde{\omega}_{\mathcal{M}_5} = \text{Holomorphic Symplectic Form on } \mathcal{M}_5 \]

3. \( \forall x \in \mathcal{M}, \exists \) Holomorphic Section

\[ s_x: \mathbb{CP}^1 \rightarrow \mathbb{Z} \text{ with normal bundle } \mathcal{O}(1)^{\oplus 2r} \]

4. \( \exists \) Anti-Holomorphic \( \sigma: \mathbb{Z} \rightarrow \mathbb{Z} \)

Covering \( \Sigma \rightarrow -\frac{1}{\Sigma} \)
Given 1, 2, 3, 4 one can reconstruct the metric:

For \( z \in \mathbb{C}^\times \):

\[
\omega = -\frac{i}{25} \omega_+ + \omega_3 - \frac{i}{2} z \omega_-
\]

Kähler form

\[
\omega_+ = \omega_1 + i \omega_2
\]

Our strategy is to construct the holomorphic sections \( S_x \) explicitly for the 3D C-model target.
\[ M \quad J_u \sim \mathbb{T}^2 / \Gamma \]

\[ M_v \quad \mu \]

- For \( \delta = 0 \), \( J_u \) is holomorphic.
- For \( \delta \neq 0 \), \( J_u \) is neither holomorphic nor anti-holomorphic.

- Holomorphic function on \( M_\delta \) is determined by restriction to some (i.e. any) fiber \( J_{u_0} \).

- A basis of \( \mathcal{C}^\infty \) functions on the torus \( J_{u_0} \) is labeled by \( y \in T_{u_0} \).

- Call the holomorphic functions in a neighborhood of \( J_{u_0} \): \( \mathcal{X}_\delta \)
THE LEADING, NO QUANTUM CORRECTIONS, APPROXIMATION:

\[ X_y^{sf} = \exp \left[ \pi \mathbb{R} S^{-1} Z_y + i \Theta_y + \pi \mathbb{R} S \bar{Z}_y \right] \]

[ A. NEITZKE \& B. PIOLINE ]

\[ \Theta_y : T^* \otimes \mathbb{R}/2\pi \mathbb{Z} \rightarrow \mathbb{R}/2\pi \mathbb{Z} \]

**CHECK**

\[ \Omega = \frac{1}{8\pi^2 R} \epsilon_{i,j} \frac{dX_i}{X_i} \wedge \frac{dX_j}{X_j} \]

\[ = \frac{1}{8\pi} \left[ \frac{i}{5} \langle dZ, d\theta \rangle + \ldots \right] \]

\[ \langle dZ, d\theta \rangle = -da^I \wedge dz_I \]

ETC.
6. SINGLE-PARTICLE CORRECTIONS

Now we include the first Q.C.

- For simplicity consider $r = 1$.
- Consider a point $u_* \in \mathcal{M}_v$

Where a single HN becomes has $m \rightarrow 0$

$\Rightarrow$ dominant contribution near $u_*$. Choose duality frame so it has charge $(q, 0)$, $q > 0$

$KK$ reduction $\Rightarrow$ target

Space metric is a Gibbons-Hawking ansatz: [Seiberg Witten; Obguri Vafa; Seiberg Shendker]

$$g = V^{-1}(x^0) \left( \frac{d\phi_m}{2\pi} + A \right)^2 + V(x^0) \, dx^2$$

$$F = * dV \quad \forall x \in \mathbb{R}^3$$
Here:

\[ \alpha = x^1 + i x^2 \]

\( \varphi_c = 2\pi R \times^3 \) \hspace{1cm} \text{PERIODIC}

\[ V(x) = \frac{g^2 R}{4\pi} \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{g^2 R^2 |a|^2 + \left( \frac{\varphi_c}{2\pi} + n \right)^2}} \]

\[ = \ V^{sf} + V^{inst} \]

\[ V^{sf} = -\frac{g^2 R}{4\pi} \left( \log \frac{\alpha}{\Lambda} + \log \frac{\bar{\alpha}}{\bar{\Lambda}} \right) \]

\[ V^{inst} = \frac{g^2 R}{2\pi} \sum_{n \neq 0} e^{i n g \varphi_c} K_0 \left( 2\pi R |\ln q a| \right) \]

\( \sim e^{-2\pi R |\ln q a|} \) \hspace{1cm} \text{INSTANTON CONTRIBUTION}
Now, what are the holo functions on twistor space?

Algebra of holo functions \( \{ X_\theta \} \) on twistor space is generated by:

\[
X_e := X_{(1,0)} = \exp \left\{ i\varphi_e + \cdots \right\}
\]

\[
X_m := X_{(0,1)} = \exp \left\{ i\varphi_m + \cdots \right\}
\]

\[
X_{(k_1, k_2)} = (X_e)^{k_1} (X_m)^{k_2}
\]

\(
(k_1, k_2) \in \mathbb{Z}^2 = \Gamma^m
\)
DETERMINE $\chi_e$ AND $\chi_m$
FROM A DIFFERENTIAL EQUATION

HK STRUCTURE: $\alpha = 1, 2, 3$:

$$\omega^\alpha = dx^\alpha \wedge \left( \frac{d\varphi_m}{2\pi} + A \right) + \frac{1}{2} V^{\alpha\beta\gamma} dx^\beta dx^\gamma$$

$\Rightarrow$ COMPUTE $\overline{\omega} = \frac{-i}{2\pi} \omega_+ + \omega_3 - \frac{i}{2\pi} \omega_-$

$$\overline{\omega} = -\frac{1}{4\pi^2 R} \frac{d\chi_e}{\chi_e} \wedge \frac{d\chi_m}{\chi_m}$$
WE FIND:

\[ \chi_e = \chi_e^{sf} = \exp \left[ \frac{\pi R}{5} a + i \varphi_e + \pi R \delta \alpha \right] \]

BUT

\[ \chi_m = \chi_m^{s.f.} \cdot \chi_m^{\text{inst.}} \]

\[ \chi_m^{sf} = \exp \left[ \frac{\pi R}{5} \cdot a_D + i \varphi_m + \pi R \delta \bar{a}_D \right] \]

\[ a_D = \frac{q^2}{2\pi i} \left( a \log \frac{a}{e^\Lambda} \right) \]

\[ \chi_m^{\text{inst.}} = \text{INSTANTON CONTRIBUTION} \]
\[ \chi^\text{inst}_{m}(s) = \exp \left\{ \frac{ie}{4\pi} \int_{l_+} d\xi' \frac{\xi' + \xi}{\xi' - \xi} \log \left( 1 - \chi_{e}(\xi')^2 \right) \right\} \]

\[ - \frac{ie}{4\pi} \int_{l_-} d\xi' \frac{\xi' + \xi}{\xi' - \xi} \log \left( 1 - \chi_{e}(\xi')^{-2} \right) \}

\[ \sum l_+ = l_{(1,0)} = a \cdot R_- \]

\[ \chi_{e}(s) \text{ EXP. SMALL} \]

\[ l_- = l_{(-1,0)} = (-a) \cdot R_- \]

\[ \chi_{e}(s) \text{ EXP. LARGE} \]
1. As a function of $\xi$, $X_m$ is discontinuous across the BPS ray:

$$\ell_\gamma := \{ \xi \mid \frac{Z_\gamma}{\xi} \in \mathbb{R}^- \}$$

For $\gamma = (\pm 1, 0)$, generators of $\Gamma_{ee}$

2. Across this ray:

$$(X_e, X_m)^{cw} = U_\gamma (X_e, X_m)^{ccw}$$

$$= (X_e, X_m (1 - X_e^{\pm 1})^{\pm 9})^{cw}$$

3. $\forall \gamma$, $X_\gamma \sim X_\gamma^{s.f.} \ (1 + O(1))$
Observation: $X_y$ are the solution of a Riemann-Hilbert problem.

R-H Problem:

Find a piecewise holomorphic function with prescribed singularities and asymptotics.
7. MULTI-PARTICLE CONTRIBUTIONS

TO TAKE INTO ACCOUNT ALL BPS PARTICLES WE CANNOT USE A LOW ENERGY EFFECTIVE LAG., BECAUSE THE PARTICLES WILL BE MUTUALLY NONLOCAL.

PROPOSAL: THE HOLOMORPHIC FUNCTIONS ARE CONSTRUCTED FROM A RIEMANN-HILBERT PROBLEM IN THE S-PLANE
To give the problem a simple formulation view the collection of functions

\[ X_\gamma = e^{i \Theta_\gamma + \ldots} \]

as a family of maps

\[ X(\gamma) : \mathcal{F} \rightarrow T = \mathbb{R}^\gamma \otimes \mathbb{C}^* \]

piecewise holomorphic in \( \mathcal{F} \).

• Recall \( T_u \) has functions

\[ X_\gamma : T_u \rightarrow \mathbb{C}^* \]

• Recall \( T_u \) is symplectic

\[ \omega^T = \frac{1}{2} \epsilon_{i j} \frac{dX_i}{X_i} \wedge \frac{dX_j}{X_j} \]
RIEMANN–HILBERT PROBLEM:

1.) $X(5)$ IS DISCONTINUOUS ACROSS BPS RAYS $\gamma_x$:

$$X^{cw} = S_\gamma(X^{ccw})$$

[RECALL: $S_\gamma = \prod_{\gamma' = \gamma} U_{\gamma'}^{S_\gamma(\gamma',\nu)}$]

2.) $X(5)$ HAS ASYMPTOTICS FOR $5 \to 0, \infty$ GIVEN BY $X^{sf}(\xi)$, UP TO $O(1)$ CORRECTIONS

$$Y := (X^{sf})^{-1} X : \mathfrak{g} \to \mathfrak{g}$$

i.e.

$$Y_0 = \lim_{5 \to 0} Y(5) \quad Y_\infty = \lim_{5 \to \infty} Y(5)$$

EXIST
SOLUTION:

\[ \chi_y(s) = \chi_{y'}^{sf}(s) . \]

\[ \exp \left\{ - \frac{i}{4 \pi} \sum_{y' \in \Gamma} \Omega(y', u) \langle y, y' \rangle \right\} \]

\[ \cdot \int \frac{d5'}{5', 5'+5} \frac{5'+5}{5'-5} \log \left[ i - \sigma(y') \chi_y(s', 5') \right] \]

ITERATING THIS EQUATION
( AS A SUM OVER TREES... )

GIVES THE FULL INSTANTON EXPANSION!

⇒ EXPLICIT CONSTRUCTION OF TWISTOR COORDS!
• WE RECONSTRUCT THE METRIC FROM
\[ \omega = \frac{1}{4\pi^2 R} \chi (\omega^\top) \]

• AS \( u \) CROSSES A WALL OF MS BPS RAYS PILE UP

\[ \Sigma \]

But the jump of \( \chi \) in the RH problem is continuous as a function of \( u \): THAT IS THE KS FORMULA
Thus: The KS formula guarantees the continuity of the HK metric across walls of Ms.!

The resulting metric passes a number of consistency tests.

But... why is our proposal the right one?

Why is the metric the right one for the physical problem?
RH IS EQUIVALENT TO A DIFF.EQ.:\[ A_5 = x^{-1} 5 \delta f \]

IS CONTINUOUS IN \( S \)-PLANE:

ACROSS \( \gamma \)

\[ x^{-1} \delta f x \rightarrow (sx)^{-1} \delta f (sx) \]
\[ = x^{-1} \delta f x \]

\( \Rightarrow A_5 \) IS HOLomorphic FOR \( S \in \mathbb{C}^* \)
\[ \Rightarrow \quad \mathcal{S}_2 \mathcal{S}_3 X = \mathcal{A}_5 X \geq 2 \]

**Structure Group:** \textit{Sympl}{\textsuperscript{IT}}

**Asymptotics** \Rightarrow

\[ \mathcal{A}_5 = \mathcal{S}^{-1} \mathcal{A}_5^{(-1)} + \mathcal{A}_5^{(0)} + \mathcal{S} \mathcal{A}_5^{(+1)} \]

Since \( \mathcal{S}_2 \) is indpt. of \( r, u, v, \ldots \)

Same argument \Rightarrow \( X \) satisfies a set of differential equations:
\[
\frac{\partial}{\partial u} x = A_u \cdot x
\]
\[
\frac{\partial}{\partial v} x = A_v \cdot x
\]
\[
\Lambda \frac{\partial}{\partial \Lambda} x = A_\Lambda \cdot x
\]
\[
\overline{\Lambda} \frac{\partial}{\partial \overline{\Lambda}} x = A_\overline{\Lambda} \cdot x
\]
\[
R \frac{\partial}{\partial R} x = A_R \cdot x
\]
\[
J \frac{\partial}{\partial J} x = A_J \cdot x
\]

\[
A_i^+ = S^{-1} A_i^{(-1)} + A_i^{(0)} + J A_i^{(+1)}
\]

**KEY POINT:** THESE EQUATIONS ALL FOLLOW FROM THE PHYSICS OF THE 4D GAUGE THEORY!!
\[
\begin{align*}
\frac{\partial}{\partial u} \chi &= A_u \cdot \chi \\
\frac{\partial}{\partial \bar{u}} \chi &= A_{\bar{u}} \cdot \chi \quad \text{HOLOMORPHY ON } M_5 \\
\wedge \frac{\partial}{\partial \wedge} \chi &= A_\wedge \cdot \chi \\
\overline{\wedge} \frac{\partial}{\partial \overline{\wedge}} \chi &= A_{\overline{\wedge}} \cdot \chi \\
\end{align*}
\]

Also holomorphy... view \( \wedge \) as background vev of a VM.

\[
\begin{align*}
R \frac{\partial}{\partial R} \chi &= A_R \cdot \chi \\
S \frac{\partial}{\partial S} \chi &= A_S \cdot \chi \quad \text{ANOMALOUS SCALE AND } R\text{-SYMMETRY}
\end{align*}
\]
STOKES PHENOMENON

The S-Diff Eq. has an irregular singular point at $S = 0, \infty$; solutions exhibit Stokes phenom.

In this interpretation $S_y$ are Stokes factors associated with $Ly$.

Remaining equations:
Isomonodromic deformation

$\Rightarrow$ Stokes factors are indp't of $R, u, \lambda, \ldots$

$\Rightarrow$ Check at large $R$ in 1-instanton approximation.
1. We construct the HK metric for circle-compactification of W = 2, D = 4 field theories.

2. Quantum corrections to the dimensional reduction metric come from BPS states.

3. Continuity of the quantum-corrected metric is equivalent to the KS WCF.

4. Use twistor transform and write holomorphic functions as an explicit sum over BPS instantons.
10. Conclusion

— Other things we have done —

- Massive HMs: generalizes KS.

- There are strong connections with the \( \mathbb{L} \mathbb{L}^* \) equations of Cecotti \& Vafa.

- The functions \( X_\gamma \) are \'t Hooft-Wilson-Maldacena loop operator VEV's; moreover there is a nice interpretation in terms of a 3D TFT [To appear]
• The moduli space \((\mathcal{M}, g)\) is a moduli space of a Hitchin system [S. Cherkis & A. Kapustin]. For \(SU(2)\) we have constructed it as a space of Stokes matrices glued by KS transformations. [To appear.]
- TO DO -

- Understand better the need for the quadratic refinement.

- Singularities at superconformal points remain to be understood.

- Relations to integrable systems.

- Relation to the work of Joyce, Bridgeland, Toledano Laredo.

- Generalization to sugra.

- Should give explicit formulation of Q.C.'s to hypermultiplet moduli spaces.