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Introduction to the Langlands Program, IV

Number Theory | Algebraic Curves/$\mathbb{F}_q$ | Riemann Surfaces | S-duality

$\downarrow$

X - curve$/\mathbb{F}_q$

$F = \mathbb{F}_q(X)$

focus on the unramified case

$$\mathbb{A}_F := \prod_{x \in X}{}' F_x \supset \prod \mathcal{O}_x =: \mathcal{O}_F$$

$$F_x \simeq \mathbb{F}_{q_x}((t)) \qquad \mathcal{O}_x \simeq \mathbb{F}_{q_x}[[t_x]]$$

$$\{\pi_1(X) \to GL_n\} \longleftrightarrow \{\text{automorphic reps of } GL_n(\mathbb{A}_F)\}$$

$$\pi_1(X) \to {}^L G \qquad G(\mathbb{A}_F)$$

$$\sigma \longleftrightarrow \pi_\sigma \simeq \bigotimes_{x \in X}{}' \pi_{\sigma, x}$$

$\pi_{\sigma,x}$ acted on by $G(F_x) \simeq G((t_x))$

$\updownarrow$

$$\pi_\sigma \subset \mathrm{Fun}(G(F) \backslash G(\mathbb{A}_F))$$

$$\pi_\sigma^{G(\mathcal{O}_F)} = \bigotimes \pi_{\sigma,x}^{G(\mathcal{O}_x)}$$

1-dim

$$\left\{ \begin{array}{c} \text{Hecke eigen-function} \\ \text{on } G(F) \backslash G(\mathbb{A}_F) / G(\mathcal{O}_F) \end{array} \right\}$$

$$\{\text{Frobenius eigenvalues}\} \longleftrightarrow \{\text{Hecke eigenvalues}\}$$

$\mathbb{F}_p \ni y \qquad y^p = y$

$\cap$

$\overline{\mathbb{F}}_p \ni y \quad \mathrm{Fr}: y \mapsto y^p$ (Frobenius automorphism)

This automorphism preserves $\mathbb{F}_p \subset \overline{\mathbb{F}}_p$

So it belongs to $\mathrm{Gal}(\overline{\mathbb{F}}_p / \mathbb{F}_p)$

$q = p^n, \ \mathbb{F}_q \subseteq \overline{\mathbb{F}}_q, \ \mathrm{Fr}: y \to y^q, \ \mathrm{Fr} \in \mathrm{Gal}(\overline{\mathbb{F}}_q / \mathbb{F}_q)$

$\mathrm{Gal}(\overline{F}/F) \twoheadrightarrow \pi_1(X)$

$\uparrow$

Fr_x

up to conjugation
have a well-defined
conjugacy class in $\pi_1(X)$
for any $x \in X$.

$$\sigma : \pi_1(X) \to {}^L G.$$

$$\sigma(\mathrm{Fr}_x) \ - \text{ conjugacy class in } {}^L G$$

Frobenius eigenvalues = spectral inv. of $\sigma(\mathrm{Fr}_x)$

$$\mathrm{Tr}(\sigma(\mathrm{Fr}_x), V) \qquad V \in \mathrm{Rep}\,{}^L G.$$

invariant attached to σ and $x \in V$

Hecke operators $\quad H_{V,x} : \mathrm{Fun}\left(G(F) \backslash G(\mathbb{A}_F) / G(\mathcal{O}_F)\right) \circlearrowleft$

$V \in \mathrm{Rep}\,{}^L G, \ x \in X.$

(Satake correspondence)

$^L G$ appears for the same reason in 't Hooft ops.

$\sigma \in \{\pi_1(X) \to {}^L G\}$ $\qquad\qquad$ $f_\sigma \in \{\text{function on } {}_{G(F)}\backslash G(\mathbb{A}_F)/G(\mathcal{O})\}$

$\downarrow$ $\qquad\qquad$ $\downarrow$

B-brane in $\mathcal{M}_{flat}({}^L G)$ $\qquad\qquad$ A-brane on $\mathcal{M}_H(G)$

Hecke operators $\to$ 't Hooft ops.

$$\boxed{H_{V,x}(f_\sigma) = \mathrm{Tr}(\sigma(Fr_x), V)\, f_\sigma}$$

Hecke eigenvalue.

Matching conditions

$\mathbb{C}[\check{T}]^W$

$\shortparallel$

$\mathrm{Rep}\,({}^L G)$

Now let X be a curve $/\mathbb{C}$. Use notation C for X.

$(G_\sigma) = G$ - complex reductive Lie group

$\{\pi_1(C) \to {}^L G\}$

$\shortparallel$

$\{(E, \nabla)$: E a principal hol. ${}^L G$-bundle on C

∇ - holo connection (auto. flat) on E $\quad (\partial_z + A_z)\}$

set of pts of $\mathcal{M}_{flat}({}^L G, C)$

A. Weil,

$G(F) \backslash G(\mathbb{A}_F) / G(\mathcal{O}_F)$ - the set of isomorphism classes of G-bundles on X (or C)

= the set of $\mathbb{F}_q$- (or $\mathbb{C}$-) points of Bun_G,
the moduli stack of G-bundles on X (or C)

Hecke is a fn on pts of Bun_G.

$$\{\pi_1(C) \to {}^L G\} \longleftrightarrow \{\text{Hecke eigensheaves on } \mathrm{Bun}_G\}$$

From functions to sheaves

Z - alg. var. $/\mathbb{F}_q$ $\rightsquigarrow$ $Z(\mathbb{F}_q)$ $\longrightarrow$ $\overline{\mathbb{Q}}_\ell$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$
(e.g. $y^2 = x^3 + ax + b$, $a, b \in \mathbb{Z}$) $\longrightarrow$ (set of solutions of eqs. defining Z in $\mathbb{F}_q$) $\longrightarrow$ $\mathbb{C}$

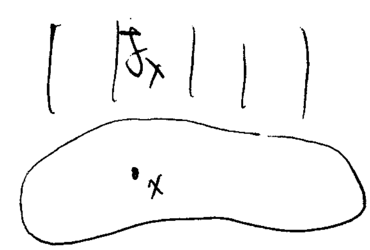
curve$/\mathbb{F}_p$

We have $Z = \mathrm{Bun}_G$.

$f_\gamma : \mathrm{Bun}_G(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell$ or $\mathbb{C}$.

$Z(\mathbb{F}_{q^n})$

$\mathcal{F}$ "ℓ-adic sheaf" (roughly, sheaf of $\overline{\mathbb{Q}}_\ell$-v. sp. on Z)
$\downarrow$
Z



Z

$x \in Z(\mathbb{F}_q) \circlearrowleft Fr_x$

Fr_x acts on $\mathcal{F}_x$ = stalk of $\mathcal{F}$ at x

$x \in Z(\mathbb{F}_q) \rightsquigarrow \mathrm{Tr}(\cancel{\ }\,\mathrm{Fr}_x, \mathcal{F}_x) \in \overline{\mathbb{Q}}_\ell$

$\mathcal{F} \rightsquigarrow$ function on $\prod_{n=1}^{\infty} Z(\mathbb{F}_{q^n})$

$$\begin{array}{c}
\{\text{Functions on } Z(\mathbb{F}_q)\} \\
\downarrow \\
\{\ell\text{-adic sheaves on } Z/\mathbb{F}_q\} \\
\downarrow \\
\{\mathbb{C}\text{-sheaves on } Z/\mathbb{C}\} \\
(\text{reg}) \downarrow \text{ Riemann-Hilbert} \\
\{\mathcal{D}\text{-modules on } Z/\mathbb{C}\} \\
\downarrow \\
\{A\text{-branes}\}
\end{array}$$

$\{\pi_1(C) \to {}^L G\} \xleftrightarrow[\text{Corr.}]{\text{Geometric Langlands}}$ {$\mathcal{D}$-modules which are Hecke eigensheaves on Bun_G}

$\mathcal{E} = (E, \nabla) \longrightarrow \mathcal{F}_\mathcal{E} = \mathcal{D}$-modules on Bun_G

$\mathcal{H}_{v,x} : \mathcal{D}\text{-mod}(\mathrm{Bun}_G) \circlearrowleft$

$\mathcal{H}_{v,x}(\mathcal{F}_\mathcal{E}) \simeq V_{\mathcal{E}_x} \otimes \mathcal{F}_\mathcal{E}$

(Nadler will talk on Monday)

What is known?

1. $G = GL_n$, $\mathcal{E}$ irreducible

   Gaitsgory - Vilonen - Frenkel
   (both over $\mathbb{F}_q$ and $\mathbb{C}$)

2. G semisimple, $\mathcal{E}$ = "oper"
   Beilinson - Drinfeld (over $\mathbb{C}$)
   $\mathcal{F}_{\mathcal{E}}$ is sheaf of conformal blocks of some "CFT"

   (more on Monday)

3. Partial results in other cases

Think of $\mathcal{E}$ as a skyscraper sheaf on $\mathcal{M}_{flat}({}^LG, C)$

$$D \left\{ \begin{matrix} \text{Category of} \\ \text{Coherent sheaves} \\ \text{on } \mathcal{M}_{flat}({}^LG, C) \end{matrix} \right\} \xleftrightarrow[\text{Corr.}]{\text{Cat. Geom Langl. } D} \left\{ \begin{matrix} \text{Category of} \\ \mathcal{D}\text{-modules on} \\ Bun_G \end{matrix} \right\}$$

Wilson ops

$W_{V,x}$

Hecke ops

$\mathcal{H}_{V,x}$

$\mathcal{E} \in \mathcal{M}_{flat}({}^LG, C)$

$\rightsquigarrow \delta_{\mathcal{E}}$ = "skyscraper sheaf" supp at $\mathcal{E}$

$$\delta_{\mathcal{E}} \longleftrightarrow \mathcal{F}_{\mathcal{E}}$$

For $G = GL_1$, this equivalence is a theorem (Fourier-Mukai equivalence)

Geom. L. C. $\equiv$ non-abelian Fourier-Mukai

$$\begin{array}{ccccc} \text{Wilson ops} \circlearrowright D(\text{Coh sheaves}) & \xleftrightarrow[\text{S-duality}]{\text{mirror symmetry}} & D\left\{\begin{array}{c}\text{cat. of} \\ \text{A-branes} \\ \text{on } \mathcal{M}_H(G,C)\end{array}\right\} \circlearrowleft \text{'t Hooft ops} & \longleftrightarrow & D\left\{\begin{array}{c}\text{cat. of} \\ \mathcal{D}\text{-modules} \\ \text{on } Bun_G\end{array}\right\} \circlearrowleft \text{Hecke ops} \\ \text{B-branes} & & \text{A-branes} & & \end{array}$$

$$\begin{array}{ccc} \mathcal{M}_{flat}({}^L G, C) & & \mathcal{M}_H(G, C) \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

SYZ mirror pair

$$\mathcal{M}_H(G,C) \simeq T^* Bun_G$$

$$\delta_\mathcal{E} \longleftrightarrow \mathcal{A}_\mathcal{E} \longleftrightarrow \mathcal{F}_\mathcal{E}$$

1. Kapustin-Witten
   unramified case, generic $\mathcal{E}$
2. Gukov-Witten
   tame ramification
3. Frenkel-Witten
   orbifold singularities $\to$ endoscopy

Ngo : Hitchin fibers / $\mathbb{F}_q$