31 July 2008
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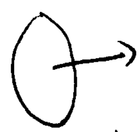
Remarks on TQFT

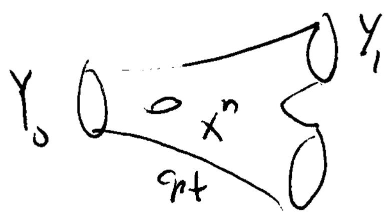
Defn: An n-dim'l TQFT is a symmetric monoidal functor

$$F: (\mathrm{Bord}_n^*, \amalg) \longrightarrow (\mathrm{Vect}_{\mathbb{C}}, \otimes)$$

objects:



closed Y^{n-1} $\longmapsto$ $F(Y)$ complex v.s.



$\longmapsto$ $F(X): F(Y_0) \to F(Y_1)$

Compare: Cohomology Theory

$$(\mathrm{Top}, \amalg) \longrightarrow (\mathrm{Vect}_{\mathbb{C}}, \oplus)$$

Extended:

$\dim(M^{k}\text{(closed)})$	$F(M)$	Category #
n	element of $\mathbb{C}$	-1
$n-1$	$\mathbb{C}$-vector space	0
$n-2$	$\mathbb{C}$-linear category	1
$n-3$	$\mathbb{C}$-linear 2-category	2

<u>Chern-Simons theory</u> $\underline{n=3}$

2-3 theory, 1-2-3 theory, 0-1-2-3 theory

$$\left.\begin{array}{ll} G & \text{compact Lie group} \\ \ell \in H^4(BG;\mathbb{Z}) & \underline{\text{level}} \end{array}\right\} \Rightarrow F_{(G,\ell)}$$

1. G finite group   hep-th/9212115
2. G torus: w/ Teleman, Lurie, Hopkins (ongoing)

G finite group

<u>Fields:</u> $\left\{ \begin{array}{c} P \\ \downarrow G \\ X \end{array} \right\}$ Galois G-cover

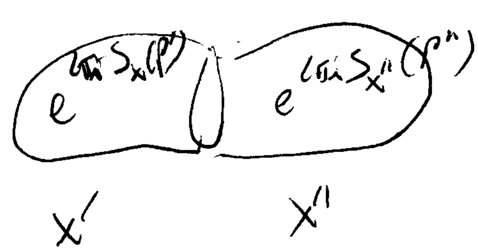
<u>Action:</u> $H^4(BG;\mathbb{Z}) \cong H^3(BG;\mathbb{R}/\mathbb{Z}) \ni \ell$

If X^3 is closed, oriented: $S_X(P) = \int_X \ell(P) \in \mathbb{R}/\mathbb{Z}$

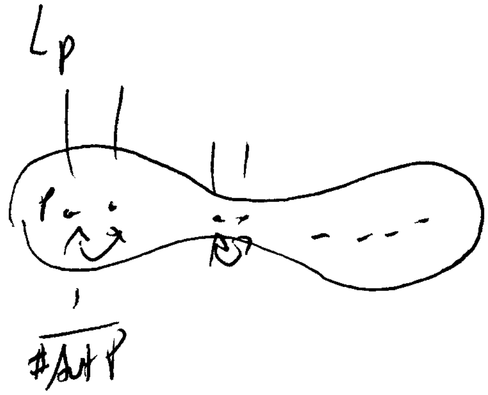
$$F(X) = \sum_{[P]} e^{2\pi i S_X(P)} \cdot \frac{1}{\# \mathrm{Aut}\, P}$$

If Y^2 is closed, oriented $\begin{array}{c} P \\ \downarrow G \\ Y \end{array} \rightsquigarrow \exp \ell(P)$

$\mathbb{T}$-torsor = Hermitian line



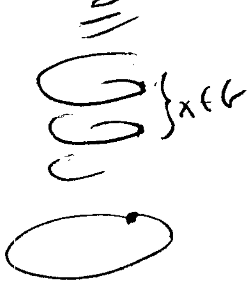
X' X''



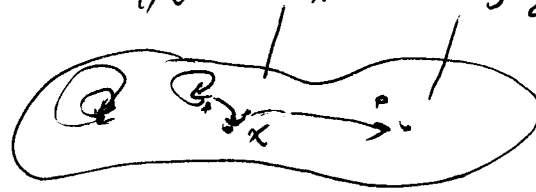
$$\left\{ \begin{matrix} p \\ \downarrow \\ y? \end{matrix} \right\}$$

S^1

$\}\, x \in G$



Fields : $\simeq G/\!/G$ $\quad W_x \longrightarrow W_{g^{-1}xg}$



$$\Big\downarrow F \qquad \ell = 0$$

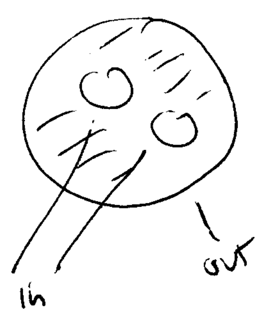
$$F(S^1) = \mathrm{Vect}_G(G) \simeq \mathbb{C}[\mathcal{G}]\text{-mod}$$

$\mathcal{G}$ = groupoid

$\mathbb{C}[\mathcal{G}]$ = path algebra

↑ quasi-Hopf algebra

<u>More structure</u>:

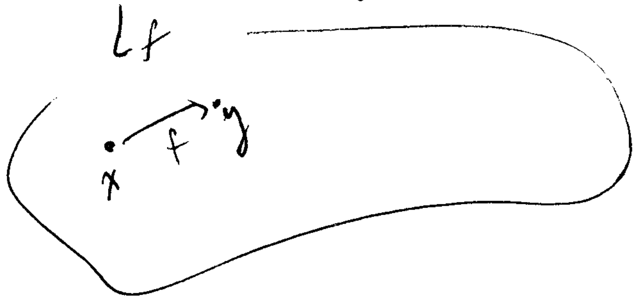


pair of pants

$$\rightsquigarrow \text{ monoidal structure } \quad (W_1 * W_2)_x = \bigoplus_{x_1 x_2 = x} (W_1)_{x_1} \otimes (W_2)_{x_2}$$

$\ell \neq 0$ Central extension

$$\mathbb{T} \to \hat{g}_\ell \to g$$



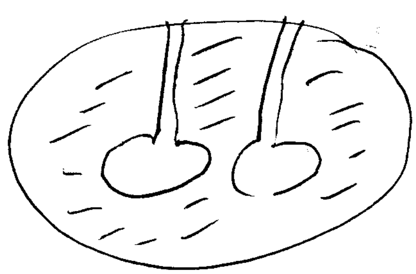
L_f hermitian line

$L_{gf} \xrightarrow{\cong} L_g \otimes L_f$

$$F_{(G,\ell)}(S^1) = \mathrm{Vect}_G^{(\ell)}(G) = \{ W_x,\ L_f \otimes W_x \to W_y \text{ for } x \xrightarrow{f} y \}$$

$$= \underbrace{A^{(\ell)}}_{\text{twisted version of } \mathbb{C}[g]}\text{-Mod}$$

Gluing law along corners:



$X = pt$ Fields $pt//G$

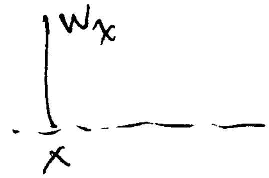
$$\ell \in H^4(BG;\mathbb{Z}) \cong H^2(BG;\mathrm{Pic})$$

$$\mathrm{Pic} = K(\mathbb{Z},2) = \text{complex lines}$$

central extension of G by cplx lines

$$\{K_{x,y}\}_{x,y\in G} \quad \lfloor \ \omega_{xyz} : K_{x,y}\otimes K_{xy,z} \xrightarrow{\cong} K_{x,yz}\otimes K_{y,z}$$

$$\mathcal{R} = \mathrm{Vect}(G) = \{W_x\}_{x\in G}$$



<u>monoidal str</u>: $(W_1 * W_2)_x = \bigoplus_{x_1x_2=x} K_{x_1,x_2}\otimes (W_1)_{x_1}\otimes (W_2)_{x_2}$

$F_{(G,\lambda)}(pt) = \mathcal{R}$-mod.

$$K \to C\times G$$

$F(pt) \rightsquigarrow F(S^1)$

<u>Def^n</u> $\mathcal{R}$ monoidal cat. The <u>Drinfeld center</u> $Z(\mathcal{R})$ consists of pairs (x,β_x) where $x\in\mathcal{R}$ and β_x natural iso

$$\beta_x(y): x\otimes y \to y\otimes x \ , \ y\in\mathcal{R}$$

and $\beta_x(y\otimes z) = (1\otimes\beta_x(z))(\beta_x(y)\otimes 1)$.

<u>Claim</u> $F(S^1) = Z(\mathcal{R})$ where

$$L_f = \frac{K_{x,y}}{K_{y,y^{-1}xy}} \quad \text{for} \quad x \xrightarrow{f} y^{-1}xy$$

$G = T$ torus

$\mathfrak{t}$ Lie algebra

$\mathfrak{t} \supset \Pi = \mathrm{Hom}(\mathbb{T}, T) \cong H_1 T \cong H_2 BT$

$\mathfrak{t}^* \supset \Lambda = \mathrm{Hom}(T, \mathbb{T}) = H^1 T = H^2 BT$

$$H^4(BT) = \mathrm{Sym}^2 \Lambda = \left\{ q : \Pi \to \mathbb{Z} \text{ quadratic} \atop q(n\pi) = n^2 q(\pi) \right\}$$

$\langle\ \rangle : \Pi \times \Pi \to \mathbb{Z}$ symmetric, even

$\langle \pi_1, \pi_2 \rangle = q(\pi_1 + \pi_2) - q(\pi_1) - q(\pi_2)$

$\langle\ \rangle$ on $\Pi \otimes \mathbb{R}$ is nondegenerate : assumption

$\tau : \Pi \to \Lambda$ injective

$\tau : \mathfrak{t} \to \mathfrak{t}^*$ $(\otimes \mathbb{R})$ isomorphism

As before, construct $K, L \to T \times T$ line bundles — K cocycle, L bihomomorphism

$$L_{x, y z} \cong L_{x,y} \otimes L_{x,z}$$

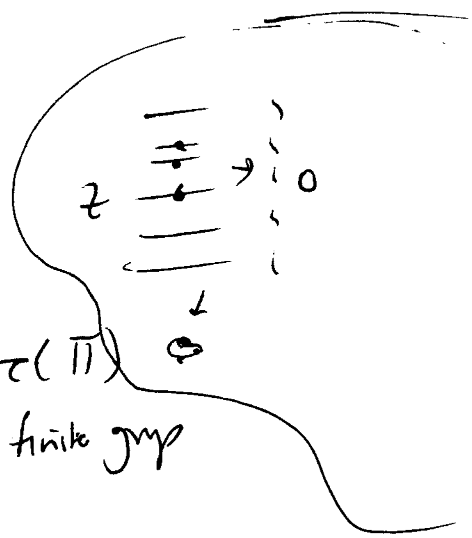
For fixed x, central extension

$$\mathbb{T} \rightleftarrows \widetilde{T}_x \to T$$

Λ_X is Λ-torsor of splittings

$$\begin{tikzcd} & C \cong \mathfrak{t} \times_\pi \Lambda \arrow[dl, "(\Lambda)"'] \arrow[dr, "(\mathfrak{t})"] & \\ T & & \Lambda/\tau(\pi) \arrow[ul, bend left, "s"] \end{tikzcd}$$

finite grp



holonomy $\tau : \pi \to \Lambda$

$$s(\lambda) = (\tau^{-1}(\lambda), \lambda)$$ project to $F \subset T$. finite grp

Idea:

$R = \{ W_x,\ x \in T : W_x = 0$ for all but finitely many $x \}$

$= FSh(T)$

$$(W_1 * W_2)_x = \bigoplus_{x_1 x_2 = x} K_{x_1 x_2} \otimes (W_1)_{x_1} \otimes (W_2)_{x_2}$$

Fact $Z(R) = FSh(C)$

Key: Replace by $Z^{\mathcal{E}}(R)$, $\mathcal{E} = FSh(\mathfrak{t})$