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Representation Theory and Topological Field Theory, IV

Motivation for homotopical/dg world

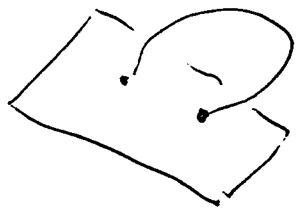
Physics TFTs "in nature"

as topological twists of supersymmetric QFT

Space of states: $\mathbb{Z}$-graded, $Q^2=0$ (BRST)

$\rightsquigarrow$ dg vector space

Likewise: open string states



again are cohomologies of complex

$\Rightarrow$ categories of branes as dg categories

Math Function theory 201

$\mathrm{Fun}(X)$ • π^* π_*

e.g. X compact oriented manifolds $\rightsquigarrow$ $H^*(X)$ de Rham

$\cup$: ring, $\pi: X \to Y$ π^*, $\pi_* = \int_\pi$

Poincaré + Künneth

$$\mathrm{Hom}(H^*(X), H^*(Y)) = H^*(X)^* \otimes H^*(Y) = H^*(X) \otimes H^*(Y) = H^*(X \times Y)$$

integral transform

integral transforms

$$f \longmapsto \int_{\pi_2} K \cup \pi_1^* f$$

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

Freed-Hopkins-Teleman, using K^*

$H^*(X)$ = cohomology of de Rham / Čech / simplicial] dg vector spaces

all quasi-isomorphic

<u>Geometric Function Theory</u>

a new kind of functional / harmonic analysis in algebraic geometry

function spaces (L^2, C^∞, distribution) $\rightsquigarrow$ dg categories

Fun $\rightsquigarrow$ Shv

codim 1 $\rightsquigarrow$ codim 2

spaces of states $\longrightarrow$ Categories of branes bounded

derived categories

unbounded

- coherent sheaves (f.g. modules / finite rank vector bundles)
- quasi-coherent sheaves (all modules / infinite rank vect bdls)
- $\mathcal{D}$-modules

e.g. Fourier-Mukai (T-duality)

$\mathcal{P}$ Poincaré line bundle

$$\begin{tikzcd} & A \times A^\vee \arrow[dl] \arrow[dr] & \\ A & & A^\vee \end{tikzcd}$$

A abelian variety, $A^\vee$ dual

e^{ixt} on $\mathbb{R} \times \mathbb{R}$

$$D(A) \to D(A^\vee)$$

$$\pi_{2*}(\pi_1^* \mathcal{F} \otimes \mathcal{P})$$

$H^0(\mathrm{Hom}^\bullet(A,B)) = \mathrm{Hom}$(deriv. catg)

$$A \longrightarrow B$$

Properties

- Refined invariant of X (variety or stack, like $Bun_G X$)

  $D(X)$ detects deformation theory

  detects X up to discrete ambiguity

  recover $H^*_{dR}(X)$ ($\mathbb{Z}/2$ graded)

  $H^*_{Dol}(X)$, Hodge structure, K-theory

- $f : X \to Y$

  $$f_* : D(X) \rightleftarrows D(Y) : f^*$$

Varieties: Toën
general (stacks): Ben-Zvi–Francis–Nadler

"Schwarz kernel theorem"

All continuous functors : $D(X) \to D(Y)$

are given by integral transform

$$\begin{tikzcd} & X \times Y \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

$$K \in D(X \times Y) = Fun(D(X), D(Y))$$

$$K * F = \pi_{2*}(\pi_1^* F \otimes K)$$

Künneth - Poincaré (BenZvi - Francis - Nadler)

$$D(X\times Y) \simeq D(X)\otimes D(Y)$$

$$D(X) = \mathrm{Hom}(D(X), \underbrace{D(\cdot)}_{= \text{dg vect}})$$

[quasicoh] $D(X)$, $\otimes$

$$X = \text{"Spec"}\, D(X)$$

[BenZvi - Francis - Nadler, in progress]

$\Rightarrow$ theory

Costello: topological B-model

Hopkins - Lurie: string topology

Cohomological (dg) TFT

<u>2 tiers</u> n Cob category (or homotopical version)

objects: closed $n-1$ manifolds N 

$\mathrm{Hom}(N_1, N_2)$ = <u>Space</u> of n-manifold cobordisms

Space $\longrightarrow$ H_0 or Π_0 Vector space

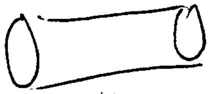
$\longrightarrow$ H_* graded vector space

$\longrightarrow$ C_* chains dg vect.

dg TFT :

$\mathrm{Chains}(n\mathrm{Cob}) \longrightarrow \mathrm{dg\ Vect}$

- $N^{n-1} \longmapsto Z(N)$ complex

N_1  N_2 is a 0-cycle in $C_*(\mathrm{Hom}(N_1, N_2))$

M

$\Rightarrow$ chain map of deg 0

$Z(M) : Z(N_1) \to Z(N_2)$

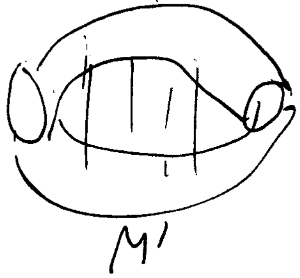
- given k-dim family $M(x)$, $x \in \mathbb{R}^k$ of cobordism $N_1 \to N_2$

$\Rightarrow$ k-chain $\in C_k(\text{cobordisms})$

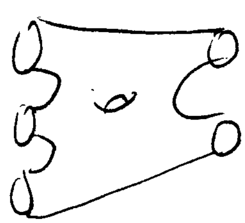
$\Rightarrow$ chain map of degree k

$Z(N_1) \to Z(N_2)$

M



M'

e.g. $n=2$ 

Space of 2-manifolds with given topology

$\simeq M_{g,n} \simeq B(\mathrm{Diff}\,\Sigma_{g,n})$

$\parallel$

space of surfaces of genus g, n punctures

3 tiers

Ob	$n-2$	manifolds	dg category
Mor	$n-1$	manifold cobordism	functors
2-mor	n	manifold ~~w/~~ cob w/corner	natural transformation

2d TFT

$\bullet \longmapsto$ dg category

Costello, Kontsevich-Soibelman

2d TFT $\Longleftrightarrow$ NC CY $\supset D(X)$ ~~x~~ CY manifold

(CY categories)

dg categories "with traces"