21 July 2008
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D-branes, I

D-branes in TQFT

A-branes $\leftarrow$ $\boxed{\text{mirror symmetry}}$ $\rightarrow$ B-branes

(derived) Fukaya category   Derived Category

2003 TASI lectures

1. Basic derivation
2. Quivers
3. Matrix factorization

Witten: hep-th/9112056.

$N=(2,2)$ SCFT $\varphi : \Sigma \to X$

Kähler manifold

	A	B	
ψ_+^{i}	$\varphi^* T_X$	$K_\Sigma \otimes \varphi^* T_X$	Anomaly forces
$\psi_+^{\bar{i}}$	$K_\Sigma \otimes \varphi^* \bar{T}_X$	$\varphi^* \bar{T}_X$	$c_1(T_X)=0$
ψ_-^{i}	$\bar{K}_\Sigma \otimes \varphi^* T_X$	$\bar{K}_\Sigma \otimes \varphi^* T_X$	(B-model)
$\psi_-^{\bar{i}}$	$\varphi^* \bar{T}_X$	$\varphi^* \bar{T}_X$	X is a CY

B-model BRST operator looks like $\bar{\partial}$

Closed string vertex ops are elts of

$$H^{0,q}_{\bar{\partial}}(X, \Lambda^p T_X)$$

finite-dim'l vector space

(A-model: de Rham cohomology)

D-branes: boundary conditions on Σ

$$\partial_z \varphi^I = R^I{}_J(\varphi)\, \bar{\partial}_z \varphi^J$$

+1 eigenvalue for R $\Rightarrow$ Neumann

−1 — — — $\Rightarrow$ Dirichlet

Try $\psi^I_+ = R^I{}_J(\varphi)\, \psi^J_-$

Split this eqn into holo + antiholo indices

<u>A-model</u>

$$R = \begin{array}{c|c} & \begin{array}{cc} \hat{\imath} & \bar{\imath} \end{array} \\ \begin{array}{c} j \\ \bar{j} \end{array} & \left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right) \end{array}$$

X has an almost complex structure $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$RJ + JR = 0$

$\Rightarrow$ J exchanges $\pm R$ eigenvals

$\Rightarrow$ D-branes are Lagrangian submanifolds

<u>B-model</u>

$$R = \begin{array}{c c} & \begin{array}{cc} i & \bar{i} \end{array} \\ \begin{array}{c} j \\ \bar{j} \end{array} & \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \end{array}$$

$RJ + JR = 0$

$\Rightarrow$ J preserves the R eigenvectors

$\Rightarrow$ D-branes are holomorphic submanifolds

So, for example, a D-brane can cover the whole of X.

Putting in Chan-Paton factors we obtain holomorphic vector bundles.

Associated to boundary terms in action

$$E \to X$$

Open string vertex ops are elts of

$$H^{0,q}_{\bar\partial}(X, \mathrm{End}(E)).$$

If we have two D-branes E, F,

open string from E to F lie in $H^{0,q}_{\bar\partial}(X, \underbrace{\mathrm{Hom}(E,F)}_{\cong\, E^\vee \otimes F})$

Ghost number of an open string in $H^{0,q}_{\bar\partial}(X, \mathrm{End}(E))$ is q.

What about an open string in $H^{0,q}_{\bar\partial}(X, \mathrm{Hom}(E,F))$?

We are free to associate a ghost number with the D-branes themselves $\mu(E)$

ghost number $q + \mu(E) - \mu(F)$.

D-brane category

Objects: D-branes (holo vector bundles)

Morphisms: Hilbert spaces of open string states

May as well restrict to $q=0$

Let us ~~assume~~ denote a ghost number shift by $[n]$.

$$\mu(E[n]) = \mu(E) + n.$$

$$\mathrm{Hom}^{0,q}_{\bar\partial}(X, \mathrm{Hom}(E,F)) = H^{0,0}_{\bar\partial}(X, \mathrm{Hom}(E, F[q]))$$

Composition of morphisms

- correlation function

$= \langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_\gamma \cdots \rangle$

a = D-brane $\qquad V_{aa} \longrightarrow \mathbb{C}$

This map is ghost number $-d$, $d = \dim X$.

given by Serre duality:

$$H^{0,d}(X, \mathrm{End}(E)) \cong H^{0,0}(X, \mathrm{End}(E))^\vee$$
$$\hspace{5em} \ni 1$$

is given by wedge product followed by Serre duality

- Perfect pairing $\quad V_{ab} \sim V_{ab}^\vee$

$V_{ab} \otimes V_{bc} \otimes V_{ca} \to \mathbb{C}$

$V_{ab} \otimes V_{bc} \longrightarrow V_{ac}$ $\quad$ Composition of morphisms

Combining D-branes

Given an open string $f \in \mathrm{Hom}(a,b)$,
can we "condense the tachyon" to bind a and b?
Let c be the resulting combination

$$\begin{tikzcd} & c \arrow[dl, "{[1]}"'] & \\ a \arrow[rr, "f"'] & & b \arrow[ul] \end{tikzcd}$$

c is b plus "anti-a"

$$\begin{tikzcd} & 0 \arrow[dl, "{[1]}"'] & \\ a \arrow[rr, "1"'] & & a \arrow[ul] \end{tikzcd}$$

~~$E \oplus F$~~

$$0 \to E \to \cancel{E \oplus F} \to F \to 0$$

G = deformation of $E \oplus F$

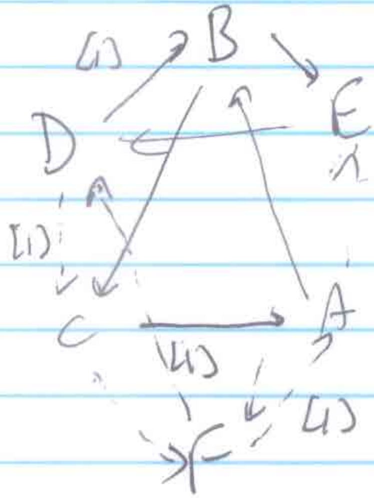
$$\mathrm{Ext}^1(F, E)$$

if c is "$b + \bar{a}$"

then b is "$a + c$", from a map $c \to a[1]$

$$\begin{tikzcd} & c \arrow[dl] & \\ a[1] \arrow[rr, "{[1]}"'] & & b \arrow[ul] \end{tikzcd}$$

Associativity of combining 3 D-branes



$E = B + D$

$B = A + C$

$E = (A+C) + D$
$= A + (C+D)$

$F = C + D$

~~Octagon~~ Octahedral axiom

X projective   Thus, the D-brane category is a "triangulated category"

We need to add more D-brane to holo. vector bundles

You obtain ~~derived~~ the bounded derived category of coherent sheaves.

Coherent sheaves:

e.g. consider $\mathcal{O}_X$ and $\mathcal{O}_X(-D)$, $c_1 = -D$

D is a divisor

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

D(X) consists of complexes of sheaves (vector bundles)

$$0 \to \mathcal{E}^{-2} \to \mathcal{E}^{-1} \to \mathcal{E}^{0} \to \mathcal{E}^{1} \to \mathcal{E}^{2} \to \cdots$$

Quasi-Isomorphism

$$\begin{tikzcd}
\to \mathcal{E}^{-2} \arrow[r] \arrow[d] & \mathcal{E}^{-1} \arrow[r] \arrow[d] & \mathcal{E}^{0} \arrow[r] \arrow[d] & \mathcal{E}^{1} \arrow[r] \arrow[d] & \mathcal{E}^{2} \to \arrow[d] \\
\to \mathcal{F}^{-2} \arrow[r] & \mathcal{F}^{-1} \arrow[r] & \mathcal{F}^{0} \arrow[r] & \mathcal{F}^{1} \arrow[r] & \mathcal{F}^{2} \to
\end{tikzcd}$$

Commutes

quasi-isomorphism if it induces the identity on cohomology

$$D(X) = \frac{\text{chain complexes}}{\text{q-iso}}$$