28 July 2008

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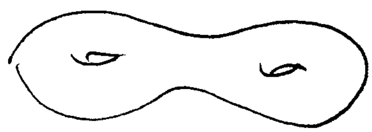
Introduction to Hecke functors and geometric Satake

Geometric Satake correspondence

Lusztig, Segal, Beilinson-Drinfeld, Mirković-Vilonen

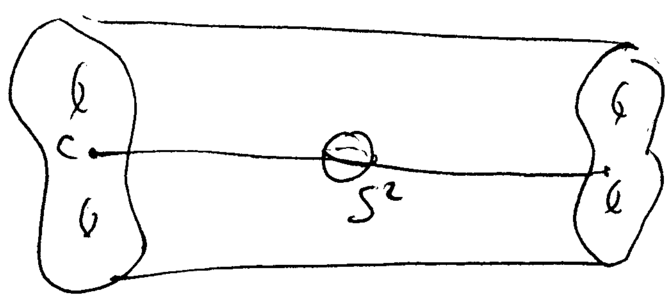
Fix C complex curve

G reductive group. $\supset G_c$ max compact



C

$$Z_G(C) = D\text{-mod}(Bun_G(C))$$
$$= A\text{-branes}(T^* Bun_G(C))$$

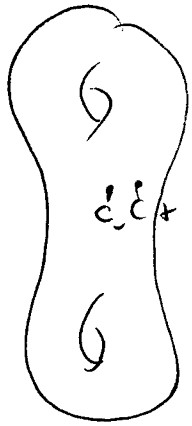


Rough picture

$$Z_G(S^2) \circlearrowright Z_G(C)$$

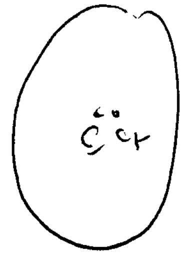
$\uparrow$

action for each $c \in C$



$\tilde{C}$

local picture



$\mathbb{D} = D_- \sqcup D_+$

away from 0

a 0_+

$$\begin{tikzcd}
 & Bun_G(\tilde{C}) = \text{Hecke} \arrow[dl, "p_-"'] \arrow[dr, "p_+"] \arrow[dd] & \\
Bun_G(C) & & Bun_G(C) \\
 & Bun_G(\mathbb{D}) &
\end{tikzcd}$$

Source of kernels

<u>Goal</u>: Study $D\text{-mod}(Bun_G(\mathbb{D}))$

$$Bun_G(\mathbb{D}) = L_+G \backslash LG / L_+G$$

$$LG = G((t))$$
$$L_+G = G[[t]]$$

<u>Ex</u> Compare to G-bundles on S^2

Combinatorics

$T \subset G$, max. torus

$$\Lambda_T = \mathrm{Hom}(\mathbb{C}^\times, T) \hookrightarrow LG$$

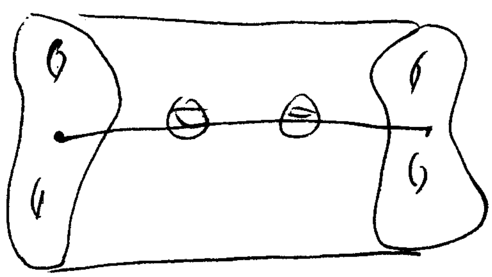
Cartan decomp $\Lambda_T/W \simeq$ Set of orbits in $L_+G \backslash LG / L_+G$

$G = GL_n$

$$\Lambda_T = \begin{vmatrix} t^{k_1} & & 0 \\ & \ddots & \\ 0 & & t^{k_n} \end{vmatrix} \hookrightarrow GL_n(\mathbb{C}(t))$$

$k_1 \geq \cdots \geq k_n$

Classical Satake correspondence



Kernels compose (by concatenation)

~~F~~ Function rather than D-mods.

$$\mathbb{C}_c\left[L_+G \backslash LG / L_+G\right] \simeq K_0(\mathrm{Rep}\, G^\vee)$$

Isom of rings

Composition of operators

$\otimes$

Geometric Satake corr   lift to $\otimes$-cats

$$D\text{-mod}(L_+G\backslash LG/L_+G) \xleftrightarrow[\text{equiv}]{} \mathrm{Rep}\, G^\vee$$

Understand $Gr = LG/L_+G$ affine Grassmannian

$$\simeq \underbrace{\Omega G_c}_{\text{based loop group}} \quad (\text{polynomial loops})$$

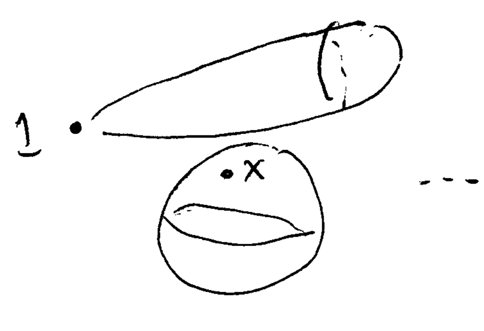
A. ΩG_c topological group

$G_c = U_n$, $G = GL_n$

$$\Omega U_n = F(\mathbb{CP}^{n-1}) \Big/ \langle \underbrace{x_1 \perp x_2 \perp \cdots \perp x_n}_{\text{orthonormal frame}} = 1 \rangle$$

Ex $G_c = \cancel{SO(3,\mathbb{R})}$ $SO_3\mathbb{R}$, $G = SO_3\mathbb{C}$

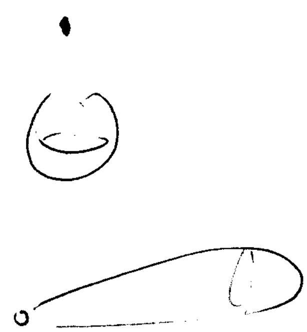
$$\Omega SO_3(\mathbb{R}) = F(\mathbb{CP}^1) \Big/ \langle x \cdot \underset{\text{antipodal}}{\alpha(x)} = 1 \rangle$$



even
____
odd

Components : word length mod 2

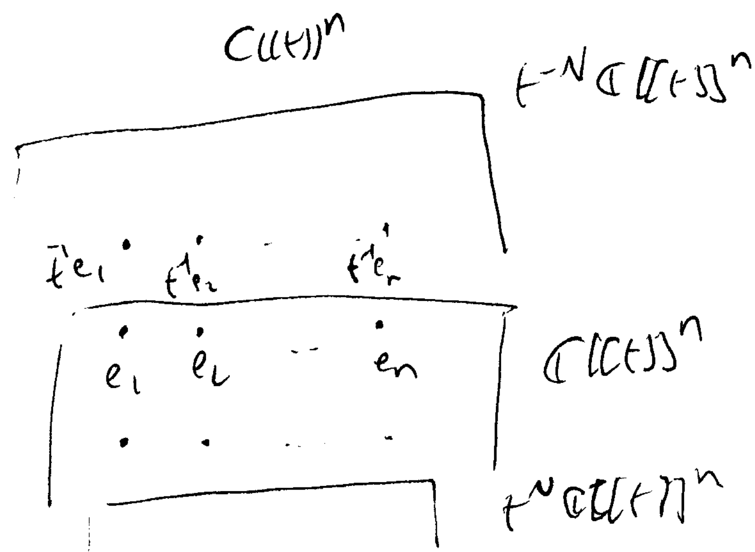
Spherical orbits (LG_+) : word length



B. Gr inf-dim Grassmannian

$G_c = U_n$, $G = GL_n$

$$Gr = \{ \underbrace{W \subset \mathbb{C}((t))^n}_{\mathbb{C}\text{-subspace}} \mid \exists \underset{N(W)}{N} \gg 0 \ \ t^N \mathbb{C}[[t]]^n \subset W \subset t^{-N} \mathbb{C}[[t]]^n, \ tW \subset W \}$$



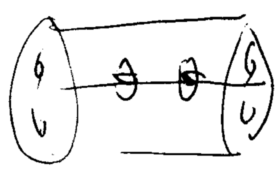
$Gr = LG/L_+G$

Components: $\dim W/W \cap \underbrace{\mathbb{C}[[t]]^n}_{W_0} - \dim W_0/W\cap W_0$

Sph. orbits: Λ_T/W_G

Geometric Satake corr revisited

't Hooft		Wilson
Spherical D-mods on Gr	$\longleftrightarrow$	Rep $G^\vee$
convolution (mult. of ΩG_c, rel pos. of Gr)	$\longleftrightarrow$	$\otimes$

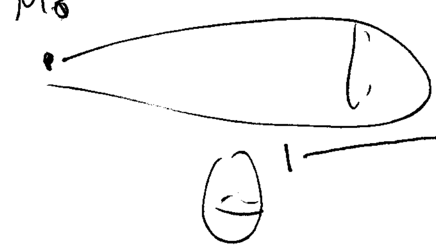


~~Hom(zero section, M)~~ $\longmapsto$ Underlying vector space

M D-module $\longmapsto$ Hom(zero section, M) $= \Gamma_{D\text{-mod}}(Gr, M)$

Ex $G_c = SO_3\mathbb{R}$

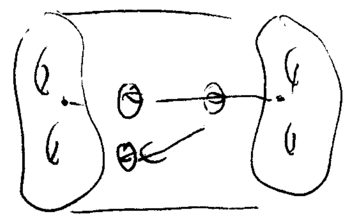
M_0 $\longrightarrow$ triv. rep $\mathbb{C}$



1 $\longrightarrow$ stand rep $\mathbb{C}^2$

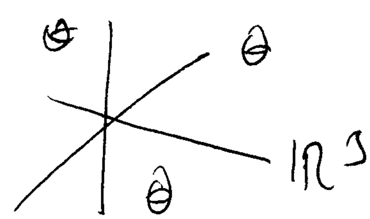
Rep $G^\vee$, $G^\vee = SL_2\mathbb{C}$

<u>Commutativity of quantities</u>



Can collide operators (locally) in $\mathbb{R}^3$ (not just along a line)

$\rightsquigarrow$ Spherical D-modules
E_3-category



Spherical D-modules is in fact E_∞-category
(completely comm.)

Twisted complexes of spherical D-modules
$\rightsquigarrow$ <u>only</u> E_3-category

<u>Derived</u> Twisted complexes $\longleftrightarrow$ $\mathrm{Conn}_{G^\vee}(S^2) = \langle 1 \rangle \cap_{G^\vee/G^\vee} \langle 1 \rangle$

triv principal

$\mathrm{Conn}_{G^\vee}(S^1) = \frac{G^\vee}{G^\vee}$

derived version of $BG^\vee$