24 July 2008
A. Kapustin

Gauge Theory and Geometric Langlands, II

A_μ (gauge field)
$\phi_\mu \in \Omega^1(\mathrm{ad}(E))$
$\sigma \in \Gamma(\mathrm{ad}_\mathbb{C}(E))$
⋮

$t = \frac{v}{u} \in \mathbb{C} \cup \{\infty\}$

$Q \sim Q_\ell + t Q_r$

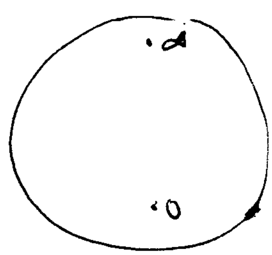
BPS equations

$$(F - \phi\wedge\phi + t\, d_A\phi)^+ = 0$$
$$(F - \phi\wedge\phi - t^{-1} d_A\phi)^- = 0$$
$$d_A * \phi = 0$$

$$S = \{Q_t, V\} + \frac{i\Psi}{4\pi}\int \mathrm{Tr}\, F\wedge F$$

$$\Psi = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}\,\frac{t^2-1}{t^2+1}$$

t



1) $t = \infty \Rightarrow \Psi = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} = \tau$

$t = 0 \Rightarrow \Psi = \bar\tau$

2) $t = \pm 1 \Rightarrow \Psi = \frac{\theta}{2\pi}$

3) $t = \pm i \Rightarrow$ no e^2 or θ-dependence

$\mathcal{A} = A + i\phi$

$\mathcal{F} = d\mathcal{A} + \mathcal{A}^2 = 0.$

$$S: \tau \mapsto \frac{-1}{n\tau}, \quad n=1,2,3$$

$$t \mapsto \frac{-\tau}{|\tau|} t$$

Special case: $\theta = 0$.

$$t \mapsto -i t$$

$$t = i \mapsto t = 1$$

Reduction to 2d

$$M_4 = C \times \Sigma$$

C: compact Riemann surface

Σ: Riemann surface

$$\phi = \phi_C + \phi_\Sigma$$

$$A = A_C + A_\Sigma$$

Vacua are described by

Hitchin eqns
$$\begin{cases} F_C - \phi_C \wedge \phi_C = 0 \\ d_{A_C} \phi_C = 0 \\ d_{A_C} * \phi_C = 0 \end{cases}$$

---

top. σ-model on Σ with target $\mathcal{M}_{Hit}(G, C)$

$\mathcal{M}_{Hit}(G, C)$ is hyperKähler

Consider the space of pairs (A_C, ϕ_C).

$$ds^2 \sim \int \mathrm{Tr}\left(\delta A_z \otimes \delta A_{\bar z} + \delta A_{\bar z} \otimes \delta A_z + \delta\phi_z \otimes \delta\phi_{\bar z} + \delta\phi_{\bar z} \otimes \delta\phi_z \right) d^2z$$

HK

I, J, K

$IJ = K$

I: $A_{\bar z}, \phi_z$, $\mathcal{A}_z$

J: $A_z + i\phi_z$, $A_{\bar z} + i\phi_{\bar z} = \mathcal{A}_{\bar z}$

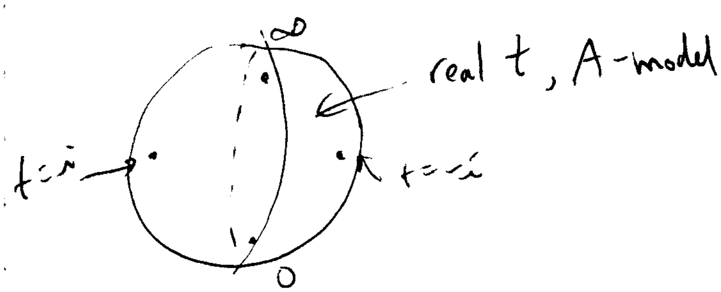
K: $A_{\bar z} - \phi_{\bar z}$, $A_z + \phi_z$

$\delta\phi, \delta A$

$$\omega_I \sim \int_C \mathrm{Tr}\; \delta A \wedge \delta A - \delta\phi \wedge \delta\phi$$

$$\omega_J \sim \int_C \mathrm{Tr}\left(\delta\phi_{\bar z}\, \delta A_z + \delta\phi_z\, \delta A_{\bar z} \right)$$

$$\omega_K \sim \int_C \mathrm{Tr}\left(\delta\phi \wedge \delta A \right)$$



$$\mathcal{A}_c = A_c + i\phi_c$$

Hitchin eqs: in complex structure I

$$F_{z\bar{z}} - [\phi_z \, \phi_{\bar{z}}] = 0 \qquad \cong \mathcal{M}_{\substack{\text{Stable} \\ \text{Higgs}}}(G, C)$$

$$\bar{\partial}_A \phi_z = 0$$

$$\Omega \sim \int_C \mathrm{Tr}\, \delta A_{\bar{z}}\, \delta\phi_z = \omega_J + i\omega_K$$

$t = i$: B-model wrt complex structure J

$$\mathcal{A} = A + i\phi$$

$$\mathcal{F} = d\mathcal{A} + \mathcal{A}^2 \qquad \mathcal{M}_{Hit}(G, C) \cong \mathcal{M}_{\substack{\text{stable} \\ \text{flat}}}(G_{\mathbb{C}}, C)$$

Hitchin: $\mathcal{F} = 0$

$$d_A * \phi = 0$$

$t = 1$; A-model in complex structure $K = IJ$

S-duality: $(t = i \to t = 1)$

B-model for $\mathcal{M}_{\substack{\text{stable} \\ \text{flat}}}(G_{\mathbb{C}}, C) \cong$ A-model for $\mathcal{M}_{Hit}({}^LG, C)_K$

A-model for $\mathcal{M}_{\substack{\text{Higgs} \\ \text{bundle}}}(G, C) \cong$ A-model for $\mathcal{M}_{\substack{\text{Higgs} \\ \text{bundle}}}({}^LG, C)$

$(t = \infty \to t = \infty)$

$$M_4 \times C \times \Sigma$$

$$\begin{tikzcd}
\text{flat } G_{\mathbb{C}}\text{-connection} \arrow[r] \arrow[d] & \text{A-brane in } \mathcal{M}_{Hit}({}^LG, C)_K \arrow[dl] \\
\text{D-modules on } \mathrm{Bun}_{{}^LG}(C) &
\end{tikzcd}$$