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Gauge Theory and Geometric Langlands, I

1) YM theory

$$A_\mu dx^\mu \in \mathrm{Conn}(E,M).$$ ← principal G-bundle

$$S = \frac{1}{2e^2}\int_M \mathrm{Tr}\, F\wedge * F + \frac{i\theta}{8\pi^2}\int_M \mathrm{Tr}\, F\wedge F$$

$$\sim \mathrm{tr}\, F_{\mu\nu}^2$$

$$F = dA + A^2$$

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$$

If $M = \mathbb{R}^4$ then $\tau \to \tau + 1$ is a symmetry

$$\theta \to \theta + 2\pi$$

$$S \to S + \underbrace{\frac{i}{4\pi}\int \mathrm{Tr}\, F\wedge F}_{2\pi i\,\#}$$

$$e^{-S} \to e^{-S}$$

It would be nice if $\tau \to -\frac{1}{\tau}$ were a symmetry

$$\left(\text{if } \theta = 0,\quad \frac{e^2}{4\pi} \to \left(\frac{e^2}{4\pi}\right)^{-1}\right.$$

Electric probes : irreducible reps of G, or hom from T to $U(1)$, modulo W

Magnetic probes (GNO): homomorphism from $U(1) \to G$, modulo conjugation, or or hom from $U(1) \to T$ modulo Weyl group

$\Lambda_{weight}(G) = \mathrm{Hom}(T, U(1))$

$\Lambda_{coweight}(G) = \mathrm{Hom}(U(1), T)$

$G \longleftrightarrow {}^L G$

$\Lambda_{weight}(G) = \Lambda_{weight}({}^L G)$

$\Lambda_{coweight}(G) = \Lambda_{weight}({}^L G)$

$G = U(N)$, ${}^L G = U(N)$

$G = SU(N)$ ${}^L G = SU(N)/\mathbb{Z}_N$

$G = SO(2N+1)$ ${}^L G = USp(N)$

$G = G_2$ ${}^L G = G_2$

2) $N=4$ SYM $M = \mathbb{R}^4$

Bose fields $\begin{cases} A_\mu \in \mathrm{Conn}(E) \\ \phi^i \in \Gamma(\mathrm{ad}\, E) \quad i = 1, \dots, 6 \end{cases}$

Fermi fields $\begin{cases} \psi^a \in \Gamma(S_+ \otimes \mathrm{ad}\, E) \quad a = 1, \dots, 4 \\ \psi_a \in \Gamma(S_- \otimes \mathrm{ad}\, E) \quad a = 1 \dots 4 \end{cases}$

$$S = \frac{1}{2e^2}\int \mathrm{Tr}\, F\wedge * F + \frac{i\theta}{8\pi^2}\int \mathrm{Tr}\, F\wedge F$$
$$+ \frac{1}{e^2}\sum_i \int \mathrm{Tr}\, d_A\phi^i \wedge * d_A \phi^i$$
$$+ \frac{1}{e^2}\sum_{i<j}\int \mathrm{Tr}\,[\phi_i,\phi_j]^2\, \mathrm{vol}$$
$$+ \text{ fermionic term}$$

call S if $n=1$

$$S : \tau \to -\frac{1}{n\tau} \qquad\qquad (ST)^3 = -1$$
$$G \to {}^L G$$

Symmetries:

$$\text{Poincaré} \xrightarrow{1} \text{Poincaré}$$
$$\text{Spin}(6) \xrightarrow{1} \text{Spin}(6)$$
$$Q_{l_*} \longrightarrow e^{i\phi} Q_l \qquad\qquad \phi = \phi(\tau)$$
$$Q_r \longrightarrow e^{-i\phi} Q_r$$

$$e^{i\phi} = \sqrt{\frac{|\tau|}{-\tau}}$$

on $\mathbb{R}^4$, the left handed ones have central charge

$$\{Q^a_\alpha, Q^b_\beta\} = \varepsilon_{\alpha\beta} Z^{ab}$$

$$Z \approx \frac{n + m\tau}{\mathrm{Im}\,\tau} \qquad\qquad \begin{array}{l} n \to m \\ m \to -n \end{array}$$

3) GL-twist

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_{\text{fields}} \mu e^{-S} \mathcal{O}_1 \cdots \mathcal{O}_n$$

Suppose in the space of fields (∞-dim supermanifold) we have an odd vector field Q s.t.

$$2Q^2 := \{Q, Q\} = 0 \qquad (Q \text{ is called a BRST operator})$$

$$Q(S) = 0$$

$$\mathcal{L}_Q \mu = 0.$$

Restrict observables $\mathcal{O}_i$ by $Q(\mathcal{O}_i) = 0$.

Then $\langle \mathcal{O} \rangle = 0$ if $\mathcal{O} = Q(A)$

$\Rightarrow$ observables are elements of Q-cohomology

Further, suppose $\frac{\delta S}{\delta g_{\mu\nu}} = \{Q, \cdot\}$

$\Rightarrow$ get a topological field theory

Typically, in this case

$$S = \underbrace{\{Q, V\}}_{\text{coupling dependent}} + \text{topological piece}$$

$\Rightarrow$ 1-loop exact

$\rho : Spin(4) \to Spin(6)$

$Q_l^a \sim (4, 2_l) \ni Q_l$

$Q_{r,a} \sim (\bar{4}, 2_r) \ni Q_r$

$\{Q_l, Q_l\} = \{Q_r, Q_r\} = \{Q_l, Q_r\} = 0$

$Q = u Q_l + v Q_r$

$(u:v) \in \mathbb{P}^1$.

$t = \frac{v}{u} \in \mathbb{C} \cup \{\infty\}$

$$\left.\begin{array}{l} A_\mu \in Conn(E, M) \\ \phi_M \in \Omega^1(ad(E)) \\ \sigma = \phi_5 + i\phi_6 \in \Gamma(ad(E) \otimes \mathbb{C}) \end{array}\right\} \text{bosonic fields}$$

$$\left.\begin{array}{l} \eta, \tilde{\eta} \in \Gamma(ad(E) \otimes \mathbb{C}) \\ \psi, \tilde{\psi} \in \Omega^1(ad(E) \otimes \mathbb{C}) \\ \chi \in \Omega^2(ad(E) \otimes \mathbb{C}) \end{array}\right\} \text{fermionic fields}$$

Supersymmetry:

$$\delta A = i(u\psi + v\tilde{\psi})$$
$$\delta \phi = i(v\psi - u\tilde{\psi})$$
$$\delta \sigma = 0$$
$$\delta \chi^+ = u(F - \phi\wedge\phi)^+ + v(d_A\phi)^+$$
$$\delta \chi^- = v(F - \phi\wedge\phi)^- - u(d_A\phi)^-$$
$$\vdots$$

$$\{Q, \operatorname{tr}_R P\exp\left(-\oint_\gamma A\right)\} \neq 0$$

$$t = i, \quad \delta A = iu(\psi + i\tilde{\psi})$$
$$\delta\phi = -u(\psi + i\tilde{\psi})$$

$$\delta(A + i\phi) = 0$$

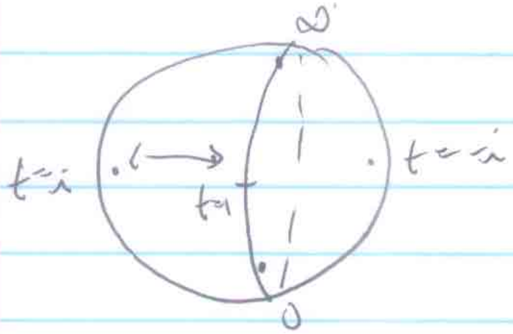
$$\mathcal{A} = A + i\phi$$

$$\{Q, \operatorname{tr}_R P\exp\left(-\oint_\gamma \mathcal{A}\right)\} = 0.$$

$$t = -i, \quad \delta(A - i\phi) = 0.$$

Path integral localizes on

$$(F - \phi\wedge\phi + t\, d_A\phi)^+ = 0$$
$$(F - \phi\wedge\phi - t^{-1} d_A\phi)^- = 0$$
$$d_A * \phi = 0.$$


∞
$t=i$
$t=1$
$t=-i$
0