24 July 2008
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Surface Operators and Ramification, II

$2+2=4$

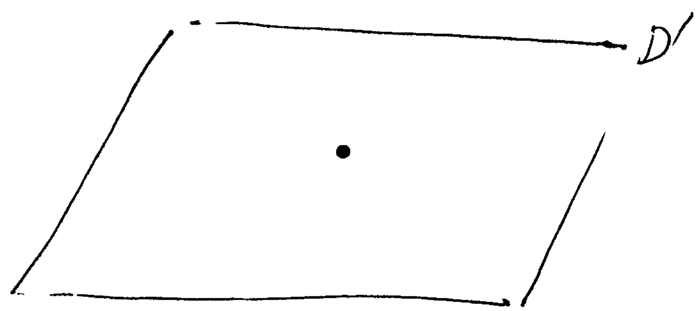
M 4-manifold

D 2-manifold

support of surf. op

D' normal bundle

$D = \mathbb{R}^2 \subseteq \mathbb{R}^4$



$$\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$$
$$\circlearrowright \quad \circlearrowright$$
$$SO(2) \quad SO(2)$$

$F_A = dA + A \wedge A$ 2-form

$F_A = 2\pi\alpha\,\delta + \ldots$

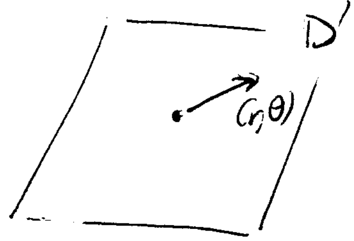
$$\langle \mathcal{O} \rangle = \int_{(\cdots)} e^{-S} (-)$$

Ex $G = U(1)$

$$\exp\left(i\eta \int_D F_A\right)$$

$\alpha, \eta \in [0,1)$, continuous parameters

$A = \alpha \, d\theta + \cdots$

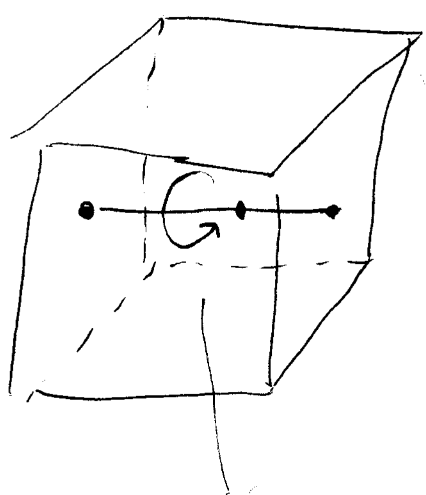


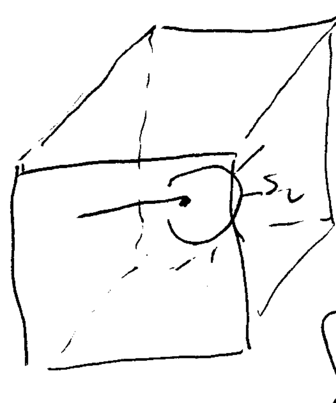
line operator

$W_\eta \,, T_\alpha$

$$W_q(\gamma) = \exp\left(iq\int_\gamma A\right)$$

$$= \exp\left(iq\int_D F\right) \qquad \gamma = \partial D$$





$\int_{S^2} \frac{F}{2\pi} = \alpha$

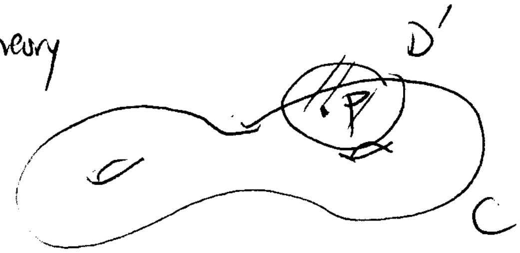
$\int_{S^1} A = 2\pi\alpha$

S-duality

surface operators

$(\alpha, \eta) \to (\eta, -\alpha)$

$\mathcal{N}=4$ gauge theory



$$M = C \times \Sigma$$

on C: Hitchin eqs.

$$\begin{cases} F_A - \phi \wedge \phi = 0 \\ d_A \phi = 0, \quad d_A * \phi = 0. \end{cases}$$

$\frac{r\,d\theta}{r}$

singularity: $A = \alpha\, d\theta + \dots$
(surface op)
$\phi = \beta \frac{dr}{r} - \gamma\, d\theta + \dots$

obey Hitchin eqs.

$\alpha, \beta, \gamma \in \mathfrak{t}$

$\mathfrak{t} = Lie(\mathbb{T})$

$\mathbb{T}$ = maximal torus of G.

$\mathcal{W}$ = Weyl gp. $\rightsquigarrow$ $\alpha, \beta, \gamma \in \mathfrak{t} \times \mathfrak{t} \times \mathfrak{t} / \mathcal{W}$

gauge transformation by a $\mathbb{T}$-valued function

$$f = \exp(\theta \cdot u)$$

$$\alpha \to \alpha + u$$

$$u \in \Lambda_{cochar} \Rightarrow \alpha \in \mathfrak{t}/\Lambda_{cochar} = \mathbb{T}$$

$\cancel{\exp(i\theta \cdot u)}$

labeled by $(\alpha, \beta, \gamma) \in (\mathbb{T} \times \mathfrak{t} \times \mathfrak{t})/\mathcal{W}$

along $D \subset M$,

$$G \to \mathbb{T}$$

$\mathbb{T}$-bundles over D are classified by $m \in \Lambda_{cochar} = \mathrm{Hom}(U(1), \mathbb{T})$

$$\exp(i \eta\, d)$$

$$\eta \in \mathrm{Hom}(\Lambda_{cochar}, U(1)) = {}^L\mathbb{T} = {}^L\mathfrak{t}/\Lambda_{char}$$

$$(\alpha, \beta, \gamma, \eta) \in (\mathbb{T} \times \mathfrak{t} \times \mathfrak{t} \times {}^L\mathbb{T})/\mathcal{W}$$

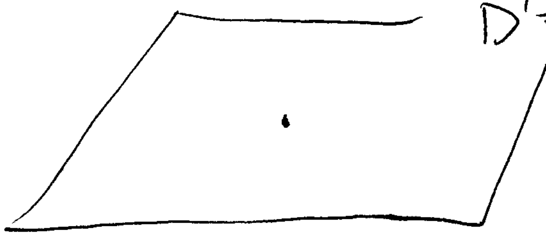
S-duality

Ex $G = SU(2)$

$\mathbb{T} = U(1)$

$$\begin{cases} A = a(r)\, d\theta + h(r) \dfrac{dr}{r} \\ \phi = b(r) \dfrac{dr}{r} - c(r)\, d\theta \end{cases}$$

$h = 0$ by gauge choice



$D' = \mathbb{R}^2 \setminus \{0\}$

$$\begin{cases} a(r) \to \alpha \\ b(r) \to \beta \\ c(r) \to \gamma \end{cases} \qquad r \to 0 \qquad \alpha = 0,\ \beta = 0,\ \gamma = 0$$

if we set $s = -\ln r$

Hitchin's eqns $\rightsquigarrow$ Nahm's eqns

$$\begin{cases} \dfrac{da}{ds} = [b,c] \\ \dfrac{db}{ds} = [c,a] \\ \dfrac{dc}{ds} = [a,b] \end{cases}$$

i) trivial solution $a(s) = b(s) = c(s) = 0$

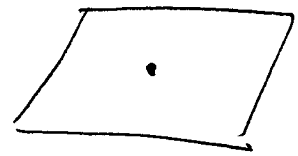
ii) non-trivial solution

$$a = \frac{-1}{s+f} R^{-1} J_1 R \qquad J_3 = [J_1, J_2] \text{ etc}$$

$$b = \frac{-1}{s+f} R^{-1} J_2 R$$

$$c = \frac{-1}{s+f} R^{-1} J_3 R$$

$f \in \mathbb{R}_+ \quad R \in SO(3)$

$\mathcal{M}_{Hit}(SU(2), C_{ram})$ $\quad C =$ 

$= \cancel{\mathbb{R}^+}\ \{0\} \cup (\mathbb{R}_+ \times S^3/\mathbb{Z}_2)$

$\cong \mathbb{R}^4/\mathbb{Z}_2 = \mathbb{C}^2/\mathbb{Z}_2 =$ nilpotent cone for $G_{\mathbb{C}} = SL(2,\mathbb{C})$

deformation / resolution $\qquad$ hyperkähler

$$\mathbb{H}^2 /\!/\!/ U(1) \qquad \vec{\mu} = (\alpha, \beta, \gamma)$$

$$\mathbb{C}^2/\mathbb{Z}_2 \rightsquigarrow \mathcal{C} \simeq T^*\mathbb{CP}^1$$

$$\left[\frac{\omega_I}{2\pi}\right] = \alpha \quad \left[\frac{\omega_J}{2\pi}\right] = \beta \quad \left[\frac{\omega_K}{2\pi}\right] = \gamma \quad \left[\frac{B}{2\pi}\right] = \eta$$