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# Introduction to the Langlands Program, III

## Langlands correspondence

Number fields $\longrightarrow$ Function fields for curves $/\mathbb{F}_q$ $\longrightarrow$ Algebraic Curves $/\mathbb{C}$ (geometric version) $\longrightarrow$ Mirror symmetry + S-duality

Taniyama–Shimura–Weil (led to proof of Fermat's Last Thm) – special case of L.C. for $GL_2$

$\mathbb{Q} \rightsquigarrow$ field of rational functions on $X$ – smooth projective curve $/\mathbb{F}_q$

$$\left[\begin{array}{l} \mathbb{F}_q \text{ – finite field with } q \text{ elements, } q = p^n, \ p \text{ prime} \\ \qquad \mathbb{F}_q = \{0, 1, \ldots, p-1\} \end{array}\right]$$

e.g. $X = \mathbb{P}^1$. $F = \mathbb{F}_q(\mathbb{P}^1) = \left\{ \frac{P(t)}{Q(t)} \,\middle|\, P, Q \text{ – relatively prime polynomials over } \mathbb{F}_q \right\}$

$\mathbb{Q} \approx$ rat. function on $2, 3, 5, 7, \cancel{9}, 11, \ldots, \infty$

$\overline{F}$ – separable closure of $F$

$\mathrm{Gal}(\overline{F}/F)$ – group of automorphisms of $\overline{F}$ preserving $F$

$$\left\{ \begin{array}{c} n\text{-dimensional} \\ \text{irreducible representation} \\ \text{of } \mathrm{Gal}(\overline{F}/F) \end{array} \right\}$$

$$\mathrm{Gal}(\bar F/F) = \varprojlim_{N} \pi_1^{et}(X \setminus \{x_1,\dots,x_N\})$$

$$\begin{tikzcd} \mathrm{Gal}(\bar F/F) \arrow[r] \arrow[d] & GL_n(\bar{\mathbb{Q}}_\ell) \\ \pi_1(X\setminus\{x_1,\dots,x_N\}) \arrow[ur] & \end{tikzcd}$$

$(\ell, p) = 1$

ramified at $x_1, \dots, x_N$

$$\left\{ \begin{array}{c} n\text{-dim irred. reps} \\ \text{of } \mathrm{Gal}(\bar F/F) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{(complex)} \\ \text{irreducible} \wedge \text{cuspidal automorphic} \\ \text{representation of } GL_n(\mathbb{A}_F) \end{array} \right\}$$

hom $\mathrm{Gal}(\bar F/F) \to {}^LG(\bar{\mathbb{Q}}_\ell)$

$G(\mathbb{A}_F)$ (reductive group)

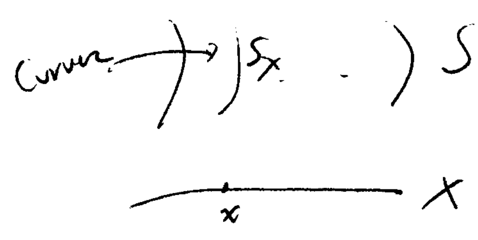
<u>Thm</u> (Drinfeld $n=2$, Lafforgue $n>2$)

$$\left\{ \begin{array}{c} \text{Frobenius eigenvalues, } Fr_x \\ \text{for all but finitely many } x \in X \end{array} \right\} \qquad \{ \text{Hecke eigenvalues} \}$$

We looked at elliptic curve $E/\mathbb{Q}$

$E \to$ 2-dim Galois rep.

Surface, which is a fibration over $X$




$H^i_{et}(S_x, \mathbb{Q}_\ell)$ - rep of $\mathrm{Gal}(\bar F/F)$.

$\mathbb{A}_F := \prod{}'$ (completion of $F$)

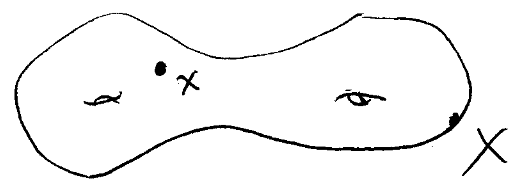
<u>Number fields</u>

$F = \mathbb{Q}$, completions = primes + $\infty$

<u>Function field</u>

$F = \mathbb{F}_q(X)$

Completions = points of $X$



$t_x$

$F \hookrightarrow \mathbb{F}_{q_x}((t_x))$ – formal Laurent power series
$= F_x$

$X = \mathbb{P}^1$

points: $\infty \amalg \mathbb{A}^1$

$\mathbb{A}^1$: spec $\mathbb{F}_q[t]$

points = irred. poly

$\mathbb{F}_q[t]/I_x \cong \mathbb{F}_{q_x}$ $\quad q_x = q^{\text{deg of poly defining } I_x}$

$$\mathbb{A}_F = \prod_{x \in X}{}' F_x = \{ (f_x)_{x \in X} : f_x \in \mathcal{O}_x \text{ for all but finitely many points} \}$$

$F_x \supseteq \mathcal{O}_x$ = formal Taylor series $\mathbb{F}_{q_x}[[t_x]]$

$$\mathrm{Fun}\left( GL_n(F) \backslash GL_n(\mathbb{A}_F) \right) \circlearrowleft GL_n(\mathbb{A}_F)$$

$\pi$ is an automorphic rep if it occurs in this space.

[irreducible] $\longleftrightarrow$ [cuspidal]

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If $\pi$ is irreducible $\pi = \bigotimes'_{x \in X} \pi_x \circlearrowleft GL_n(F_x)$

$GL_n(\mathbb{A}_F) = \prod'_{x \in X} GL_n(F_x)$

$GL_n(F_x) \cong GL_n(\mathbb{F}_{q_x}((t_x)))$

$\forall x, \pi_x \exists K_x \subseteq GL_n(\mathcal{O}_x)$ s.t. $\dim \pi_x^{K_x} = 1$.

$K_x = GL_n(\mathcal{O}_x)$ for almost all $x \in X$, more precisely, for $x \notin \{x_1, \dots, x_N\}$

(if $x = x_i$, $\pi_x^{GL_n(\mathcal{O}_x)} = 0$ and we need to take $K_x$ to be smaller)

Say that $\sigma : \mathrm{Gal}(\overline{F}/F) \to GL_n$ is <u>unramified</u> if $\{x_1, \dots, x_N\} = \emptyset$

i.e. $\mathrm{Gal}(\overline{F}/F) \to GL_n$, $\searrow \pi_1(X) \nearrow$

The corresponding $\pi$ will have $K_x = GL_n(\mathcal{O}_x)$ $\forall x \in X$

Kapustin-Witten

Tame ramification: Gukov–Witten

Focus on unramified case:

$$\dim \Pi_x^{GL_n(\mathcal{O}_x)} = 1.$$

$$GL_n(\mathcal{O}_F) = \prod_{x \in X} GL_n(\mathcal{O}_x)$$

$$\dim \Pi^{GL_n(\mathcal{O})} = 1.$$

$$\overset{\Psi}{V} \longrightarrow \text{function on } GL_n(F)\backslash GL_n(\mathbb{A}_F)/GL_n(\mathcal{O})$$

<u>Lemma</u> (A. Weil)

$GL_n(F)\backslash GL_n(\mathbb{A}_F)/GL_n(\mathcal{O})$ is the set of isomorphism classes of rank $n$ vector bundles on $X$.

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$$GL_n(F)\backslash GL_n(\mathbb{A})/\prod_{x \neq x_i} GL_n(\mathcal{O}_x) \times \prod_{i=1}^{N} K_{x_i}$$

$$K_{x_i} = \text{Iwahori} = \begin{pmatrix} t_x & & * \\ & \ddots & \\ t_x & & t_x \end{pmatrix}$$

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principal $G$-bundles