Geometric Langlands & non-abelian Hodge

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Langlands Workshop
KITP
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[–, Pantev] Langlands duality for Hitchin systems 0604617

–, Simpson

–, Pantev

GLC review:

- roots, chars, weights
- Bun, Loc, Db
- Hecke correspondences
- Hecke operators, automorphic sheaves
- stacks & gerbes
- Examples: GL(n), GL(1).

Abelianization:

- Higgs bundles, Hitchin system
- spectral + cameral covers
- Ancient approach: abelianized Hecke eigensheaves
- $\rightsquigarrow$ duality: Higgs v. ^{L}Higgs.

Classical limit:

- definition, conjecture, theorem, proof:
- duality along base
- " " fibers
- transcendental ingredient: Hodge theory
- duality for gerbes

non-abelian Hodge:

- review
- Big Picture
- Compatibility

Wobbly bundles:

- setup
- examples

Geometric Langlands Conjecture

$\exists$ natural equivalence of categories:

$$\boxed{c: D^b(\mathcal{L}oc) \xrightarrow{\sim} D^b({}^L\mathcal{B}un, \mathcal{D})}$$

sending structure sheaves of points $\mathbb{V}$ in $\mathcal{L}oc$ to automorphic $\mathcal{D}$-modules on ${}^L\mathcal{B}un$:

$$\boxed{H^{\mu}(c(\mathcal{O}_{\mathbb{V}})) = c(\mathcal{O}_{\mathbb{V}}) \boxtimes \rho^{\mu}(\mathbb{V})}$$

Notation:

C ; $G, T, \mathfrak{g}, \mathfrak{t}$; ${}^LG, {}^LT$, etc.

$\text{roots}_{\mathfrak{g}}$ $\subset$	char_G $\subset$	$\text{weights}_{\mathfrak{g}}$
$\text{coroots}_{\mathfrak{g}} = \text{roots}_{{}^L\mathfrak{g}}$ $\subset$	$\text{cochar}_G = \text{char}_{{}^LG}$ $\subset$	$\text{coweights}_{\mathfrak{g}} = \text{weights}_{{}^L\mathfrak{g}}$

Bun, ${}^LBun$: mod. sp. of semistable principal G, ${}^LG$ bundles on C

Loc, ${}^LLoc$: mod. sp. of semistable G, ${}^LG$ local systems $\mathbb{V} = (V, \nabla)$ on C

$\mathcal{B}un$, $\mathcal{L}oc$ etc.: the corresponding moduli stacks

Hecke correspondence :=

moduli space of quadruples $(V, V'; x, \beta)$:

- V, V': principal ${}^LG$-bundles on C
- $x \in C$
- $\beta : V|_{C \setminus x} \xrightarrow{\sim} V'|_{C \setminus x}$ an isomorphism

$$\begin{tikzcd} & \mathrm{Hecke} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ {}^L\mathrm{Bun} & & {}^L\mathrm{Bun} \times C \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{Hecke}^{\mu} \arrow[dl, "p^{\mu}"'] \arrow[dr, "q^{\mu}"] & \\ {}^L\mathrm{Bun} & & {}^L\mathrm{Bun} \times C \end{tikzcd}$$

$\lambda \in \mathrm{char}^+_{{}^LG} = \mathrm{cochar}^+_G$ dominant character of ${}^LG$

$\rho = \rho_\lambda$ irrep of ${}^LG$ with h.w. $= \lambda$

$\mu \in \mathrm{cochar}^+_{{}^LG} = \mathrm{char}^+_G$ dom. char. of G

$$\mathrm{Hecke}^{\mu} := \left\{ (V, V'; x, \beta) \,\middle|\, \begin{array}{l} \forall \lambda \in \mathrm{char}^+_{{}^LG} \text{ w/ } \rho = \rho_\lambda, \\ \rho(\beta) : \rho(V) \to \rho(V') \otimes \mathcal{O}_C(\langle \mu, \lambda \rangle x) \end{array} \right\}$$

- Fibers of p, q: ∞-dim'l <u>ind-schemes</u>

  $q^{-1}(V', x)$ = affine Grassmannian
- $\mathrm{Hecke}^{\mu} \subset \mathrm{Hecke}$: f.d. closed subscheme
- Hecke^{μ} is smooth $\Leftrightarrow$ μ minuscule weight
- $\mathrm{Hecke} = \varinjlim_{\mu} \mathrm{Hecke}^{\mu}$
- p^{μ}, q^{μ}: proper, loc. trivial fibrations.

$\boxed{\text{Hecke operators}}$ = integral transforms

$$\boxed{\begin{aligned} H^{\lambda} : D^b({}^L\mathrm{Bun}, \mathcal{D}) &\longrightarrow D^b({}^L\mathrm{Bun} \times C, \mathcal{D}) \\ \mathcal{M} &\mapsto q_!^{\lambda}\left(p^{\lambda *}\mathcal{M} \otimes IC^{\lambda}\right)[d_{\lambda}] \end{aligned}}$$

Similarly: H^{λ}_{x}, for $x \in C$.

$\Rightarrow$ Hecke algebra $\mathcal{A}$ generated by the H^{λ}_{x}.

abelian.

A $\mathcal{D}$-module $\mathcal{M}$ on ${}^L\mathrm{Bun}$ is a | Automorphic sheaf / Hecke eigensheaf

w/ eigenvalue $\mathbb{V} = (V, \nabla)$ a G-loc. sys. on C if

$$\boxed{H^{\lambda}(\mathcal{M}) = \mathcal{M} \boxtimes \rho^{\lambda}(\mathbb{V})} \quad \forall \lambda.$$

With this setup, GLC makes sense, except...

Discrepancy: ${}^L\mathrm{Bun}$ is disconnected.

$$\boxed{\begin{gathered}\pi_0({}^L\mathrm{Bun}) = H^2(C, \pi_1({}^LG)) = \pi_1({}^LG) = Z(G)^\wedge \\ D^\ell({}^L\mathrm{Bun}, \mathcal{D}) = \coprod_{\gamma \in Z(G)^\wedge} D^\ell({}^L\mathrm{Bun}^\gamma, \mathcal{D})\end{gathered}} \Rightarrow$$

while Loc is irreducible $\Rightarrow D^\ell(\mathrm{Loc})$ indecomposable.

So: replace Loc by

$$\boxed{\mathcal{L}oc := \text{moduli stack of semistable } G\text{-local systems}}$$

$\mathcal{L}oc$ is an alg. stack w/ coarse moduli space = Loc.

$\exists$ open substack $\mathcal{L}oc^{rs}$ of "regularly stable" local systems

$\mathcal{L}oc^{rs} \to \mathrm{Loc}^{rs}$: banded $Z(G)$-gerbe

$$\boxed{D^\ell(\mathcal{L}oc^{rs}) = \coprod_{\gamma \in Z(G)^\wedge} D^\ell(\mathcal{L}oc^{rs}; \gamma).}$$

Example: $G = {}^LG = GL(n)$

$\mathbb{V} = (V, \nabla) =$ rank n loc. sys. / C

Hecke algebra $\mathcal{H}$ is generated by H^i:

Hecke$^i = \{(V, V', x) \mid V \subset V' \subset V(x),\ i = \ell(V'/V)\}$

fiber $= Gr(i, n)$.

Example: $G = GL(1)$, Bun $=$ Pic

$H^1 : C \times \mathrm{Pic}^d(C) \longrightarrow \mathrm{Pic}^{d+1}(C)$

$x, L \longmapsto L(x)$ abel-Jacobi

$\mathbb{V} = (L, \nabla)$

$c(\mathbb{V}) =$ the unique loc. sys. on $\mathrm{Pic}(C)$ restricting to (L, ∇).

$(\pi_1(\mathrm{Pic}^0 C) = \pi_1(C) / [\,,\,]\ .)$

$Higgs = Higgs_{C,G} = \{(V,\varphi) \mid \varphi : V \to V \otimes \omega_C\}$

$\boxed{\text{Hitchin map:}}$

$$\boxed{\begin{array}{l} h : Higgs_{C,G} \to B_{C,G} := H^0(C, (\omega_C \otimes \underline{t})/W) = \bigoplus_{i=1}^{r} H^0(C, \omega_C^{\otimes d_i}) \\ (V,\varphi) \mapsto \quad \text{"}\varphi \bmod W\text{"} \leftrightarrow (I_1(\varphi), \dots, I_r(\varphi)) \end{array}}$$

Cameral covers:

$$\begin{tikzcd} C_b \arrow[r, phantom, "\subset"] \arrow[d] & \tilde{C} \arrow[r] \arrow[d] & Tot(C, \omega_C \otimes \underline{t}) \arrow[d] \\ C \times \{b\} \arrow[r, phantom, "\subset"] & C \times B \arrow[r] & Tot(C, (\omega_C \otimes \underline{t})/W) \end{tikzcd}$$

$h^{-1}(b) \sim Prym(C_b / C)$

$h : Higgs \to B$ integrable system

$Higgs \supset T^* Bun$

$\forall \mu \Rightarrow$ associated spectral covers $\bar{C}_b^{\mu}$, $\bar{C}^{\mu} \to C \times B$.

e.g. $G = GL(n) \Rightarrow \deg(\bar{C}/C) = n$.

[Do Gaitsgory '00]: $\pi : \tilde{C} \to C \times B$ determines an abelian group scheme J over $C \times B$. $h : Higgs \to B$ is a principal homogeneous stack over the ab. group stack $Tors_J$ of J-torsors.

$=$ "abelianization". (Hitchin, Atiyah, Beilinson–Kazhdan, Faltings, ...)

Ancient approach

$(G = GL(n))$

$$\begin{tikzcd} \overline{\mathcal{C}} & \overline{\mathcal{C}}_V \arrow[l] \arrow[d, "\pi_V"] \\ C\times B \arrow[u, leftarrow, "\pi"] & C\times T^*_V \mathrm{Bun} \arrow[l] \end{tikzcd}$$

$\mathbb{V} = (V, \nabla) \Rightarrow$

L on $\overline{\mathcal{C}}_V$, $\pi_{V*} L \cong V \otimes \omega_C^{-(n-1)/2}$

∇ induces holo. conn. on L

(Depends on $\sqrt{\omega_C}$: use $\mathrm{Ram}_\varphi \cong \pi_\varphi^* \omega_C^{\otimes(n-1)}$
The natural induced connection ∇' has residue $= \frac{1}{2}$ along Ram_φ. Set $\nabla = \nabla' + \frac{-1}{2} d\log f_\varphi$, where $(f_\varphi) = -\mathrm{Ram}_\varphi + 2(n-1)\pi_\varphi^* \xi$, ξ a section of $\sqrt{\omega_C}$)

Use case $G = GL(1)$: spread (L, ∇) to $(\widetilde{L}, \widetilde{\nabla})$ on $\mathrm{Pic}(\overline{\mathcal{C}}_V / C\times T^*_V\mathrm{Bun}) =: \widetilde{\mathrm{Higgs}} = \mathrm{Higgs} \times_B T^*_V \mathrm{Bun}$

Now push to Higgs $\Rightarrow$

$(\widetilde{V}, \widetilde{\nabla})$ on Higgs, mero loc. sys along fibers over B.

Fiber of $\widetilde{V}$ at $(E,\psi) \in \mathrm{Higgs}$: $\left((E,\psi) \leftrightarrow (\overline{C}_\psi, M_\psi)\right)$, $\overline{C}_\psi = \overline{C}_\varphi$

$$\boxed{\widetilde{V}_{(E,\psi)} = \bigoplus_{\substack{\varphi \in T^*_V \mathrm{Bun} \\ h(\varphi) = h(\psi)}} \langle L_\varphi, M_\psi \rangle_{\overline{C}_b}}$$

fiber of Poincaré bundle

$$\begin{tikzcd} & \mathrm{Hecke} \arrow[dl] \arrow[dr] & \\ \mathrm{Bun} & & \mathrm{Bun} \times C \end{tikzcd}$$

$$\begin{tikzcd} & \mathrm{Ab}\ \widetilde{\mathrm{Hecke}} \arrow[dl] \arrow[dr] & \\ \mathrm{Higgs} & & \mathrm{Higgs} \times C \end{tikzcd}$$

$(V, V', x, \beta) \ldots$

$$\left((V,\varphi), (V',\varphi'), x, \begin{tikzcd} V|_{C-x} \arrow[r, "\beta"] \arrow[d, "\varphi"'] & V'|_{C-x} \arrow[d] \\ V \otimes \omega_C \arrow[r, "\beta \otimes \mathrm{id}"] & V' \otimes \omega_C \end{tikzcd} \right)$$

$Gr(i, n)$

finite collection of φ_x-eigenspaces

$\widetilde{V}$ has an automorphic property w.r.t. $\widetilde{\mathrm{Hecke}}$.

To state this for all G, need: a duality between Higgs_G, $\mathrm{Higgs}_{{}^L G}$.

To finish:
* Average over all $\varphi \in T^*_V \mathrm{Bun}$?
* Or: deform
* Or: use Simpson's.

classical limit

λ-connections: $D(fs) = fDs + \lambda\, df$

$D : V \longrightarrow V \otimes \Omega^1$

$\lambda = 1$: connection

$\lambda \neq 0$: ditto

$\lambda = 0$: Higgs field.

Simpson: diffeomorphism.

As $\lambda \to 0$: $\quad (X = \mathrm{Bun}_{C, {}^L G})$

$\boxed{\begin{array}{l} \mathrm{Loc}_X \rightsquigarrow \mathrm{Higgs}_X \\ \mathcal{D}_X\text{-modules} \rightsquigarrow \mathrm{Sym}^{\bullet} T_X\text{-modules} = \text{coherent sheaves on } T^*_X. \end{array}}$

classical limit of GLC:

$$\boxed{D^b_{coh}(\mathcal{H}iggs^{\circ}_{C,G}) \simeq D^b_{coh}(\mathrm{Higgs}_{C,{}^L G})}$$

note: need to understand $\lim\limits_{\lambda \to 0} IC^{\lambda}$.

cf. Arinkin.

Classical limit conjecture:

$\exists$ natural equivalence of categories

$$\boxed{c : D^b(\mathcal{H}iggs) \xrightarrow{\sim} D^b({}^L\mathcal{H}iggs)}$$

inducing

$$\boxed{c^\circ : D^b(\mathcal{H}iggs^\circ) \xrightarrow{\sim} D^b({}^L\mathcal{H}iggs)}$$

c° sends str. sheaves of points in $\mathcal{H}iggs^\circ$ to Hecke eigensheaves:

$(V, \varphi) \in \mathcal{H}iggs^\circ \Rightarrow$

$$\boxed{H^\mu\,(c^\circ(V,\varphi)) = c^\circ(V,\varphi) \boxtimes (\rho^\mu(V), \rho^\mu(\varphi))}$$

D & Pantev 06/04/617: true over $B \setminus \Delta$.

Δ = discriminant, parametrizes singular cameral covers.

Underlying geometry: $\mathcal{H}iggs$, ${}^L\mathcal{H}iggs$ are dual integrable systems.

Hausel + Thaddeus: cases $GL(n)$, $SL(n)$.

Hitchin: G_2

Arinkin, Ngo: some info $/ \Delta$

Ngo: $\Rightarrow$ "Fundamental Lemma"

steps

$\boxed{\text{. Duality along the base:}}$

$\exists$ isomorphism $\ell : B \xrightarrow{\sim} {}^L B$

$\ell(b) = {}^L b$

lifts to : $\ell : \tilde{\mathcal{C}} \xrightarrow{\sim} {}^L\tilde{\mathcal{C}}$

interchanges short $\leftrightarrow$ long roots.

A, D, E, F, G self dual algebras $\Rightarrow$

$\ell : B \xrightarrow{\sim} B$.

A, D, E : $\ell = id$

F, G : $\ell \neq id$.

$$\boxed{\begin{tikzcd}
\tilde{\mathcal{C}} \arrow[r] \arrow[d] & \omega_C \otimes \underline{t} \arrow[r, "\sim"] \arrow[d] & \omega_C \otimes {}^L\underline{t} \arrow[d] & {}^L\tilde{\mathcal{C}} \arrow[l] \arrow[d] \\
B \times C = H^0(C, (\omega_C \otimes \underline{t})/W) \times C \arrow[r, "\text{eval}"] & (\omega_C \otimes \underline{t})/W \arrow[r, "\sim"', "\kappa"] & (\omega_C \otimes {}^L\underline{t})/W & \cdots B \times C \arrow[l]
\end{tikzcd}}$$

κ = any killing form $\underline{t} \xrightarrow{\sim} {}^L\underline{t}$

Duality along fibers

$T_{\mathbb{C}} = \Lambda \otimes \mathbb{C}^*$
$\Lambda = \mathrm{cochar}_G$

[D+ Gaitsgory]:

$p : \tilde{C} \to C$ cameral $\Rightarrow$

$$\overline{T} := p_*(\Lambda \otimes \mathcal{O}^*_{\tilde{C}})^W$$

$$T := \{ t \in \overline{T} \mid \alpha(t)|_{D^\alpha} = 1 \ \forall \text{ root } \alpha \text{ of } \mathfrak{g} \}$$

$\alpha : T \to \mathbb{C}^* \qquad \alpha^\vee : \mathbb{C}^* \to T$

$D^\alpha \subset \tilde{C}$: fixed divisor for reflection ρ^α

$$T^0 := \text{connected component of } T$$

$T^0 \subset T \subset \overline{T}$ sheaves of ab. gps. on C

[DG]: <u>$h^{-1}(b)$ is a torsor over $H^1(T)$</u>

<u>Real versions:</u> $\qquad \mathcal{O}^*_{\tilde{C}} \leftrightarrow S^1$

$$\overline{T}_{\mathbb{R},S} = \{ \lambda \otimes z \mid \alpha^\vee(z^{\langle\alpha,\lambda\rangle}) = 1 \text{ in } T_{\mathbb{R}} \}$$
$$\cup$$
$$T_{\mathbb{R},S} = \{ \lambda \otimes z \mid z^{\langle\alpha,\lambda\rangle} = 1 \text{ in } S^1 \}$$
$$\cup$$
$$T^0_{\mathbb{R},S} = \{ \lambda \otimes z \mid \langle\alpha,\lambda\rangle = 0 \text{ in } \mathbb{Z} \}$$

Hodge theory from [DDP 2005]:

holo + real sheaves have same H!

reason: Hodge theory $\Rightarrow$ finite ker, coker but the cone is a complex of $\mathbb{R}$ v.s.'s.

$\Rightarrow$ topological calculations.

$$\boxed{\begin{tikzcd}
\mathrm{Compo}^\circ & \mathrm{Compo} & \overline{\mathrm{Compo}} \\
H^1(C,T^\circ) \arrow[u] \arrow[r] & H^1(C,T) \arrow[u] \arrow[r] & H^1(C,\bar{T}) \arrow[u] \\
P^\circ \arrow[u, phantom, "\cup"] \arrow[r, two heads] & P \arrow[u, phantom, "\cup"] \arrow[r, two heads] & \bar{P} \arrow[u, phantom, "\cup"] & \text{ab. var's}
\end{tikzcd}}$$

$$\mathrm{Compo}^\circ = \begin{cases} \mathbb{Z}/2 & G = Sp(N) \\ \pi_1(G) & \text{else} \end{cases}$$

$\mathrm{Compo} = \pi_1(G)$ always

so:

compo's of Hitchin fiber $\leftrightarrow$ compo's of Higgs.

Key: $$\boxed{\mathrm{Compo}^\circ = H^1(\mathbb{Z}, \Lambda^\vee)^{\wedge}_{tor} = \left(\frac{(\Lambda^\vee)^\ell}{(1-\rho_1, \dots, 1-\rho_\ell)\Lambda^\vee} \right)_{tor}}$$

$\Rightarrow$ cochar(P) = cochar$({}^LP)^\vee$; $\hat{P} = {}^LP$.

Duality for Higgs gerbes

Higgs is a banded $\mathcal{T}$-gerbe

= over B, Higgs is a torsor over the comm. gp. stack $Tors_{\mathcal{T}}$.

$\exists$ Hitchin section $B \to$ Higgs $\Rightarrow$ it's the trivial torsor.

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Langlands duality and non-abelian Hodge theory

(Started with transparencies)

Non-abelian Hodge

Corlette - Simpson:

M compact Kähler

$$\mathrm{Higgs}^0(M) \xleftrightarrow{\text{diffeo}} \mathrm{Loc}(M)$$

non compact?

Simpson: OK for M = curve

parabolic bundles

	(E, θ)	(V, ∇)
jump	a	$a-2b$
eigenvalue	$b+ci$	$a+2ci$

~~Boalch~~ : all dim, D smooth

Biquard

condition
"locally abelian"

$c_1 = c_2 = 0$

parabolic chern classes

Sabbah
Jost-Kang-Zuo

Functoriality:

$\oplus, \ldots$
pullback OK
direct images

$z \mapsto z^n$

$a, b, c \qquad \frac{a}{n}, \frac{b}{n}, \frac{c}{n}$

<u>Big Picture</u>

$\mathcal{M} = \mathrm{Bun}$

$\widetilde{\mathcal{M}} = \mathrm{Higgs} \sim T^* \mathrm{Bun}$

$H \subset \mathcal{M} \times \mathcal{M}$: Hecke

$\widetilde{H} \subset \widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}}$: abelianized Hecke

<u>Equivalence</u>

$$\begin{tikzcd} \text{Sheaves on } \widetilde{M} \text{ finite}/M \arrow[r, leftrightarrow, "\text{spectral corresp.}"'] & \text{Higgs sheaves on } M \arrow[r, leftrightarrow, "\text{n.a.H}"'] & \text{loc. syst. on } M \\ \widetilde{H} \text{ acts} \arrow[u] & H \text{ acts} \arrow[u] \arrow[ur] & \end{tikzcd}$$

~~Compatibility with spectral~~

<u>Lemma</u> $\widetilde{H} \subset \widetilde{M} \times \widetilde{M}$

$$\widetilde{H} = N^*_H / T^*M \times T^*M$$

General compatibility:

N^*_H on spec-data $\Leftrightarrow$ H on Higgs

<u>proof</u>: Koszul

$$H(X, (E, \theta)) = H^{\cdot}(X, E \xrightarrow{\theta} E \otimes \Omega^1 \xrightarrow{\theta} E \otimes \Omega^2 \to \cdots), \quad X \supset X_0$$

exact where θ is injective

$\mathcal{S}$: supported where $\theta = 0$.

$$
\begin{tikzcd}
\text{quasi finite} \arrow[r, leftrightarrow] & \text{mero Higgs} & & \\
& & \text{parabolic Higgs} \arrow[ul] & \\
& \{c_1 = c_2 = 0\} \arrow[uu] \arrow[ur, hook] \arrow[rr, leftrightarrow] & & \text{parabolic } \mathcal{D}\text{-module}
\end{tikzcd}
$$

$$\mathrm{Higgs} \supsetneq T^* \mathrm{Bun}$$

$\exists$ unstable V

θ

(V, θ) stable.

$$
\begin{tikzcd}
\mathrm{Prym} \arrow[r, hook] \arrow[dr, dashed] & \mathrm{Higgs} \arrow[d, dashed] \\
& \mathrm{Bun}
\end{tikzcd}
$$

Blow up, get $\tilde{P} \to \mathrm{Bun}$.

Image?

<u>Thm</u> image = non very stables.

$$
\begin{tikzcd}
T^* \mathrm{Bun} \supset N = \bigcup_i \not\equiv N^* S_i \arrow[d] \arrow[dr] & \\
\mathrm{Bun} \supset & \bigcup_{i \neq 0} S_i
\end{tikzcd}
$$

non very stables or "wobbly".

Key: need wobblies to see

* NCD
* [illegible]
* $c_1 = c_2 = 0$

$SL(2)$

$(\mathbb{P}^1, 5 \text{ points})$

Bun depends on weights

wt < 0.4 $\quad$ $Bun = d P_4$

$0.4 < wt < 0.6$

$Bun = d P_5$

wobbly