

# Geometric Langlands & non-abelian Hodge

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Langlands Workshop

KITP

8/7/08

[-, Panter] Langlands duality for Hitchin systems 06 04 617

- , Simpson

- , Panter

## GLC review:

- roots, chars, weights
- Bun, Loc, dt
- Hecke correspondences
- Hecke operators, automorphic sheaves
- stacks & gerbes
- Examples:  $GL(n)$ ,  $GL(1)$ .

## Abelianization:

- Higgs bundles, Hitchin system
- Spectral & canonical covers
- Ancient approach: abelianized Hecke eigen sheaves
- $\rightarrow$  duality: Higgs " " Higgs.

## Classical limit:

- definition, conjecture, theorem, proof:
- duality along base
- " " fibers
- transcendental ingredient: Hodge theory
- duality for gerbes

## non-abelian Hodge:

- review
- Big Picture
- Compatibility

## Wobly bundles:

- setup
- examples

# Geometric Langlands Conjecture

$\exists$  natural equivalence of categories:

$$c: D^b(\text{Loc}) \xrightarrow{\sim} D^b({}^L\text{Bun}, \mathcal{D})$$

sending structure sheaves of points  $V$  in  $\text{Loc}$  to automorphic  $\mathcal{D}$ -modules on  ${}^L\text{Bun}$ :

$$H^m(c(\mathcal{O}_V)) = c(\mathcal{O}_V) \boxtimes \rho^m(V)$$

Notation:

$C; G, T, \mathfrak{g}, \mathfrak{t}; {}^L G, {}^L T, \text{etc.}$

roots $_{\mathfrak{g}}$	$\subset$ char $_G$	$\subset$ weights $_{\mathfrak{g}}$
coroots $_{\mathfrak{g}}$	$\subset$ cochar $_G$	$\subset$ coweights $_{\mathfrak{g}}$
 roots $_{\mathfrak{g}}$	 char $_{\mathbb{C}}$	 weights $_{\mathfrak{g}}$

$\text{Bun}, {}^L\text{Bun}$ : mod. sp. of semistable principal  $G, {}^L G$  bundles on  $C$

$\text{Loc}, {}^L\text{Loc}$ : mod. sp. of semistable  $G, {}^L G$  local systems  $V = (V, \nabla)$  on  $C$

$\text{Bun}, \text{Loc}$  etc.: the corresponding moduli stacks

# Hecke correspondence :=

moduli space of quadruples  $(V, V', x, \beta)$ :

- $V, V'$ : principal  $L_G$ -bundles on  $C$
- $x \in C$
- $\beta: V|_{C, x} \xrightarrow{\sim} V'|_{C, x}$  an isomorphism



$\lambda \in \text{char}_{L_G}^+ = \text{cochar}_G^+$  dominant character of  $L_G$

$\rho = \rho_\lambda$  irrep of  $L_G$  with h.v.  $\Rightarrow$

$\mu \in \text{cochar}_{L_G}^+ = \text{char}_G^+$  dom. char. of  $G$

$$\text{Hecke}^m := \left\{ (V, V', x, \beta) \mid \begin{array}{l} \forall \lambda \in \text{char}_{L_G}^+ \text{ w/ } \rho = \rho_\lambda \\ \rho(\beta): \rho(V) \rightarrow \rho(V') \otimes \mathcal{O}_C((m, \lambda)x) \end{array} \right\}$$

- Fibers of  $p, q$ :  $\infty$ -dim'l ind-schemes  
 $q^{-1}(V', x) = \text{affine Grassmannian}$

$\text{Hecke}^m \subset \text{Hecke}$ : f.d. closed subscheme

- $\text{Hecke}^m$  is smooth  $\Leftrightarrow m$  minuscule weight

$$\text{Hecke} = \varinjlim_m \text{Hecke}^m$$

- $p^m, q^m$ : proper, loc. trivial fibrations.

Hecke operators = integral transforms

$$H^m : D^b(\text{Loc Bun}, \mathcal{D}) \rightarrow D^b(\text{Loc Bun} \times \mathbb{C}, \mathcal{D})$$

$$m \mapsto \rho^m (p^m \otimes m \otimes \mathcal{I}(\mathbb{C}^m)) [d_m]$$

Similarly:  $H_x^m$ , for  $x \in \mathbb{C}$ .

$\Rightarrow$  Hecke algebra  $A$  generated by the  $H_x^m$ .  
Abelian.

A  $\mathcal{D}$ -module  $m$  on  $\text{Loc Bun}$  is a Automorphic sheaf / Hecke eigen sheaf w/ eigenvalue  $V = (V, \rho)$  a G-loc. sys. on  $\mathbb{C}$  if:

$$H^m(m) = m \otimes \rho^m(V)$$

With this setup, GLC makes sense, except...

Discrepancy:  ${}^L\text{Bun}$  is disconnected.

$$\pi_0({}^L\text{Bun}) = H^2(C, \pi_1({}^L\mathbb{G})) = \pi_1({}^L\mathbb{G}) = \mathbb{Z}/C^A \Rightarrow$$
$$D^L({}^L\text{Bun}, \mathcal{D}) = \coprod_{\gamma \in \mathbb{Z}/C^A} D^L({}^L\text{Bun}^\gamma, \mathcal{D})$$

while  $\text{Loc}$  is irreducible  $\Rightarrow D^L(\text{Loc})$  indecomposable.

So: replace  $\text{Loc}$  by

$\text{Loc} :=$  moduli stack of semistable  $G$ -local systems

$\text{Loc}$  is an alg. stack of course moduli space =  $\text{Loc}$ .

$\exists$  open substack  $\text{Loc}^{\text{rs}}$  of "regularly stable" local systems

$\text{Loc}^{\text{rs}} \rightarrow \text{Loc}^{\text{rs}} : \text{banded } \mathbb{Z}/C^A\text{-gerbe}$

$$D^L(\text{Loc}^{\text{rs}}) = \coprod_{\gamma \in \mathbb{Z}/C^A} D^L(\text{Loc}^{\text{rs}}, \gamma).$$

Example:  $G = {}^L G = GL(n)$

$V = (V, \sigma) = \text{rank } n \text{ loc. sys. / } C$

Hecke algebra  $A$  is generated by  $H^i$ :

Hecke  $i = \{(V, V', x) \mid V \subset V' \subset V(x), i = \mathcal{L}(V'/V)\}$   
fiber =  $Gr(i, n)$ .

Example:  $G = GL(1)$ ,  $\text{Bun} = \text{Pic}$

$H^i: C \times \text{Pic}^d(C) \rightarrow \text{Pic}^{d+i}(C)$

$x, L \mapsto L(x)$  Abel-Jacobi

$V = (L, \sigma)$

$c(V) = \text{the unique loc. sys. on } \text{Pic}(C)$   
restricting to  $(L, \sigma)$ .

$(\pi, (\text{Pic } C) = \pi, (C) / (C, \sigma))$

$$\text{Higgs} = \text{Higgs}_{C, G} = \{ (V, \varphi) \mid \varphi: V \rightarrow V \otimes \omega_C \}$$

Hitchin map:

$$h: \text{Higgs}_{C, G} \rightarrow B_{C, G} := H^0(C, (\omega_C \otimes \mathbb{Z}) / \omega) = \bigoplus_{i=1}^r H^0(C, \omega_C^{\otimes d_i})$$

$$(V, \varphi) \mapsto \text{"}\varphi \text{ mod } \omega\text{"} \leftrightarrow (I_1(\varphi), \dots, I_r(\varphi))$$

Commutative covers:

$$C \subset \tilde{C} \rightarrow \text{Tot}(C, \omega_C \otimes \mathbb{Z})$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$C \times B \subset C \times B \rightarrow \text{Tot}(C, (\omega_C \otimes \mathbb{Z}) / \omega)$$

$$h^{-1}(L) \sim \text{Frym}(C_2 / C)$$

$h: \text{Higgs} \rightarrow B$  integrable system

$$\text{Higgs} \supset T^* B_{\text{un}}$$

$B_{\text{un}} \Rightarrow$  associated spectral covers  $\tilde{C}_n^{\text{un}}, \tilde{E}^{\text{un}} \rightarrow C \times B$ .

e.g.  $G = \text{GL}(n) \Rightarrow \text{dy}(\tilde{C}/C) = n$ .

[De Gaiety '00]:  $\pi: \tilde{E} \rightarrow C \times B$  determines an abelian group scheme  $T$  over  $C \times B$ .  $h: \text{Higgs} \rightarrow B$  is a principal homogeneous stack over the ab. group stack  $\text{Tors}_T$  of  $T$ -torsors.

= "abelianization".

(Hitchin, Atiyah, Beilinson-Kazhdan, Faltings, ...)



# Ancient approach

$$(G = GL(n))$$

$$V = (V, \sigma) \Rightarrow$$

$$L \text{ on } \bar{E}_V, \pi_{V*} L \cong V \otimes W_C^{-\frac{(n-1)}{2}}$$

$\sigma$  induces hol. conn. on  $L$

(Depends on  $\sqrt{W_C}$ : use  $\text{Ram}_\sigma \cong \pi_\sigma^* W_C \otimes (n-1)$ )

The natural induced connection  $\sigma'$  has residue  $= \frac{1}{2}$  along  $\text{Ram}_\sigma$ . Set  $\sigma = \sigma' + \frac{1}{2} d \log f_\sigma$ , where

$$(f_\sigma) = -\text{Ram}_\sigma + 2(n-1) \pi_\sigma^* f, \quad f \text{ a section of } \sqrt{W_C}$$

Use case  $G = GL(1)$ : spread  $(L, \sigma)$  to  $(\tilde{L}, \tilde{\sigma})$  on

$$\text{Pic}(\bar{E}_V / C \times T_V^* \text{Bun}) =: \widetilde{\text{Higgs}} = \text{Higgs} \times_B T_V^* \text{Bun}$$

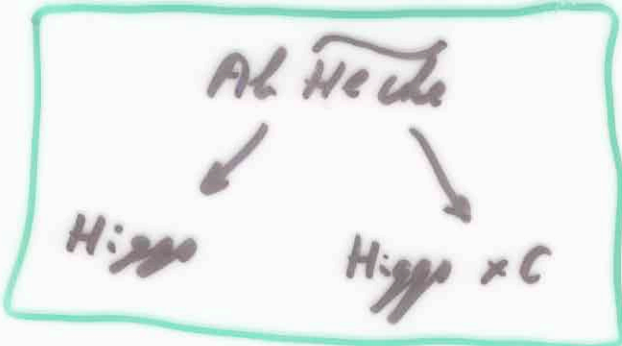
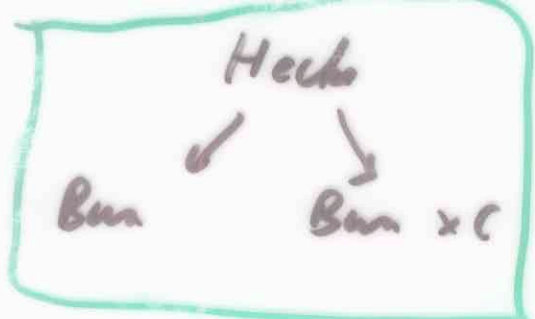
Now push to Higgs  $\Rightarrow$

$(\tilde{V}, \tilde{\sigma})$  on Higgs, more loc. exp along fibers over  $B$ .

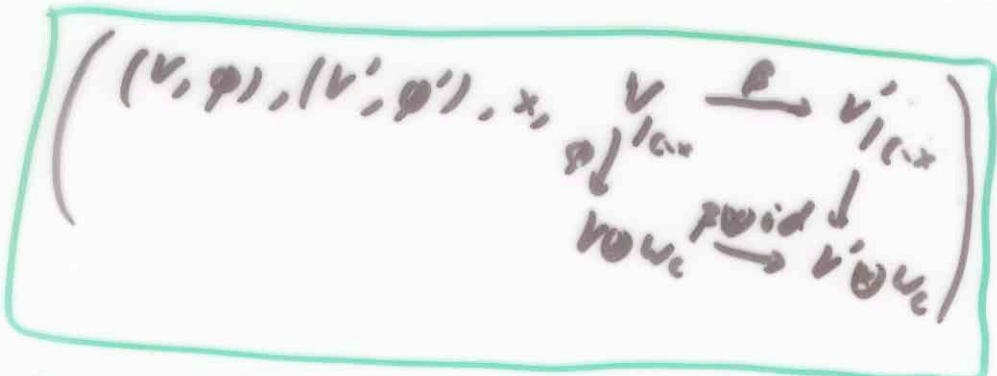
Fiber of  $\tilde{V}$  at  $(E, \psi) \in \text{Higgs}$ :  $(E, \psi) \leftrightarrow (\bar{E}_\psi, M_\psi) =: \bar{E}_\psi$

$$\tilde{V}_{(E, \psi)} = \bigoplus_{\substack{\varphi \in T_V^* \text{Bun} \\ h(\varphi) = h(\psi)}} \langle L_\varphi, M_\psi \rangle \bar{E}_\psi$$

↑  
fiber of Poincaré bundle



$(V, V', x, \rho) \dots$



$G \times (i, \nu)$

finite collection of  $\varphi_i$ -eigenspaces

$\bar{V}$  has an automorphic property w.r.t. Hecke.

To state this for all  $G$ , need: a duality between  $\text{Higgs}_G$ ,  $\text{Higgs}_{G^*}$ .

To finish:

- \* Average over all  $\varphi \in T^* \text{Bun}$ ?
- \* Or: reform
- \* Or: use Simpson's.

## classical limit

$\lambda$ -connections:  $D(V_S) = \rho_S \times \lambda d/$

$$D: V \rightarrow V \otimes \Omega^1$$

$\lambda=1$ : connection

$\lambda \neq 0$ : dilts

$\lambda=0$ : Higgs field.

Simpson: diffeomorphism.

As  $\lambda \rightarrow 0$ :

( $X = \text{Bun}_C, \mathcal{L}_C$ )

$\text{Loc}_X \rightarrow \text{Higgs}_X$

$\mathcal{D}_X$ -modules  $\rightarrow \text{Sym}^* T_X$ -modules = coherent sheaves on  $T_X^*$ .

classical limit of GLC:

$$D_{\text{coh}}^b(\text{Higgs}_{C, \mathcal{L}_C}^0) \cong D_{\text{coh}}^b(\text{Higgs}_{C, \mathcal{L}_C})$$

note: need to understand  $\lim_{\lambda \rightarrow 0} \mathbb{Z}^n$ .  
cf. Arinkin.

## Classical limit conjecture:

$\exists$  natural equivalence of categories

$$c: D^b(\mathcal{Higgs}) \xrightarrow{\cong} D^b({}^L\mathcal{Higgs})$$

inducing

$$c^0: D^b(\mathcal{Higgs}^0) \xrightarrow{\cong} D^b({}^L\mathcal{Higgs}^0)$$

$c^0$  sends str. sheaves of points in  $\mathcal{Higgs}^0$  to Hecke eigen sheaves:

$$(V, \rho) \in \mathcal{Higgs}^0 \Rightarrow$$

$$H^m(c^0(V, \rho)) \cong c^0(V, \rho) \boxtimes (\rho^{-1}(V), \rho^{-1}(\rho))$$

D & Panter 06/04/617: true over  $B \setminus \Delta$ .

$\Delta$  = discriminant, parametrizes singular conical covers.

Underlying geometry:  $\mathcal{Higgs}, {}^L\mathcal{Higgs}$  are dual integrable systems.

Hammel & Thaddeus: cases  $GL(n), SU(n)$ .

Hitchin:  $G_2$

Arinkin, Ngo: some info /  $\Delta$

Ngo:  $\Rightarrow$  "Fundamental Lemma"

# Steps

## Duality along the base:

$$\exists \text{ isomorphism } \ell: \mathfrak{B} \xrightarrow{\sim} {}^L \mathfrak{B}$$
$$\ell(\alpha) = \alpha^\vee$$

$$\text{lifts to: } \ell: \tilde{\mathfrak{E}} \xrightarrow{\sim} {}^L \tilde{\mathfrak{E}}$$

interchanges short & long roots.

A, D, E, F, G self dual algebras  $\Rightarrow$

$$\ell: \mathfrak{B} \xrightarrow{\sim} \mathfrak{B}.$$

$$A, D, E: \ell = \text{id}$$

$$F, G: \ell \neq \text{id}.$$

$$\begin{array}{ccccccc} \tilde{\mathfrak{E}} & \xrightarrow{\quad} & \mathfrak{W}_C \otimes \mathfrak{t} & \xrightarrow{\sim} & \mathfrak{W}_C \otimes \mathfrak{t}^\vee & \xleftarrow{\quad} & {}^L \tilde{\mathfrak{E}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathfrak{B} \times \mathfrak{C} = H^0(C, (\mathfrak{W}_C \otimes \mathfrak{t})/\mathfrak{w}) \times \mathfrak{C} & \xrightarrow{\text{incl}} & (\mathfrak{W}_C \otimes \mathfrak{t})/\mathfrak{w} & \xrightarrow{\sim} & (\mathfrak{W}_C \otimes \mathfrak{t}^\vee)/\mathfrak{w} & \xrightarrow{\sim} & \mathfrak{B} \times \mathfrak{C} \end{array}$$

$\mathfrak{w}$  = any Killing form  $\mathfrak{t} \subset \mathfrak{g} \subset \mathfrak{t}^\vee$

# Duality along fibers

$$\begin{aligned} T_C &= \Lambda^2 C^* \\ \Lambda &= \text{coker } \rho \end{aligned}$$

[D+ Gaiitsgorj]:

$$p: \tilde{C} \rightarrow C \quad \text{canonical} \quad \Rightarrow$$

$$\tilde{T} := p_* (\Lambda \otimes \mathcal{O}_{\tilde{C}}^*)^{\vee}$$

$$T := \{t \in \tilde{T} \mid \langle t, t \rangle_{D^*} = 1 \text{ } \forall \text{ root } \alpha \text{ of } \rho\}$$

$$\alpha: T \rightarrow C^* \quad \alpha^\vee: C^* \rightarrow T$$

$D^* \in \tilde{C}$ : fixed divisor for reflection  $\rho^\vee$

$T^0 :=$  connected component of  $T$

$T^0 \subset T \subset \tilde{T}$  sheaves of ab. gps. on  $C$

[DG]:  $L^\vee(\mathcal{F})$  is a torsor over  $H^1(\mathcal{F})$

Real versions:  $\mathcal{O}_{\tilde{C}}^* \leftrightarrow S'$

$$\tilde{T}_{R,S} = \{\lambda \otimes \tau \mid \langle \lambda, \tau \rangle_{\mathcal{O}_{\tilde{C}}^*} = 1 \text{ in } T_{R,S}\}$$

$$\begin{aligned} \cup \\ T_{R,S} &= \{\lambda \otimes \tau \mid \langle \lambda, \tau \rangle_{S'} = 1 \text{ in } S'\} \\ \cup \end{aligned}$$

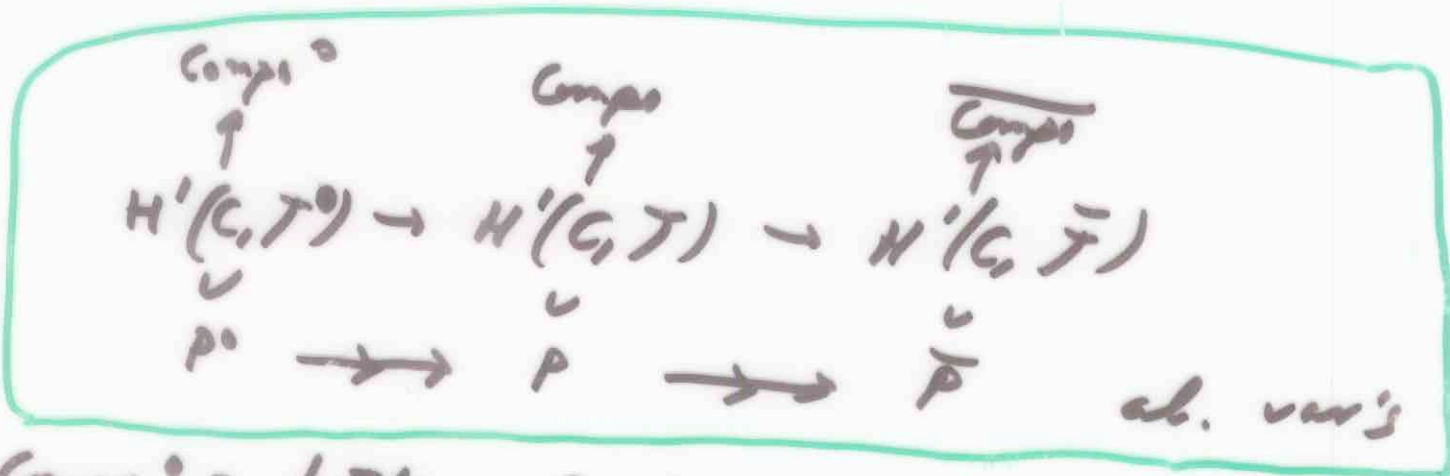
$$\begin{aligned} \cup \\ T^0_{R,S} &= \{\lambda \otimes \tau \mid \langle \lambda, \tau \rangle = 0 \text{ in } \mathcal{O}\} \end{aligned}$$

Hodge theory from [DDP 2005]:

holo + real sheaves have same  $H^i$

reason: Hodge theory  $\Rightarrow$  finite her, coher  
but the cone is a complex of  $\mathbb{R}$  v.s.  $\mathbb{C}$ .

$\Rightarrow$  topological calculations.



$$\text{Comp}^0 = \begin{cases} \mathbb{R}/\mathbb{Z} & G = Sp(N) \\ \pi_1(G) & \text{else} \end{cases}$$

$$\text{Comp} = \pi_1(G) \text{ always}$$

so:

Comp's of Hitchin fiber  $\leftrightarrow$   
Comp's of Higgs.

Key:  $\text{Comp}^0 = H^1(\mathbb{R}, \mathbb{A}^1) \hat{=} \text{tor}$   
 $= \left( \frac{(\mathbb{A}^1)^{\mathbb{Z}}}{(1-p_1, \dots, 1-p_r) \mathbb{A}^1} \right) \text{tor}$

$$\Rightarrow \text{coker}(P) = \text{coker}(P)^{\vee}, \quad \hat{P} = \text{cp.}$$

Duality for Higgs gerbes

Higgs is a banded  $T$ -gerbe

= over  $B$ , Higgs is a torsor over  
the comm. gp. stack  $Tors_T$ .

$\exists$  Hitchin section  $B \rightarrow \text{Higgs} \Rightarrow$   
it's the trivial torsor.



7 August 2008  
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## Langlands duality and non-abelian Hodge theory

(Started with transparencies)

### Non-abelian Hodge

Carlette - Simpson:

$M$  compact (Kähler  
diff)

$$Higgs^0(M) \longleftrightarrow \text{Loc}(M)$$

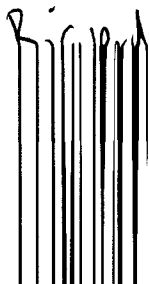
non compact?

Simpson: OK for  $M = \text{curve}$

parabolic bundles

	$(E, \theta)$	$(V, \nabla)$
Jump	$a$	$a - 2b$
eigenvalue	$b + ci$	$a + 2ci$

Boatch : all dim,  $D$  smooth



Conditions

"locally abelian"

$$c_1 = c_2 = 0$$

Parabolic Chern classes

Sabbah

Jost-Kang-Zins

Functoriality:

$$\oplus, \dots$$

pullback OK

direct images

$$Z \mapsto Z^n$$

$$\otimes a, b, c \quad \frac{a}{n}, \frac{b}{n}, \frac{c}{n}$$

### Big Picture

$$M = \text{Bun}$$

$$\tilde{M} = \text{Higgs} \sim T^* \text{Bun}$$

$$H \subset M \times M: \text{Hedge}$$

$$\tilde{H} \subset \tilde{M} \times \tilde{M}: \text{abelianized Hedge}$$

Equivalence



~~Compatibility with spectral~~

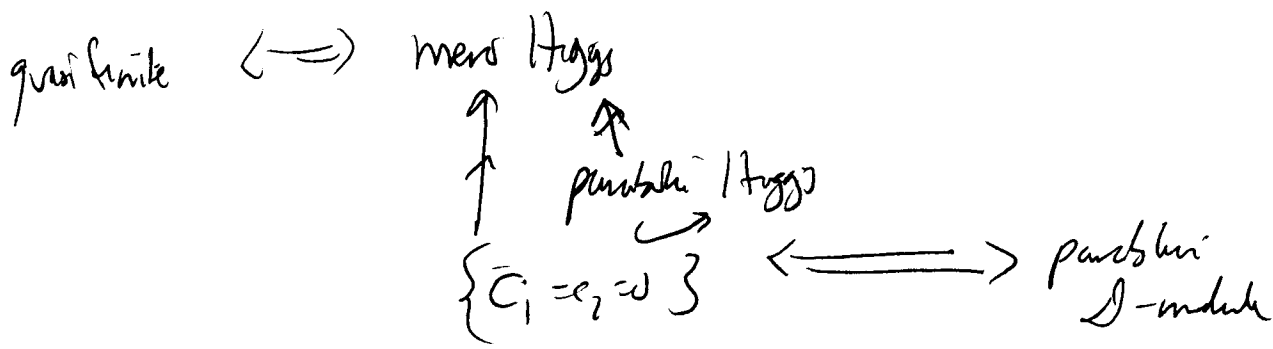
Lemma  $\widetilde{H} \subset \widetilde{M} \times \widetilde{M}$   
 $\parallel$   
 $N_{H^*} / T^*M \times T^*M$

General compatibility:

$N_{H^*}$  on spec-data  $(=) H$  on Higgs

Proof: Koszul

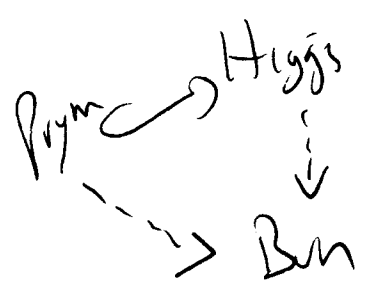
$H(x, (E, \theta)) = H(x, E \xrightarrow{\theta} E \otimes \mathcal{O}(1) \xrightarrow{\theta} E \otimes \mathcal{O}(2) \rightarrow \dots)$   
 exact when  $\theta$  is injection  
 $\mathcal{E} =$  supported when  $\theta = 0$ .



$Higgs \supsetneq T^*Ben$

$\exists$  unstable  $V$   
 $\emptyset$

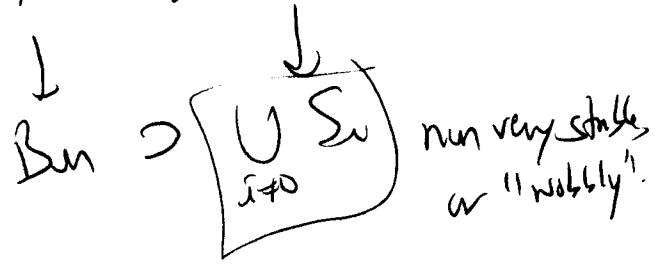
(V) stable.



Blur up, get  $\tilde{p} \rightarrow Ben$ .  
 Image?

Then image = non very stable.

$T^*Ben \supset N = \cup_i N^*S_i$



Key: need weights to be

\* NCD

\* kind abelian

\*  $c_1 = c_2 = 0$

SL(2)

$(\mathbb{P}^1, 5 \text{ points})$

Ben depends on weights

$w_1 < 0.4$  Ben =  $dP_4$

$0.4 < w_1 < 0.6$   
Ben =  $dP_3$

