6 August 2008
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Topological Strings and D-modules

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X CY_3, Ω $(3,0)$-form

<u>A-model</u> $\Sigma_g \xrightarrow{d} X$

$$GW_{g,d} = \int_{[\bar{M}_g(X,d)]^{vir}} 1 \in \mathbb{Q}$$

$$t \in H^{1,1}(X,\mathbb{C})$$

$$F_g(t) = \sum_d GW_{g,d}\, e^{-dt} + \begin{cases} \frac{1}{6}\int t^3 & g=0 \\ \frac{1}{12}\int c_2 \wedge t & g=1 \end{cases}$$

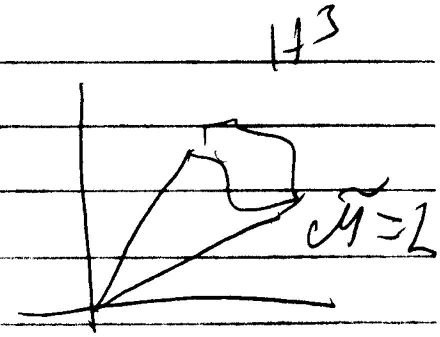
$$Z(t,\lambda) = \exp \sum_{g \geq 0} \lambda^{2g-2} F_g(t)$$

$$\lambda^2 \sim \hbar$$

<u>B-model</u>

$$\tilde{\mathcal{M}}_X \xrightarrow{\text{period map}} H^3(X,\mathbb{C})$$

$$\Omega \longmapsto [\Omega]$$



$t^i = \oint_{A^i} \Omega, \quad p_i = \oint_{B^i} \Omega$ $\qquad \omega = dt^i \wedge dp_i$

$\qquad\qquad\qquad\qquad \shortparallel \qquad\qquad\qquad\qquad L = \text{graph}(d\mathcal{F}_0)$

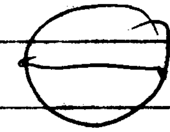
$\qquad\qquad\qquad \dfrac{\partial \mathcal{F}_0}{\partial t_i}$

$Z \sim e^{\mathcal{F}_0/\lambda^2}$ ~~WKB approx. quantization of $\mathcal{M}_X$~~

$\mathcal{F}_1$ = analytic torsion $= \sum \frac{1}{2} pq(-1)^{p+q} \log \det{}' \Delta_{p,q}$

$\mathcal{F}_g$: given cubic graphs $\quad \mathcal{F}_2$

$g \geq 2$



Kodaira-Spencer

$$Z = \int D\phi \, e^{\int \partial\phi \wedge \bar\partial\phi + \lambda A^3}$$

CS-like

~~$\phi \in \Omega^{1,1}(X)$~~

$A = t + \partial\phi \in \Omega^{2,1}$

$t \in H^{2,1}(X)$ harmonic

$\mu \in \Omega^{0,1}(T_X)$

$A = \iota_\mu \Omega$

$$A^3 = \int \underbrace{A}_{b,1} \wedge \underbrace{(\mu \wedge \mu)}_{b,2} \cdot \Omega$$

Natural action $\mathrm{Diff}(X)$, factors through $Sp(\cancel{2b}\, b_3, \mathbb{Z})$

act in the metaplectic

3. Donaldson-Thomas : branes in B-model

$\mathcal{E}$ coherent sheaf, $\mu = [\mathcal{E}] \in K^0(X)$

$$DT(\mu) = \int_{[\overline{\mathcal{M}}_{\mathcal{E}}]^{vir}} 1 \in \mathbb{Z}, \quad \text{"BPS invariant"}$$

rank $\mathcal{E} = 1$, Ideal sheaves

$$ch_2 = d, \quad ch_3 = n$$

$$Z \sim \sum DT_{d,n} e^{-dt} q^n, \quad q = e^{\lambda}$$

4. Gopakumar-Vafa

$\mathcal{E}$ sheaf loc. on a curve $Z \subset X$

$[\mathcal{E}] = d$, $ch_3 = n \Rightarrow GV_{d,n} \in \mathbb{Z}$

$$Z = \prod_{n_1, n_2 \geq 0, n, d} \left(1 - q^{n_1 + n_2 + n} e^{-dt}\right)^{-GV_{d,n}}$$

ex $X = \mathbb{C}^3$

$$\cancel{Z = \prod (1-q^n)^{-n}}$$

$$Z = \prod (1-q^{n_1 + m c})^{-1}$$

$$= \sum_{\pi} q^{|\pi|}$$

plane partition

$$= \exp \sum_{g \geq 1} \underbrace{\frac{B_{2g} B_{2g-2}}{2g(2g-2)(2g-2)}}_{\int_{\overline{M}_g} c_{g-1}^3} \lambda^{2g-2}$$

Z be a τ-function of (underlying) integrable hierarchy.

X based on a local curve C.

Base $B = E \times E' \rightsquigarrow \mathbb{C}^* \times \mathbb{C}^* \rightsquigarrow \mathbb{C} \times \mathbb{C}$

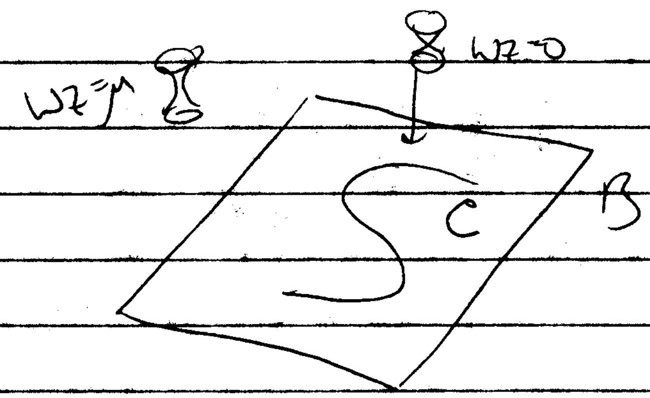
HK, hol. sympl. form $dx \wedge dy$ $(x \bmod 2\pi i, 2\pi i \tau)$

$\mathcal{C} \subset B$ $\quad H(x,y)=0.$

hol. curve

CY_3 X: hypersurface $\mathbb{C}^2\times B$

$$WZ + H(x,y) = 0.$$



$$\Omega = \frac{dw\wedge dx\wedge dy}{w}.$$

"Seiberg-Witten" case

$$\omega = dx\wedge dy$$

locally, on $\mathcal{C}$: $\omega = d\eta$

$$\eta = y\,dx$$

SW-differential

Periods of Ω $\Rightarrow$ periods of η on $\mathcal{C}$

Consider B-model on X ~ $\mathcal{C}$

[exact solution, corresponding to random matrix models —]

$$\int_{N\times N} dX\, e^{\frac{1}{\lambda}\mathrm{Tr}\, W(X)} = Z$$

$H(x,y) = y^2 + V(x) = 0$

$V(x) = W'(x)^2 + \dots$

Moduli space $(\mathcal{C}, \eta)$

g-dim^l

Eynard - Orantin

KS - theory on $\mathcal{C}$

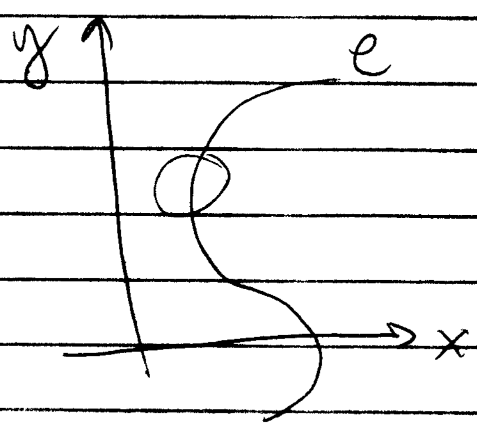
Field $\varphi : \mathcal{C} \to \mathbb{C}$ $\quad [\varphi \in \Omega^{1,1} \text{ on } CY]$

$$S = \int_{\mathcal{C}} \partial\varphi\,\bar\partial\varphi + \frac{\eta}{\lambda}\,\bar\partial\varphi + \frac{\lambda}{\eta}(\partial\varphi)^2\bar\partial\varphi$$

$A^3 \sim \frac{1}{\Omega}$

$$Z = \int \mathcal{D}\varphi\, e^{-S}$$

issues when $\eta = y\,dx = 0$.



$y\,dx = 0$

$\swarrow$ $y = 0$ do not contribute

$\searrow$ "$dx = 0$" branch points do contribute

$F_g \Rightarrow$ computed in terms of branch pts.

Boson - Fermion correspondence $\Rightarrow$ integrable hierarchies

$$\partial\varphi = \sum \alpha_n x^{-n-1}\, dx = \psi^{+}\psi(x)$$

$$\psi = e^{\varphi},\ \psi^{+} = e^{-\varphi}$$

KP-formalism

$\mathcal{F}$ = Fock space of free fermion

semi-∞ wedge repr.

H, basis $\{z^n\}_{n\in\mathbb{Z}}$
$\cup$
H_+ basis, $\{z^n\}_{n\geq 0}$

$W \subset H$, comp. to H_+

$|0\rangle \in \mathcal{F}$, $|0\rangle = z^0 \wedge z^1 \wedge z^2 \wedge \cdots$

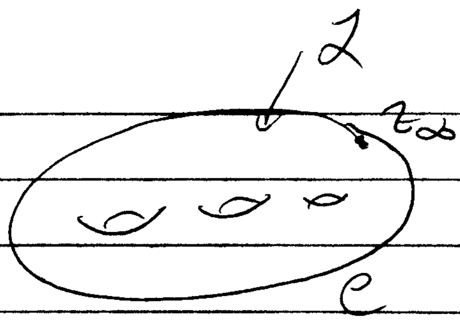
$|W\rangle = w^0 \wedge w^1 \wedge \cdots$

τ-function of KP

$GL(\infty, \mathbb{C})$ in $\mathcal{F}$

Cartan subgroup

$$\langle 0 | e^{\sum t_n \alpha_n} | W\rangle = \tau_W(t)$$



$$\psi \in \underbrace{H^0(K^{1/2} \otimes \mathcal{L}, \mathcal{C} \setminus \{\infty\})}_{W}$$

$\Rightarrow$ geom. solution to KP

$$Z_W(t) = \det(\bar{D}_{\mathcal{L}})$$
$$= \int \mathcal{D}\varphi \, e^{\int \partial\varphi \bar{\partial}\varphi}$$

Where is $\lambda = \hbar$?

KdV solution

$$P = \partial_x^2 + a(x)$$

$$P_+^{n+1/2}$$

$$Q = P_+^{3/2}$$

geom. solution $[P, Q] = 0 \Rightarrow$

top. strings/matrix models $[P, Q] = \lambda$

$$P = z, \quad Q = \lambda \frac{\partial}{\partial z}$$

$$\frac{\eta}{\lambda} \bar{\partial}\varphi \sim \frac{F_0}{\lambda^2}$$

$$\partial\varphi = \frac{1}{\lambda}\eta + \partial\varphi^{qu}$$

<u>dualities</u> $\Rightarrow$ closed top. string on X (based on $\mathcal{C}$)

D-branes in $\mathbb{C}^3 = \mathbb{C}\times B$

IIA string

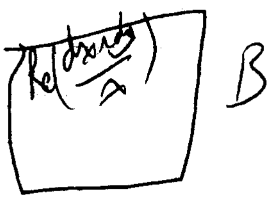
$\mathbb{R}^3 \times B \times \tilde{\mathbb{R}}^3$

$\cap = \mathcal{C}$

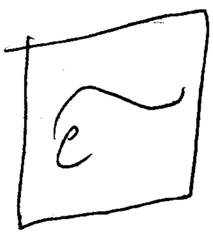
(i) D6-brane $B \times \tilde{\mathbb{R}}^3$

(ii) D4-brane $\mathcal{C} \times \mathbb{R}^3$

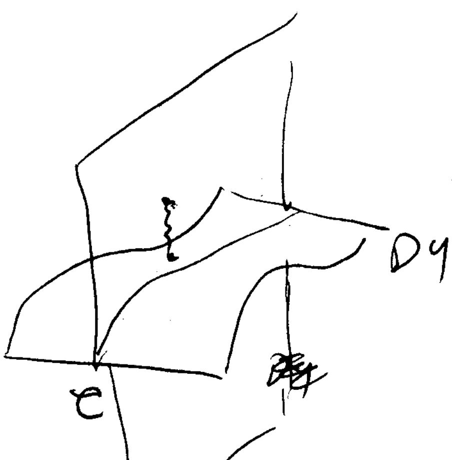
(iii) B-field on B given by $\frac{dx\wedge dy}{x}$



$\rightsquigarrow B_{cc}$



$\rightsquigarrow B'$



$\mathrm{Hom}(B', B_{\mathcal{C}})$ = functions on $\mathcal{C}$, or sections of $K_{\mathcal{C}}^{1/2}$

$\Rightarrow$ 2d QFT on the curve $\mathcal{C}$

$\Rightarrow$ Free Fermions

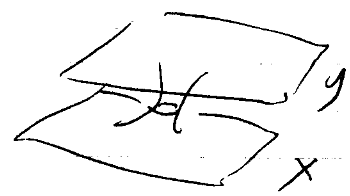
$$\int_{\mathcal{C}} dz\, \psi^{+} \bar{D} \psi$$

However, ψ are module $\mathrm{Hom}(B_{cc}, B_{cc})$

$$\mathcal{O}_B = \mathbb{C}[x,y] \rightsquigarrow \mathcal{D}_{\mathcal{I}}$$

$$[x,y] = \lambda$$

Curve $\mathcal{C}$ $\quad xy = \mu$



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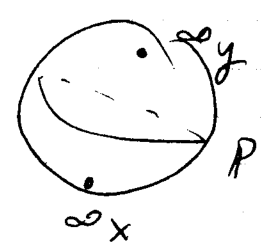
$\mathcal{D}$-module $\quad D = x\frac{\partial}{\partial x} - \mu$ $\qquad \tilde{t}_n$



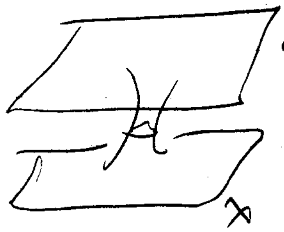
t_n

$\Rightarrow$

$$|t\rangle = \exp \sum t_n \alpha_{-n} |0\rangle$$

$$Z = \langle \tilde{t} | \mu^{L_0} | t \rangle \qquad y = \frac{\mu}{x}$$

$$= \exp \sum n t_n \tilde{t}_n \mu^n$$



$y \leftarrow \tilde{\psi}(y)$

$x \leftarrow \psi(x)$

$(x,y) \to (y,-x)$

$[x,y] = \lambda$

Fourier transform

$\tilde{\psi}(y) = S\psi(y) = \int e^{\frac{x\cdot y}{\lambda}} \psi(x)$

$\psi(x) = \psi_{cl}\cdot\psi_{qu}$, $\psi_{cl} = e^{\varphi} = x^{\mu}$

$\partial\varphi = y\,dx = \mu\frac{dx}{x}$

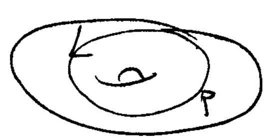
$\tilde{\psi}_{qu}(y)\, y^{\mu} = \int dx\, e^{\frac{xy}{\lambda}}\, x^{\mu}\, \psi_{qu}(x)$

$S_{\mu} : \mathcal{F} \to \mathcal{F}$

$\lambda \to 0$ $\tilde{\psi}_{qu}(y) = \psi_{qu}(x(y))$ $x(y) = \frac{\mu}{y}$

$Z = \langle \tilde{F} | S_{\mu} | t \rangle$

Sol to Toda eqn.



$A = \eta\lambda\,\partial\varphi^{qu}$